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A Simple Counterexample to Kouchnirenko's Conjecture

Bertrand Haas

*Department of Mathematics, Wells Hall, Michigan State University
East Lansing, Michigan 48824-1027, U.S.A.
e-mail: haasb@math.msu.edu*

Abstract. In the late 70's A. Kouchnirenko posed the problem of bounding from above the number of *positive* real roots (that is, with positive coordinates) of a system of k polynomial equations in k variables not in function of the degrees of the polynomials (like in Bezout Theorem), but in function of the number of terms involved, and he conjectured an upper bound. Although he never wrote this conjecture (as far as I know), many references to it can be found ([3] for one of the first and [7] for one of the most recent). I recently learned that Kouchnirenko himself was "100% sure" that his conjecture was false, since a colleague of his once presented him a simple counterexample, a system of two *threenomial* (polynomials with three terms) equations in two variables with 5 positive roots while the conjecture predicts at most 4. His colleague tragically died soon afterward. The counterexample was lost and Kouchnirenko never found it again. Here such a counterexample is presented, maybe the lost counterexample found again ...

Kouchnirenko's Conjecture

Let $f_1 = f_2 = \cdots = f_k = 0$ be a system of polynomial equations in k variables and m_i be the number of terms of f_i. The number of non-degenerate isolated positive roots of this system is less or equal to $(m_1 - 1)(m_2 - 1)\cdots(m_k - 1)$.

Remark 1. It is necessary to assume the roots to be non-degenerate in the sense that the (real) zero set of each polynomial must have pure codimension one. Indeed, if we omit this condition, then counterexamples where the zero set of one polynomial has codimension

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$k-1$, where $k > 2$ are already known (B. Sturmfels [7] mentions one due to W. Fulton). Such counterexamples use the well known equivalence over the reals of a system of equations $f_1 = f_2 = \cdots = f_k = 0$ with one equation $f_1^2 + f_2^2 + \cdots + f_k^2 = 0$.

Remark 2. Of course all the roots, not only the positive ones, of a system of polynomials $f_1 = f_2 = \cdots = f_k = 0$ are worth interest, but notice that the roots in any open octant are equal to the roots in the open first octant of the system obtained by replacing in the polynomials each variable x_i by $\pm x_i$, accordingly, and the roots in any coordinate hyperplane are the roots of the system obtained by replacing the corresponding x_i by 0. This is one main reason why Koushnirenko's problem focuses on the positive roots only.

In particular the conjecture predicts that a system of two threenomial equations in two variables has no more than four non-degenerate isolated positive roots.

Theorem. *There exists a real number $t_0 > 0$ such that, for any positive integers a and b for which $0 < a/b \leq t_0$, the following system of polynomial equations has five non-degenerate isolated positive roots.*

$$\begin{aligned} y^{2b} + 1.1x^b - 1.1x^a &= 0 \\ x^{2b} + 1.1y^b - 1.1y^a &= 0 \end{aligned}$$

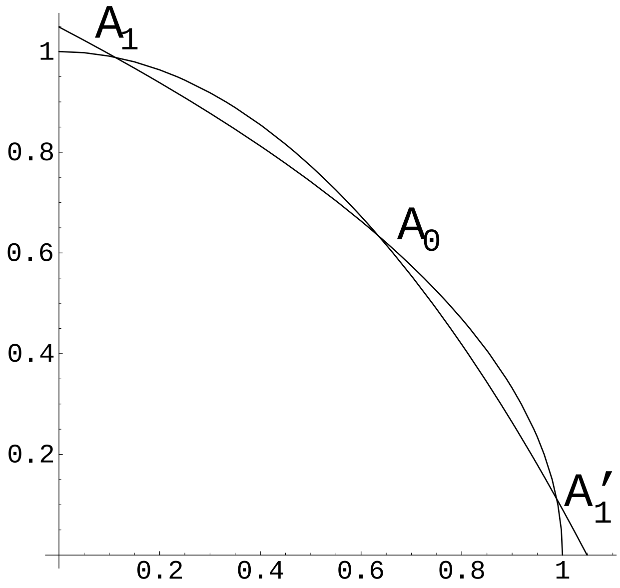

Figure 1. The two initial arcs of parabolas drawn by *Mathematica*

1. The idea of the proof

The zero sets C_1 and C_2 in $\mathbb{R}^2_{\geq 0}$ of the following two threenomials

