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On Perfect 4-Polytopes

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Abstract. The concept of perfection of a polytope was introduced by S. A. Robertson. Intuitively speaking, a polytope P is perfect if and only if it cannot be deformed to a polytope of different shape without changing the action of its symmetry group $G(P)$ on its face-lattice $F(P)$. By Rostami's conjecture, the perfect 4-polytopes form a particular set of Wythoffian polytopes. In the present paper first this known set is briefly surveyed. In the rest of the paper two new classes of perfect 4-polytopes are constructed and discussed, hence Rostami's conjecture is disproved. It is emphasized that in contrast to an existing opinion in the literature, the classification of perfect 4-polytopes is not complete as yet.

1. Perfect polytopes

First we briefly summarize some basic notions about polytopes and symmetry [2, 16, 21, 22]. A (*convex*) *n-polytope* P is an intersection of finitely many closed half-spaces in a Euclidean space, which is bounded and n-dimensional. A *supporting hyperplane* of an n-polytope in $\mathbb{E}^n$ is an affine $(n-1)$-plane H such that $H \cap P$ is non-empty and P lies in one of the closed half-spaces bounded by H. A *proper face* F of P is the non-empty intersection of P with a supporting hyperplane H. A proper face of dimension $0, 1, k$ and $n-1$ is called a *vertex*, *edge*, *k-face* and *facet*, respectively. The empty set $\emptyset$ and P itself are *improper faces* of dimension -1 and n, respectively. For an n-polytope P, let $f_i(P)$ denote the number of i-faces of P. Then the n-tuple $f(P) = (f_0(P), f_1(P), \dots, f_{n-1}(P))$ is called the *f-vector* of P.

The set $F(P)$ of all faces of a polytope P ordered by inclusion forms a lattice, the *face-lattice* of P. We say that the polytopes P and Q are *combinatorially equivalent* if and only if there is a lattice isomorphism $\lambda : F(P) \to F(Q)$. On the other hand, in case $F(P)$ and $F(Q)$ are anti-isomorphic, i.e. there is an order-reversing bijection between them, then P and Q are said to be *dual* to each other (or duals of each other).

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By a *symmetry transformation* of an n-polytope P we mean an isometry of $\mathbb{E}^n$ keeping P setwise fixed. The group $G(P)$ of all symmetry transformations of P is called the *symmetry group* of P.

The action of $G(P)$ on P induces an action α_P on $F(P)$, $\alpha_P : G(P) \times F(P) \to F(P)$. Following Robertson [11, 21], we define an equivalence relation on the set of all n-polytopes.

Definition 1.1. *Two n-polytopes P and Q are said to be symmetry equivalent if and only if there exists an isometry φ of $\mathbb{E}^n$ and a face-lattice isomorphism $\lambda : F(P) \to F(Q)$ such that for each $g \in G(P)$ and each $A \in F(P)$, $\lambda\big(g(A)\big) = (\varphi g \varphi^{-1})\big(\lambda(A)\big)$. Each symmetry equivalence class is called a symmetry type.*

We are now ready to define perfection of polytopes.

Definition 1.2. *A polytope P is said to be perfect if and only if all polytopes symmetry equivalent to P are similar to P.*

(We note that here similarity is meant in the usual geometric sense.)

We have two remarks here. First, the notions of symmetry equivalence and perfection have been elaborated by Robertson not only for the restricted class of polytopes, but for studying the much wider class of n-solids [11, 20, 21].

On the other hand, in the particular class of 3-polytopes the symmetry types are well known and applied for a long time. Namely, the types of "simple closed crystal forms" are just the symmetry types (in the sense just defined) belonging to the crystallographic point groups. These usually are listed in standard crystallographic textbooks (see e.g. [3, 18]) and have been determined even for the non-crystallographic point groups as well [12, 17]. Moreover, in [10, p. 144] a definition of the type of crystal forms appears which is consistent with Definition 1.1.

There are some known constructions by which new perfect polytopes can be obtained from given perfect polytopes. These are the *polar* P^* of P, and the binary operations $P \mathbin{\square} Q$ and $P \diamond Q$.

The polar of P is defined by $P^* = \{y \in \mathbb{E}^n : \forall x \in \mathrm{P}, \langle x, y\rangle \le 1\}$, where 0 is in the interior of P [2, 16]. The additional condition, i.e. that the origin is an interior point of P, guarantees that P^* is a polytope (actually, the dual of P). For our purposes, however, it is more appropriate to choose a sharper form of this notion, as used by Robertson and his co-workers [11, 19, 20]. Namely, we require that the origin coincide with the centroid of P. (We note that, equivalently, it is also said that P and P^* are *reciprocal* of each other [1, 6].) In this case $G(P)$ and $G(P^*)$ are equal.

$P \mathbin{\square} Q$ is called the *rectangular product* by Pólya [6, 21], or simply the *product* by Robertson et al. [11], and is defined as follows. Let P and Q an m-polytope in $\mathbb{E}^m$ and an n-polytope in $\mathbb{E}^n$, respectively. Then let $\mathbb{E}^m \times \mathbb{E}^n$ be identified with $\mathbb{E}^{m+n}$ by mapping $(x, y) \in \mathbb{E}^m \times \mathbb{E}^n$ to $z \in \mathbb{E}^{m+n}$, where $z_i = x_i$ for $i = 1, \ldots, m$, and $z_{m+j} = y_j$ for $j = 1, \ldots, n$. Thus $\mathbb{E}^m$ and $\mathbb{E}^n$ are embedded as the images of $\mathbb{E}^m \times 0$ and $0 \times \mathbb{E}^n$, as orthogonal complements. Then $P \mathbin{\square} Q$ is taken as the image of the Cartesian product $P \times Q$ under this mapping.

The *co-product* [11] $P \diamond Q$ is defined using the same identification of $\mathbb{E}^m \times \mathbb{E}^n$ with $\mathbb{E}^{m+n}$ as above. Then $P \diamond Q$ equals the convex hull $\operatorname{conv}\big(P \times \{d\} \cup \{c\} \times Q\big)$, where c and d denotes the centroid of P and Q, respectively.

An n-polytope which is not isometric to a product or co-product of polytopes of dimension less than n is called *prime*. In fact, either of the first two conditions is sufficient, since we have the equality $(P \,\square\, Q)^* = P^* \diamond Q^*$ for any polytopes P and Q with common centroid [11].

Now it is known [19] that P^* is perfect if P perfect. On the other hand, $P \,\square\, Q$ and $P \diamond Q$ are perfect only if P and Q are isometric and perfect.

Perfection is a kind of generalization of the notion of regularity. In fact, every regular polytope is perfect [21]. Recall that by one of the usual definition an n-polytope P is regular in case for all k, $0 \leq k \leq n-1$, the symmetry group $G(P)$ of P is transitive on the k-faces of P. In other words, for all k, $0 \leq k \leq n-1$, there is a single orbit of k-faces of P under the action of $G(P)$. Thus, in the study of perfect polytopes it is useful to introduce the following notion [19, 21].

Definition 1.3. *The orbit vector of an n-polytope P is $\theta(P) = \big(\theta_0, \ldots, \theta_{n-1}\big)$, where θ_i is the number of orbits of i-faces of P, for each $i = 0, \ldots, n-1$, under the action of $G(P)$.*

Non-regular perfect polytopes appear first in dimension 3 (perfect 2-polytopes coincide with the regular polygons). These are the cuboctahedron, the icosidodecahedron, and their polars, which are the rhombic dodecahedron and the rhombic triacontahedron, respectively [21].

Naturally arises the following

Problem 1.4. *Characterize the orbit vectors of perfect n-polytopes.*

2. Nodal polytopes

The following notion is borrowed in a slightly modified form from Engel [10].

Definition 2.1. *Let G be a finite group of isometries of $\mathbb{E}^n$. Then the symmetry scaffolding of G is the union of the fixed point sets of all transformations in G and is denoted by* $\operatorname{scaf} G$.

Sometimes we will use the same term for the intersection of this set with the unit sphere $\mathbb{S}^{n-1}$ (centered at the origin) as well; however, when the distinction is important, the attribute *spherical* will be used for the latter.

Likewise, we may replace a polytope with its spherical variant in the following sense. For a given n-polytope P, take a unit sphere $\mathbb{S}^{n-1}$ centered at the centroid of P. Then project radially P to $\mathbb{S}^{n-1}$. We refer to the image under this projection as the *spherical image* of P.

Definition 2.2. *For a given group G and a point A in* $\operatorname{scaf} G$*, the fixed point set of A is* $\operatorname{fix}_A = \{x \in \mathbb{E}^n : g(x) = x,\ \forall g \in G_A\}$*, where G_A is the stabilizer of A in G. Then* $\dim(\operatorname{fix}_A)$*, the dimension of* fix_A*, is called the degree of freedom of A. A point in the spherical symmetry scaffolding of G is called a node in case it has zero degree of freedom. A vertex of a polytope P is called nodal if in the spherical image of P it coincides with a node in* $\operatorname{scaf} G(P)$*. A nodal polytope is a polytope whose vertices are all nodal.*

Theorem 2.3. *Every vertex-transitive nodal polytope is perfect.*

Proof. Let P be a vertex-transitive nodal n-polytope and let Q a polytope different from P but belonging to the same symmetry type. Then we have an isometry φ and a lattice isomorphism λ such that the condition in Definition 1.1 holds. Choose an arbitrary vertex V of P. Now it is easy to check the following two direct consequences of the symmetry equivalence of P and Q:

(a) $\varphi G_V \varphi^{-1} = G_{\lambda(V)}$, i.e. the stabilizer in $G(P)$ of any vertex of P is transformed by φ to the stabilizer in $G(Q)$ of the corresponding vertex of Q;
(b) vertex-transitivity of P implies vertex-transitivity of Q.

Because of the latter, we may assume that both P and Q is inscribed in the unit sphere $\mathbb{S}^{n-1}$. On the other hand, the following equality holds as well: $\varphi G_V \varphi^{-1} = G_{\varphi(V)}$. Thus $G_{\lambda(V)} = G_{\varphi(V)}$. But since P is nodal, G_V fixes just one pair of diametrally opposite points on $\mathbb{S}^{n-1}$, and so does its transform $\varphi G_V \varphi^{-1}$. Hence $\lambda(V) = \pm\varphi(V)$ for all vertices V of P. We assume that P is not centrally symmetrical, since otherwise the proof would be complete by this equality. Now if for all vertices $\lambda(V)$ of Q we have the plus sign, then Q is congruent to P, and we are done. The same is true in case minus sign is assumed everywhere. We show that the case of mixed signs cannot occur.

Suppose the contrary, say $\lambda(A) = \varphi(A)$ and $\lambda(B) = -\varphi(B)$ for two distinct vertices A and B of P. Since inversion in the centre of $\mathbb{S}^{n-1}$ commutes with all isometries mapping this sphere to itself, the latter equality can be written as $\lambda(B) = \varphi(-B)$. Because of transitivity, we have $B = g(A)$ with some $g \in G(P)$. Using symmetry equivalence, we obtain: $\lambda(B) = \lambda\big(g(A)\big) = (\varphi g \varphi^{-1})\big(\lambda(A)\big)$. Substituting $\lambda(A)$ and $\lambda(B)$, we obtain $\varphi(-B) = (\varphi g \varphi^{-1})\big(\varphi(A)\big)$, whence $\varphi(-B) = \varphi\big(g(A)\big)$. Hence $-B = g(A)$, which together with $B = g(A)$ implies $B = -B$, a contradiction.