$$\begin{aligned} P_1(x,y) &= y^2 + 1.1x - 1.1 \\ P_2(x,y) &= x^2 + 1.1y - 1.1 \end{aligned}$$

are two arcs of parabolas which intersect transversally in three points as can be seen on Figure 1 drawn by *Mathematica*. One can actually replace all the coefficients 1.1 by $1+\epsilon$ for ϵ sufficiently small. Observe that C_1 is symmetric to C_2 with respect to the diagonal line $y = x$.

The zero sets $C_{1,t}$ and $C_{2,t}$ in $\mathbb{R}^2_{\geq 0}$ of the following deformations of P_1 and P_2:

$$\begin{aligned} P_{1,t}(x,y) &= y^2 + 1.1x - 1.1x^t \\ P_{2,t}(x,y) &= x^2 + 1.1y - 1.1y^t \end{aligned}$$

form two continuous one-parameter families of diffeomorphic curves in the complement of a small neighborhood of the coordinate axis if t remains small enough. Under that condition $C_{1,t}$ and $C_{2,t}$ still intersect transversally in 3 points in this region. Moreover they also intersect transversally in 2 points near the coordinate axis, bringing the number of intersection points to 5. Taking $t = a/b$ rational numbers, and rescaling $(x,y) \mapsto (x^b, y^b)$ yields systems of 2 threenomial equations with 5 positive roots.

2. The proof

2.1. Near the coordinate axis

To follow this idea, we must describe more precisely the neighborhood of the y-axis and the behavior of $C_{1,t}$ in it. By symmetry we get a similar description for the neighborhood of the x-axis and for $C_{2,t}$ in it. Let

$$h_{1,t}(x) = \sqrt{1.1x^t - 1.1x} \quad \text{defined on the closed interval} \quad [0,1].$$

So $C_{1,t}$ is the graph of $y = h_{1,t}(x)$ (see Fig. 2), and let

$$g(t) = t^{1/1-t} = \exp\frac{\ln t}{1-t} \quad \text{defined for} \quad t \geq 0 \quad \text{with} \quad g(0) = 0\,.$$

Clearly $g(t)$ is continuous and increasing.

Lemma 1. *For any $t > 0$, the function $x \mapsto h_{1,t}(x)$ is continuous and has a unique maximum at $x = g(t)$ and two minima at $x = 0$ and $x = 1$* (see Fig. 2)*.*

The function $h_{1,t}(x)$ is clearly continuous for $0 \leq x \leq 1$, differentiable for $0 < x < 1$ and $h'_{1,t}(x) = 1.1(tx^{t-1} - 1)/2\sqrt{1.1x^t - 1.1x}$. So $h'(x) = 0$ for $tx^{t-1} - 1 = 0$, that is for $x = g(t)$, which remains in the open interval $0 < x < 1$ if $t > 0$ remains small enough. The lemma follows since $h'_{1,t}(x) > 0$ for $0 < x < g(t)$ and $h'_{1,t}(x) < 0$ for $g(t) < x < 1$.

Lemma 2. *There exists a $t_1 > 0$ such that $h_{1,t}(g(t_1)) > 1$ for all $0 \leq t \leq t_1$.*

Indeed $x \mapsto h_{1,0}(x) = \sqrt{1.1}\sqrt{1-x}$ is continuous at $x = 0$ and $h_{1,0}(0) = \sqrt{1.1} > 1$. So $h_{1,0}(x) > 1$ for x small enough. Since $x = g(t)$ is continuous, increasing with t and $g(0) = 0$, we get that $h_{1,0}(g(t)) > 1$ for t small enough, that is, for all $0 \leq t \leq t'_1$ for some $t'_1 > 0$. Let $x'_1 = g(t'_1)$. The function $t \mapsto h_{1,t}(x'_1) = \sqrt{1.1{x'_1}^t - 1.1x'_1}$ is continuous for $t \geq 0$ and