3. Wythoffian perfect 4-polytopes

First we recall some well-known facts concerning Wythoffian polytopes in general. Let W be a finite *Coxeter group*, i.e. a finite group generated by reflections. We assume that W is *essential* relative to $\mathbb{E}^n$, i.e. it acts on $\mathbb{E}^n$ with no nonzero fixed points. Let scaf W denote the symmetry scaffolding of W, and let D denote the closure of a connected component of the set $\mathbb{S}^{n-1}\backslash$scaf W. Then D is a fundamental domain of W (with respect to its action on $\mathbb{S}^{n-1}$) and, by a theorem of Coxeter [6, 9], it is a (spherical) simplex. Scaf W can actually be formed as the union of all mirror hyperplanes belonging to the reflections in W. The n mirror hyperplanes passing through the facets of D belong to the reflections which generate W. (It is customary to call the cone determined by the union of these n mirrors an n-dimensional *kaleidoscope*). W is presented per definitionem by these generators σ_i and by the defining relations

$$(\sigma_i\sigma_j)^{p_{ij}} = 1, \quad (i, j = 0, \dots, n-1),$$

where $p_{ii} = 1$ and $p_{ij} \geq 2$ if $i \neq j$. The dihedral angles, i.e. the angles between the ith and jth facets of D are just equal to π/p_{ij}.

A concise description of the fundamental domain D (or, equivalently, of the group W itself) is given by its *Coxeter diagram*. This has n vertices or *dots* (here we adopt the terms used by Coxeter [6, 7]) representing the n facets of D (or the n generators of W), and has an edge or a *link* between two of its dots whenever $p_{ij} > 2$. The link is unmarked when $p_{ij} = 3$, but it is marked with the value of p_{ij} otherwise. W is *irreducible* or *reducible* according as the underlying graph of its Coxeter diagram is connected or disconnected.

The following construction is well known and is the basis of Coxeter's work on uniform polytopes (see [4–8], especially [8], p. 3]).

Construction 3.1. (Wythoff's construction) *Form the convex hull of the orbit of a suitable point for one of the finite reflection groups or for the rotatory subgroup of such a group.*

By suitable choice of the initial point of the construction certain conditions are meant which ensure that the polytope obtained will be uniform. More concretely, for a 4-dimensional kaleidoscope this means that the point is in one of the following positions in the corresponding fundamental spherical tetrahedron:

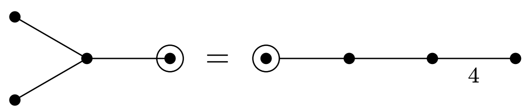
(1) at a vertex of the tetrahedron,
(2) on an edge, where it would be cut by the internal bisector of the dihedral angle at the opposite edge,
(3) on a facet, equidistant from the remaining 3 facets,
(4) in the interior of the tetrahedron, at the centre of the inscribed sphere.

Following Coxeter (see e.g. [4], p. 327), we have:

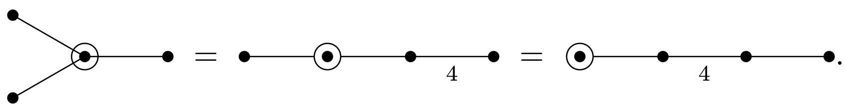
Definition 3.2. *A polytope which can be obtained by Wythoff's construction is called Wythoffian.*

The usual Coxeter symbol of a Wythoffian polytope is obtained from the Coxeter diagram of the group applied in the construction by placing rings around one or more dots of the diagram. The location and number of the rings indicate the location of the initial point as follows. This point may be at a vertex or in the relative interior of a face determined by 2, 3 or 4 vertices of the fundamental domain. Thus, the rings are placed around just the dots which symbolize the facets (or *walls*) opposite to the respective vertices.

Different constructions may result in the same Wythoffian polytope, as the following examples show [7, p. 575]:



(1)



(2)

These equalities will be utilized in the next section.

3.1. Wythoffian perfect 4-polytopes of the first kind

Theorem 3.3. *Let P be a Wythoffian polytope whose vertex set can be obtained as the W-orbit of a vertex of the fundamental domain of a finite reflection group W. Then P is perfect.*

This is a known and proven statement (see [19], p. 263 and the references therein), but in our context its proof is very simple. In fact, it is a consequence of some basic properties of the finite Coxeter groups.

Namely, the stabilizer of a vertex V of the fundamental domain D is generated by reflections in the mirror hyperplanes passing through V. This "parabolic" subgroup of W has no fixed point other than V (when action of W is considered on the unit sphere $\mathbb{S}^{n-1}$). Hence V is a node in the spherical symmetry scaffolding of W. Since the vertex set of P is the orbit $W(V)$, by Theorem 2.3 P is perfect.

Definition 3.4. *A perfect polytope that satisfies the condition in Theorem 3.3 is called a Wythoffian perfect polytope of the first kind.*

Separating this particular class of perfect polytopes is important in order to properly discuss Rostami's conjecture. It is formulated in [20, p. 370] as follows.

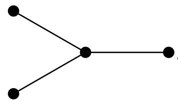
Rostami's Conjecture. *Any perfect 4-polytope P is either a square $Q \,\square\, Q$ or $Q \diamond Q$ of some regular polygon Q, or, for some irreducible finite reflection group W with fundamental domain D, either P or its polar P^* is the convex hull of the orbit under W of a vertex of D.*

Thus, apart from the polygon quadrates, it says that up to polarity the only prime 4-polytopes are Wythoffian polytopes of the first kind. If this were true, we would have a classification of perfect 4-polytopes. Just this is stated in [19], where an erroneous proof of the conjecture is given. In fact, Madden states in his Classification Theorem 4.4 [19, p. 277] as follows: "*Let P be a prime perfect 4-polytope, then P or P^* is Wythoffian*". We remark that in this formulation "Wythoffian" is meant as "Wythoffian of the first kind" in our sense, which is clear from the context of the paper in question.

In what follows we shall see (in Section 3.3 and thereafter) that Madden's classification theorem is false, since we have several new classes. In fact, our opinion is that the perfect 4-polytopes are far from being classified, for further new classes may arise from a more careful investigation in the future.

We note that Wythoffian perfect polytopes of the first kind form a class which usually is treated in terms of uniform polytopes due to Coxeter. In particular, in dimension 4 such a polytope is either regular, or is a "simple truncation" of a regular polytope [6, 7]. Equivalently, in our context we have the following

Proposition 3.5. *Every Wythoffian polytope of the first kind can be obtained by Wythoff's construction using the symmetry group of a regular polytope.*

In fact, since the only four-dimensional reflection group which is not the symmetry group of a regular polytope is , the equalities (1–2) imply our statement.

3.2. Kepler polytopes

The notion of a Kepler polytope was introduced in [13]. The name relates to the fact that the first non-regular examples, namely the rhombic dodecahedron and the rhombic triacontahedron, were discovered by Kepler. In the present context it can be defined as follows.

Definition 3.6. *An n-polytope P is called a Kepler polytope if and only if the following conditions hold:*

(1) *P^* is a Wythoffian perfect polytope of the first kind.*
(2) *$G(P)$ is equal to the symmetry group of a regular polytope.*

For $n = 4$ an n-polytope satisfying (1) satisfies (2) as well, cf. Proposition 3.5. Thus in our case the first condition is sufficient.

By polarity we have at once:

Proposition 3.7. *Kepler polytopes are perfect.*

They can be constructed independently of Wythoff's construction as follows [13].

Construction 3.8. *Reconstruct a polytope from its spherical image that is obtained as a factor set of the fundamental tessellation of a finite Coxeter group W.*

We describe it in somewhat more detail. Given a finite Coxeter group W with a fundamental domain D, consider the spherical tessellation which consists of all transforms of D by W, i.e. the *fundamental tessellation* belonging to W. Choose a vertex V of D, and form the union of all tiles having V in common. Perform this at each transform of V. Basic properties of Coxeter groups imply that a factor set of the original tessellation is obtained this way, which may be called a factor tessellation. It is actually the spherical image of a facet-transitive polytope. The final step is to reconstruct this polytope from its spherical image. This can be performed by placing tangent hyperplanes to each transform of V and taking the intersection of all closed half-spaces containing the sphere in question and bounded by these hyperplanes.

This construction provides a method by which for all k, $0 \leq k \leq n-1$, the shape and number of k-faces of such an n-polytope can be determined. Given a group W with its Coxeter diagram, only the following input data are necessary: the order of W and of its subgroups that are Coxeter groups in themselves as well as the choice of the initial vertex V [13].

We note that the construction works equally well with an arbitrarily chosen initial point instead of a vertex (in fact, any of the four possibilities will do which were considered for Wythoff's construction). But in this case, in general, it does not yield a perfect polytope. On the other hand, by any choice, using the same point in the two constructions results polytopes that are polars of each other.

A complete list of the 4-dimensional examples are in Table 1 at the end of this section. In the notation $\mathrm{f}_i\mathrm{X}_4$, where $\mathrm{X} = \mathrm{A}, \mathrm{B}, \mathrm{F}$ or H, X_4 is the usual symbol of the Coxeter group (or its fundamental domain) from which the polytope is obtained, and f_i ($i = 0, 1, 2$ or 3) shows which of the properly numbered vertices of the fundamental simplex is chosen for an initial point of the factorization construction. (To avoid ambiguities that sometimes occur in the literature, we note that we use Coxeter diagram of the form $\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet\!\!\underset{p}{-\!\!\!-\!\!\!-}\!\!\bullet$ where $p = 4$ for B_4 and $p = 5$ for H_4; moreover, numbering is made from left to right.)

It is seen the following rule (more generally, it is a consequence of Proposition, p. 125 in [13]):

Proposition 3.9. *Every non-regular Kepler polytope of dimension 4 is dipyramidal, i.e. its facets are dipyramids.*

These dipyramids are as follows. In case of a Coxeter group $[p, q, r]$ they are p-gonal and r-gonal dipyramids, i.e. their base is a regular p-gon and r-gon, respectively. In the particular case of [4, 3, 3], however, the tetragonal dipyramids exhibit higher symmetry than merely dipyramidal; they become regular octahedra and we obtain the regular 24-cell (this is the reason of the coincidence $\mathrm{f}_1\mathrm{B}_4 = \mathrm{f}_0\mathrm{F}_4$).

Proposition 3.9, together with Proposition 3.5 and Definition 3.6, implies the following

Corollary 3.10. *The polar of a Wythoffian perfect 4-polytope of the first kind is either regular or dipyramidal.*

Finally, we note that the orbit vector of all the non-regular examples is $\theta = (2, 2, 1, 1)$.

SYMBOL	NAME	f-VECTOR
$\mathrm{f}_0\mathrm{A}_4 = \mathrm{f}_3\mathrm{A}_4$	regular 4-simplex	$(5, 10, 10, 5)$
$\mathrm{f}_1\mathrm{A}_4 = \mathrm{f}_2\mathrm{A}_4$	dipyramidal 10-cell	$(10, 30, 30, 10)$
$\mathrm{f}_0\mathrm{B}_4$	hyper-cube	$(16, 32, 24, 8)$
$\mathrm{f}_1\mathrm{B}_4$	regular 24-cell	$(24, 96, 96, 24)$
$\mathrm{f}_2\mathrm{B}_4$	dipyramidal 32-cell	$(24, 88, 96, 32)$
$\mathrm{f}_3\mathrm{B}_4$	regular 16-cell	$(8, 24, 32, 16)$
$\mathrm{f}_0\mathrm{F}_4 = \mathrm{f}_3\mathrm{F}_4$	regular 24-cell	$(24, 96, 96, 24)$
$\mathrm{f}_1\mathrm{F}_4 = \mathrm{f}_2\mathrm{F}_4$	dipyramidal 96-cell	$(48, 240, 288, 96)$
$\mathrm{f}_0\mathrm{H}_4$	regular 120-cell	$(600, 1200, 720, 120)$
$\mathrm{f}_1\mathrm{H}_4$	dipyramidal 720-cell	$(720, 3600, 3600, 720)$
$\mathrm{f}_2\mathrm{H}_4$	dipyramidal 1200-cell	$(720, 3120, 3600, 1200)$
$\mathrm{f}_3\mathrm{H}_4$	regular 600-cell	$(120, 720, 1200, 600)$

Table 1

3.3. Wythoffian perfect 4-polytopes of the 2nd kind

In this section we give a new class of prime perfect 4-polytopes. These are counter-examples to Rostami's conjecture. They can be constructed using reflection groups whose Coxeter diagram exhibits bilateral symmetry.