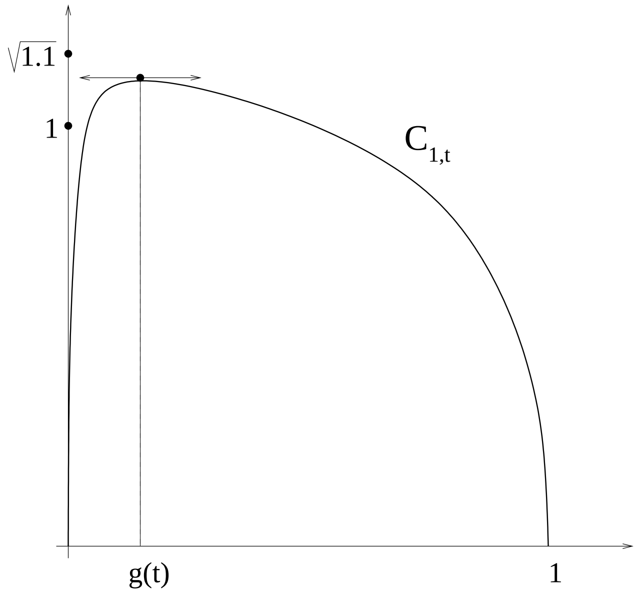

Figure 2. A curve $C_{1,t}$

$h_{1,0}(x_1') > 1$ so we get that $h_{1,t}(x_1') > 1$ for t small enough, that is, for all $0 \leq t \leq t_1''$ for some $t_1'' > 0$. Let $t_1 = \inf(t_1', t_1'')$ and $x_1 = g(t_1)$. For any $t > 0$ the function $x \mapsto h_{1,t}(x)$ continuously decreases on $g(t) < x \leq 1$ (from Lemma 1). Hence we get

$$0 \leq t \leq t_1 \leq t_1' \Rightarrow 0 \leq g(t) \leq g(t_1) \leq g(t_1') \Rightarrow 1 < h_{1,t}(x_1') \leq h_{1,t}(x_1) \leq h_{1,t}(g(t))$$

which proves the Lemma.

Let $q_2(x) = 1 - x^2/1.1 \geq 0$ for $0 \leq x \leq \sqrt{1.1}$, so C_2 is the graph of $y = q_2(x)$. Since $q_2'(x) = -2x/1.1 < 0$ the function q_2 is strictly decreasing. Also let $U_1 = \{(x, y) \in \mathbb{R}^2_{\geq 0} : x \leq g(t_1)\}$ (the t_1 of Lemma 2).

Lemma 3. *For $0 < t \leq t_1$ the curves $C_{1,t}$ intersect exactly once and transversally the curve C_2 in the interior of U_1* (see Fig. 3)*.*

From Lemma 1 we know that $h_{1,t}(x)$ is continuous and strictly increasing on $0 < x \leq g(t)$. Moreover $q_2(x)$ is continuous and strictly decreasing and $h_{1,t}(0) = 0$, $q_2(0) = 1$, $h_{1,t}(g(t)) > 1$ (from Lemmas 1 and 2) and $q_2(g(t)) < 1$ ($q_2(x)$ decreases). This proves that the graphs $C_{1,t}$ and C_2 intersect transversally exactly once in the interior of $U(t) = \{(x, y) \in \mathbb{R}_{\geq 0} : x \leq g(t)\} \subset U_1$. Moreover $h_{1,t}(x)$ decreases on $g(t) < x < x_1$ continuously but stays greater than 1 (Lemma 2) while $q_2(x)$ stays smaller than 1. Therefore their graphs do not intersect in $U_1 \setminus U(t)$. This proves the Lemma.

2.2. Counting transversal intersections

Let $V_1 = \{(x, y) \in \mathbb{R}^2_{\geq 0} : x \geq g(t)\}$ (the t_1 of Lemma 2).