To start with, take the group •——•——•——• $= [\,3, 3, 3\,]$. Apply Wythoff's construction such that let the symmetry of the diagram be preserved by the Coxeter symbol of the uniform polytope obtained, e.g. let it be $\mathrm{t}_{1,2}\,\alpha_4 =$ •——⊙——⊙——•. The geometric content of this formal symmetry is as follows. Consider the usual presentation of $[\,3, 3, 3\,]$ expressed by its Coxeter diagram. The defining relations are:

$$\begin{gathered}\sigma_0^2 = \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1,\\ (\sigma_0\sigma_1)^3 = (\sigma_1\sigma_2)^3 = (\sigma_2\sigma_3)^3 = (\sigma_0\sigma_2)^2 = (\sigma_0\,\sigma_3)^2 = (\sigma_1\,\sigma_3)^2 = 1,\end{gathered} \tag{3}$$

where σ_i $(i = 0, 1, 2, 3)$ are the generating reflections. This presentation is clearly invariant to the permutation $\rho = (\sigma_0, \sigma_3)(\sigma_1, \sigma_2)$. Geometrically ρ is realized by a half-turn about the join of the midpoints of two opposite edges of the fundamental domain of $[\,3, 3, 3\,]$: the edges V_0V_3 and V_1V_2, where V_i is the vertex opposite to the wall corresponding to the i-th generating reflection [7, p. 566]. Denote these midpoints M_{03} and M_{12}, respectively.

ρ induces an automorphism of the group $[3,3,3]$. Hence, adjoining this half-turn to the group we obtain a larger group which in Coxeter's notation is $[[3,3,3]]$, and which is isomorphic to the semi-direct product $[3,3,3] \rtimes \langle\rho\rangle$. With this extension, the original symmetry scaffolding has been extended as well. Namely, in the spherical image the great circle passing through the points M_{03} and M_{12} as well as all its transforms have been added. Thus it is directly seen that in scaf $[[3,3,3]]$ the midpoints M_{03} and M_{12} are nodes.

Taking into account the convention for uniform polytopes mentioned above, we see that the initial point of Wythoff's construction of the polytope $\bullet\!-\!\!-\!\odot\!-\!\!-\!\odot\!-\!\!-\!\bullet$ is exactly the midpont M_{12}. Therefore our polytope is nodal. Since it is vertex-transitive as well, due to Theorem 2.3 we obtain:

Proposition 3.11. *The polytope* $\mathrm{t}_{1,2}\,\alpha_4 = \bullet\!-\!\!-\!\odot\!-\!\!-\!\odot\!-\!\!-\!\bullet$ *is perfect.*

We are now ready to define the notion in the title of this section.

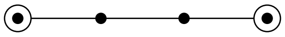
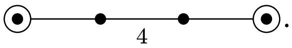
Definition 3.12. *Let P be a perfect polytope obtained by Wythoff's construction using a reflection group W with fundamental domain D. P is called a Wythoffian perfect polytope of the second kind if and only if the following conditions hold:*

1. *P is not a Wythoffian perfect polytope of the first kind.*
2. *the initial point for the construction of P can be chosen to be a relative interior point of an edge of D.*

The polytope $\mathrm{t}_{1,2}\alpha_4$, which we shall call the perfect 10-cell, has 5 facets $\odot\!-\!\!-\!\odot\!-\!\!-\!\bullet$ and 5 facets $\bullet\!-\!\!-\!\odot\!-\!\!-\!\odot$, thus it is bounded by 10 Archimedean truncated tetrahedra (here we use the technique developed by Coxeter [4] to determine the shape and number of the various faces of a uniform polytope). It can be conceived as a suitable (vertex) truncation of the regular 5-cell, or equivalently, as the intersection of 2 congruent reciprocal regular 5-cells. Its orbit vector is $\theta = (1,1,2,1)$.

A close analogue arises from the fact that the Coxeter diagram of the symmetry group of the regular 24-cell is symmetrical as well. All our above considerations remain valid in the case of this group $\bullet\!-\!\!-\!\bullet\underset{4}{-\!\!-}\bullet\!-\!\!-\!\bullet = [3,4,3]$. Thus we have an extended group $[[3,4,3]] \cong [3,4,3] \rtimes \langle\rho\rangle$ likewise. Here the polytope $\bullet\!-\!\!-\!\odot\underset{4}{-\!\!-}\odot\!-\!\!-\!\bullet$ has 24 facets $\odot\!-\!\!-\!\odot\underset{4}{-\!\!-}\bullet$ and 24 facets $\bullet\!-\!\!-\!\odot\underset{4}{-\!\!-}\odot$, thus there are altogether 48 Archimedean truncated cube facets. The orbit vector of this perfect 48-cell is the same $(1,1,2,1)$.

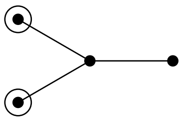
We obtained 2 types of (non-regular) perfect polytopes which are clearly prime. We remark that the existence of them directly contradicts Theorem 3.2 in [19], which is one of the main statements in the cited paper. It states that "A prime perfect $(0,3)$-transitive 4-polytope is regular". (Here $(0,3)$-transitivity means that the symmetry group of the polytope is transitive on the vertices as well as on the facets.) However, as we pointed out, there are some errors in its proof. Namely, the 10-cell is discarded due to some calculation error in applying Euler's relation [19, p. 272]. On the other hand, the possibility of 48-cell is overlooked since the facet of its polar is regarded to be regular tetrahedron instead of tetragonal disphenoid [19, p. 273] (cf. the construction in the next section).

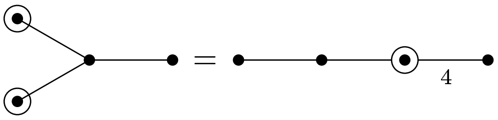
By simple analogy, we obtain 2 further examples of Wythoffian (prime) perfect polytopes of the 2nd kind. These are as follows: $t_{0,3}\,\alpha_4 =$  and $t_{0,3}\{3,4,3\} =$ . An essential difference from the former examples is that these are not $(0,3)$-transitive: their orbit vector is $\theta = (1,1,2,2)$ (i.e., it equals that of the non-regular Wythoffian perfect polytopes of the 1st kind, cf. the note at the end of Section 3.2). The first has 10 tetrahedra and 20 trigonal prisms as facets. The second has 48 octahedra and 192 trigonal prisms as facets.

Some data of these polytopes are summarized in the first part of Table 2.

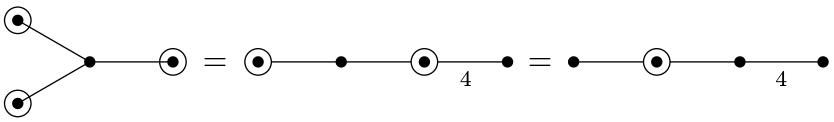
SYMBOL	FACETS	f-VECTOR	ORBIT VECTOR
$t_{1,2}\alpha_4$	10 Archimedean truncated tetrahedra	$(30,60,40,10)$	$(1,1,2,1)$
$t_{1,2}\{3,4,3\}$	48 Archimedean truncated cubes	$(288,576,336,48)$	$(1,1,2,1)$
$t_{0,3}\alpha_4$	10 regular tetrahedra + 20 trigonal prisms	$(20,60,70,30)$	$(1,1,2,2)$
$t_{0,3}\{3,4,3\}$	48 regular octahedra + 192 trigonal prisms	$(144,576,672,240)$	$(1,1,2,2)$
$f_{1,2}A_4$	30 tetragonal disphenoids	$(10,40,60,30)$	$(1,2,1,1)$
$f_{1,2}F_4$	288 tetragonal disphenoids	$(48,336,576,288)$	$(1,2,1,1)$
$f_{0,3}A_4$	20 rhombohedra	$(30,70,60,20)$	$(2,2,1,1)$
$f_{0,3}F_4$	144 tetragonal streptohedra	$(240,672,576,144)$	$(2,2,1,1)$

Table 2

We note that Condition 1 rules out e.g. the polytope  owing to the equality



[7, p. 575]. Likewise, the following equality shows that extending our idea to a Coxeter symbol with trigonal symmetry does not result in new example other than considered:



(4)

Therefore this polytope cannot be regarded Wythoffian perfect polytope "of the third kind", and not even that of the second kind.

Finally, we note that the squares $P \square P$ of regular p-gons are Wythoffian perfect polytopes of the second kind as well. For, they all are given in the form ⊙—p—•   •—p—⊙.

3.4. Polars of Wythoffian perfect 4-polytopes of the 2nd kind

These polytopes are summarized in the second part of Table 2. In this section we show how the facets of them can effectively be constructed. It is found that these polytopes are neither regular nor dipyramidal, besides, each of them is prime. Hence using Corollary 3.10 we conclude:

Proposition 3.13. *Table 2 contains Wythoffian prime perfect polytopes of the second kind and the polars of them.*

The polar of the polytope $\mathrm{t}_{1,2}\,\alpha_4$ and of $\mathrm{t}_{1,2}\{3,4,3\}$ is $\mathrm{f}_{1,2}\mathrm{A}_4$ = •——┼——┼——• and $\mathrm{f}_{1,2}\mathrm{F}_4$ = •——┼—4—┼——•, respectively. We note that in this notation the place and number of the vertical bars indicates the location of the initial point of the factorization process, cf. Construction 3.8 [13]. Actually, this agrees with the place and number of rings in the Coxeter symbol of the polar counterpart.

We determine the shape of a tile of the factor tessellation. (Since the dimension number is sufficiently low, our method here is intuitive instead of the abstract process based on Coxeter graphs in [13].) Choose first a (spherical) fundamental tetrahedron D corresponding to •——•——•——•. Its schematic drawing is shown in Figure 1 such that each edge is labelled by the value of the dihedral angle appearing at that edge. We form the union of 4 copies of it having the edge V_1V_2 in common. First take the union of the 2 copies that are adjacent by the side of the triangle $V_1V_2V_3$ (recall that they are mirror images of each other). Then instead of a body with 6 facets, we obtain a tetrahedron, since both at the edge V_1V_2 and V_1V_3 two adjacent triangles unite to an isosceles triangle. (Observe that V_1 is the fixed point of a subgroup •——•   • $\cong D_{3h}$, thus the edge V_0V_1