Lemma 4. *The curves $C_{1,t}$ in V_1 form a continuous one-parameter family of diffeomorphic curves for $0 \leq t \leq t_1$.*

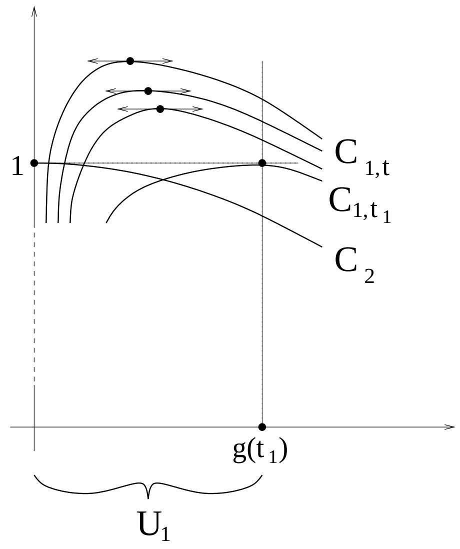

Figure 3. The curves $C_{1,t}$ intersect C_2 in U_1.

Clearly the curves are all diffeomorphic. Moreover $t \mapsto \sqrt{1.1x^t - 1.1x} = h_{1,t}(x)$ is continuous for any $x > 0$, and since x varies in $0 < g(t_1) \leq x \leq 1$, a closed interval, $(x,t) \mapsto h_{1,t}(x)$ is uniformly continuous in t. This proves the Lemma.

Recall the well known fact that small isotopies preserve transversal intersections (see for instance [1]), that is, if $\Gamma_{1,t}$ and $\Gamma_{2,t}$ are two continuous one-parameter families of diffeomorphic curves for $t \geq 0$, and $\Gamma_{1,0}$ intersects transversally $\Gamma_{2,0}$ at some point P, then $\Gamma_{1,t}$ and $\Gamma_{2,t}$ intersect transversally in any given neighborhood of P provided $t > 0$ is small enough.

Lemma 5. *There exists a $t_0 > 0$ such that the curves $C_{1,t}$ and $C_{2,t}$ intersect transversally in $\mathbb{R}^2_{>0}$ at exactly five points for all* $0 < t \leq t_0$ (see Fig. 4).

Let $U_2 = \{(x,y) \in \mathbb{R}_{\geq 0} : y \leq g(t_1)\}$ and $V_2 = \{(x,y) \in \mathbb{R}^2_{\geq 0} : y \geq g(t_1)\}$ (symmetric to U_1 and V_1 with respect to the diagonal line $y = x$). By symmetry the curves $C_{2,t}$ in V_2 form a continuous one-parameter family of diffeomorphic curves for $0 \leq t \leq t_1$. We assume that t_1 is small enough so that the intersections of C_2 with the $C_{1,t}$ are all in the interior of V_2. Since $C_2 = C_{2,0}$, the $C_{2,t}$ intersect transversally each $C_{1,t'}$ exactly once in the interior of $U_1 \cap V_2$ for $t \geq 0$ small enough. Hence $C_{1,t}$ and $C_{2,t}$ intersect transversally exactly once in the interior of $U_1 \cap V_2$, and by symmetry once in the interior of $U_2 \cap V_1$ for t small enough. Since they intersect in disjoint open sets, $C_{1,t}$ and $C_{2,t}$ intersect transversally exactly twice near the coordinate axis, that is, in the interior of $U_1 \cup U_2$ for $0 < t \leq t'_0$ for some t'_0 (they intersect also in $(0,0) \notin \mathbb{R}^2_{>0}$). We also assume that t_1 is small enough so that the three intersection points of C_1 and C_2 are in the interior of $V = V_1 \cap V_2$. Hence $C_{1,t}$ intersects transversally $C_{2,t}$ exactly three times in V for $0 \leq t \leq t''_0$ for some $t''_0 > 0$. Since $\mathbb{R}^2_{>0}$ is the union of the interior of $(U_1 \cup U_2)$ with V, the lemma simply follows by taking $t_0 = \inf(t'_0, t''_0)$.

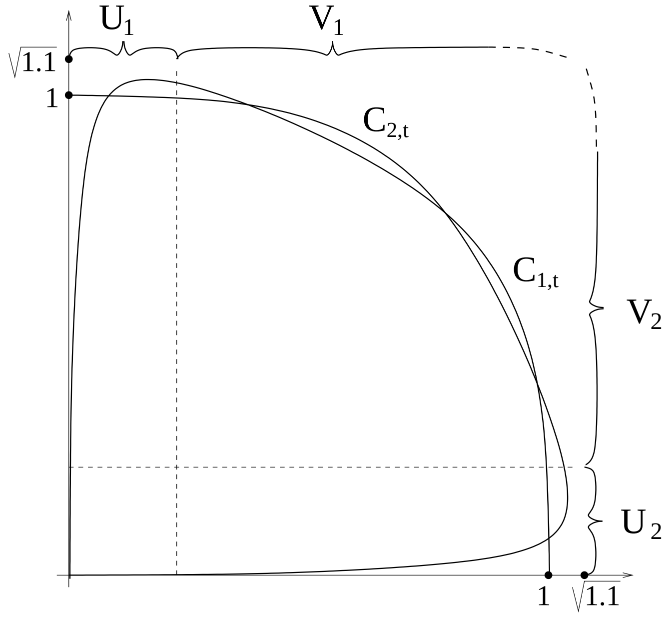

Figure 4. How $C_{1,t}$ and $C_{2,t}$ intersect.

2.3. Concluding the proof of the theorem

Let a/b be an arbitrary positive integer smaller than t_0 (from Lemma 5). The map $(x, y) \mapsto (x^b, y^b)$ is clearly a diffeomorphism $\mathbb{R}^2_{\geq 0} \to \mathbb{R}^2_{\geq 0}$. Composing with $P_{1,a/b}(x, y)$ and $P_{2,a/b}(x, y)$ yield the two threenomials of the theorem. Their zero sets intersect transversally at five points as $C_{1,a/b}$ and $C_{2,a/b}$ do by Lemma 5. This proves the theorem since transversal intersections correspond to non-degenerate isolated roots.

3. Concluding remarks

Remark 3. The first exponent t of the form $1/n$, where n is a positive integer, for which the system has five positive roots is $t = 1/53$. Jan Verschelde computed by homotopy methods [8] only one root for $t \leq 1/52$ and indeed five roots for $t = 1/53$ (with errors less than 10^{-16}):

```
x =  9.91327281341155E-01,     y =  9.91327281341155E-01
x =  9.23271265717941E-01,     y =  9.99996305511136E-01
x =  9.99990265492731E-01,     y =  9.31747195213204E-01
x =  9.99996305511136E-01,     y =  9.23271265717941E-01
x =  9.31747195213205E-01,     y =  9.99990265492731E-01
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Remark 4. Notice that the 2-dimensional polynomial system of the Theorem easily generalizes to $2k$-dimensional systems which pass Kouchnirenko's bound by $5^k - 4^k$. However it does not generalize trivially to 2-dimensional systems which would pass Kouchnirenko's bound by a function of the number of monomials.

Kouchnirenko's conjecture is true when $k = 1$. This is a direct consequence of Descartes rule which gives a sharp upper bound on the number of positive roots of a polynomial in one variable in function of the sign of the terms. The construction of polynomials with a number of positive roots equal to Descartes bound has been generalized in some sense [2] to the construction of a system of polynomial equations with some large number of positive roots (using Sturmfels-Viro method [6]). But it turned out that this number is not the right candidate for a multivariate Descartes bound as shown by T-Y. Li and X. Wang in [5], or by this simple system:

$$x + y - 2.1 = 0$$
$$x^2 + y^2 - 4 = 0$$

which has two positive roots while the candidate for the multivariate Descartes bound was one.

A. Khovanskii [3] [4] found an upper bound on the number of real positive roots of a system of k polynomial equations in k variables which depends on the number m of different monomials involved. The bound is $(k+1)^m 2^{m(m-1)/2}$ and is, as he wrote himself, "apparently considerably overstated" (it is equal to $248,832$ for a system of two threenomial equations like ours). Therefore the Kouchnirenko problem, to find a sharp bound which depends on the number of monomials involved, and the Descartes problem, to find a sharp bound which depends on the signs of the terms involved remain actively open.

According to Maurice Rojas (preprint CO/0008069 on xxx.lanl.gov server), the upper bound on the number of positive roots of a system of two threenomial equations in 2 variables is precisely 5. More recently T-Y. Li showed me a (independent) proof of this result.

Thanks to B. Sturmfels and M-F. Roy who have launched me on Kouchnirenko's conjecture and to Omer Pelman and Edwin O'Shea for interesting discussions.

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