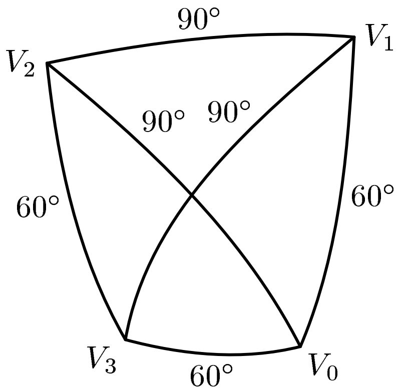


Figure 1

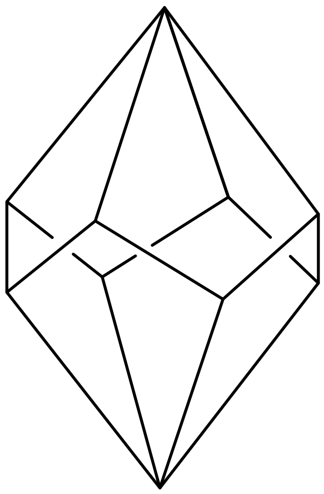
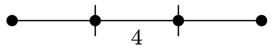
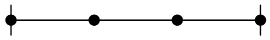
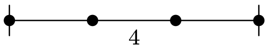


Figure 2

and its mirror image are lying on a common great circle, hence they are uniting to form the common base of the new triangles $V_0V_2V_0'$ and $V_0V_3V_0'$.) Now the union of the 2 other copies of D forms a mirror image of the larger tetrahedron just obtained. The two double tetrahedra are adjacent by the side of the mirror wall $V_0V_2V_0'$. The whole union of the 4 copies of D thus forms again a tetrahedron. In addition, it is seen that for symmetry reasons each of the four facets of this tetrahedron is an isosceles triangle. In other words, what is obtained is a *tetragonal disphenoid* [6], spherical of course.

The subgroup of the extended group $[[3,3,3]]$ that fixes setwise this disphenoid is just equal to its symmetry group and is isomorphic to $D_{2d} \cong [4,2^+]$. Its fixed point is the midpoint M_{12}. Therefore this point is to be chosen for the point of tangency of the hyperplane used in the final step of the construction of our polytope. Then this "local symmetry" D_{2d} is preserved and the facets of the polytope will be tetragonal disphenoids as well.

These arguments apply almost word by word for the case of . Thus in this case we obtain likewise tetragonal disphenoid facets.

The polars of $t_{0,3}\alpha_4$ and $t_{0,3}\{3,4,3\}$ are $f_{0,3}A_4$ =  and $f_{0,3}F_4$ = , respectively. Finding the shape of these polytopes goes along the same lines as above, thus we omit the details. The facets of $f_{0,3}A_4$ are 20 oblate rhombohedra. On the other hand, $f_{0,3}F_4$ has 144 facets. The symmetry group of such a facet (here coinciding with its stabilizer) is isomorphic to $D_{4d} \cong [8,2^+]$. This is an isohedral polyhedron bounded by 8 deltoids (or kites). A polyhedron of this type is called a *tetragonal (isosceles) trapezohedron* (also called, in geometric crystallography, a *tetragonal streptohedron* [17, p. 776]). It is illustrated in Figure 2.

3.5. An infinite class of perfect 4-polytopes

With the polytopes constructed in this section we go even more away from the Wythoffian perfect polytopes of the first kind. We start from a reflection group $[p,2,p] \cong$ , where $p > 2$. Take its rotation subgroup $[p,2,p]^+$ of index 2. It is well known [9, p. 125], that if the reflections generating $[p,2,p]$ are σ_0, σ_1, σ_2, σ_3, then $[p,2,p]^+$ is generated by $\rho_0 = \sigma_0\sigma_1$, $\rho_1 = \sigma_1\sigma_2$, $\rho_2 = \sigma_2\sigma_3$, and the defining relations are

$$\rho_0^p = \rho_1^2 = \rho_2^p = (\rho_0\rho_1)^2 = (\rho_0\rho_1\rho_2)^2 = (\rho_1\rho_2)^2 = 1. \tag{5}$$

We seek the shape of the fundamental domain of $[p,2,p]^+$. Denote D the fundamental domain of $[p,2,p]$ and V_i $(i = 0,1,2,3)$ the vertices of D. It is clear that switching from $[p,2,p]$ to $[p,2,p]^+$ the mirror walls of D are removed, while the edges corresponding to the products (5) are preserved (together with the vertices). Choose an interior point A and reflect it to each wall of D. Let the mirror images be A_0, A_1, A_2, A_3, respectively. Then the convex hull $\operatorname{conv}\{V_0,V_1,V_2,V_3,A_0,A_1,A_2,A_3\}$ may serve as a fundamental domain for $[p,2,p]^+$. (Note that the convex hull is meant here in spherical sense.) We denote this fundamental domain by D^+.

We have the following two observations here. First, it is implied by the symbol $[p,2,p]$ that D is a (spherical) tetragonal disphenoid. The dihedral angles at four of its edges equal $\pi/2$, and the remaining two are equal to π/p. Secondly, due to the reflections sending A

to A_i, the dihedral angles at every edge V_iV_j are doubled when forming the domain D^+. It is implied that D^+, instead of having 12 triangular facets, has 4 quadrilaterals and 4 triangles as facets.

Recall that the symmetry group of a tetragonal disphenoid is isomorphic to $D_{2d} \cong [4, 2^+]$. Moreover, it is seen that the symmety group of D^+ remains to be D_{2d} if and only if the point A is chosen to be the fixed point of this group. We denote this special form of D^+ by D_0^+ and this special choice of A by $A_{(0)}$.

Thus we choose $A_{(0)}$, and we regard the fundamental tessellation obtained for $[\,p, 2, p\,]^+$ as the spherical image of a polytope P. In this case the reconstruction of P is very simple. We place a tangent hyperplane H to $A_{(0)}$, form all the transforms of H by $[\,p, 2, p\,]^+$, and take the intersection of the closed half-spaces bounded by these hyperplanes and containing $\mathbb{S}^3$. (We note that as it can be shown [15], for an A in general position, the proper choice of the point of tangency of H is the centre of the circumsphere of the tetrahedron $A_0A_1A_2A_3$. In our particular case, however, this coincides with the common centre of the circumsphere and insphere of the tetrahedron $V_0V_1V_2V_3$, and thus with the fixed point of the symmetry group of D.)

Since the symmetry group of D_0^+ is D_{2d}, and the presentation of $[\,p, 2, p\,]^+$ is invariant to this group, the symmetry group of P is isomorphic to the semi-direct product $[\,p, 2, p\,]^+ \rtimes D_{2d}$. In the spherical symmetry scaffolding of this extended group the point $A_{(0)}$ is clearly node. On the other hand, a direct consequence of the construction is that P is facet-transitive. To see the other transitivity properties, call the vertices of P corresponding to the vertices V_i and A_j of D_0^+ vertices of type V and A, respectively. We observe that either of these types forms a transitivity class in itself under the action of $G(P)$. In fact, the stabilizer of a vertex of type A is isomorphic to D_{2d} and determines a conjugate class in $G(P)$. On the other hand, the stabilizer of a vertex of type V is isomorphic to $D_p \cong [p, 2]^+$ and determines another conjugate class in $G(P)$. The edges are of two distinct types as well, type VV and type VA. We find that the set of edges also decomposes to 2 transitivity classes corresponding to these types. A simple consequence is that the quadrilateral 2-faces are rhombuses and the triangular faces are isosceles triangles. In particular, by some calculation it is obtained that for $p = 4$ these latter are just equilateral triangles. Thus, a facet of P can be conceived as an intersection of a tetragonal prism and a tetragonal disphenoid, as in Figure 3 it is shown for $p = 4$ (when the disphenoid is a regular tetrahedron). It is found that the orbit vector is $\theta(P) = (2, 2, 2, 1)$.

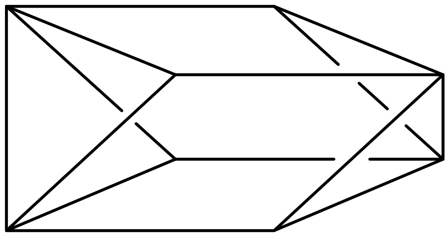


Figure 3

Taking the polar P^*, we observe that this is not only vertex-transitive but is a Wythoffian polytope. The initial point of Wythoff's construction in this case is just A_0. In this construction the rotation subgroup $[p, 2, p]^+$ of $[p, 2, p]$ is applied. Thus, in Coxeter's notation, this is a polytope of the form $\bigcirc\!\underset{p}{\text{—}}\!\bigcirc \quad \bigcirc\!\underset{p}{\text{—}}\!\bigcirc$. (Here the lack of dots from the rings indicates the removal of the mirror hyperplanes belonging to the generators of $[p, 2, p]$. On the other hand, the number of the rings indicates that this is a type of polytopes belonging to case 4 of Construction 3.1.) Finally, it is clearly perfect, which follows from the vertex-transitivity and from that the vertices are nodal (cf. Theorem 2.3). Thus we obtained a class, parametrized by p, such that its members may be called Wythoffian perfect 4-polytopes.

To establish the shape of the facets of P^*, we use the stabilizers found above and consider the orbit of a suitably chosen facet centre of P under their action. We obtain that a facet of P^* with centroid of type V is a p-gonal antiprism, and a facet with centroid of type A is a tetragonal disphenoid.

Knowing the facets of both P and P^*, it is seen that we obtained an infinite class of perfect 4-polytopes which are additional counter-examples to Rostami's conjecture (it is enough to note that neither P nor P^* is regular or dipyramidal, cf. Corollary 3.10). Here we remark that the case $p = 2$ is excluded on account of that it yields regular polytope. Namely, P is the hyper-cube (since D in this case is a regular tetrahedron with dihedral angles $\pi/2$) and P^*, one might say, has 8 regular tetrahedron facets and 8 "digonal antiprism" facets, i.e. it is the regular 16-cell.

This latter observation suggests another approach to P^*. Recall that the regular 16-cell can be conceived as a 4-dimensional "half-measure polytope" [6]. Such kind of polytopes can be obtained as follows. Select alternate vertices of an n-cube in such a way that every edge has one end selected and one end rejected. Then take the convex hull of the selected vertices. On the other hand, a 4-cube can be conceived as the product $\{4\} \square \{4\}$ of 2 equal squares. More generally, one may take the product $\{2p\} \square \{2p\}$ of 2 equal regular $2p$-gons. This is a (non-prime) "prismatic" perfect polytope. But perfectness is preserved when its alternate vertices are selected and the convex hull of the selected vertices is formed. For, what is obtained is just our polytope P^*. (Observe that the construction is made possible by that every 2-face of the product polytope has an even number of sides.) By this kind of partial truncation the $2p$-gonal prism facets are truncated to p-gonal antiprisms.

For this characteristic feature, we shall call the infinite series of such perfect polytopes P^* (p-gonal) *antiprismatic perfect polytopes.*

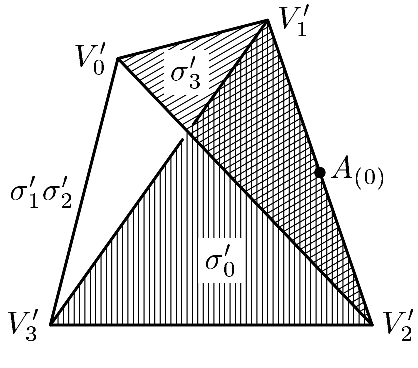


Figure 4

We show that they can be obtained by Wythoff's construction from another group. This group is $[\,p,2^+,p\,]$, generated by σ'_0, $\sigma'_1\sigma'_2$, σ'_3, where σ'_i, $(i=0,1,2,3)$ are the generators of (a new copy of) $[\,p,2,p\,]$ (cf. [7], p. 569). A characteristic part of its spherical symmetry scaffolding is illustrated in Figure 4. Here it is indicated that two of the generators of $[\,p,2,p\,]$ are removed while their product $\sigma'_1\sigma'_2$ is preserved. Choose the midpoint $A_{(0)}$ of the edge $V'_1V'_2$ for the initial point for the construction. Then P^* is obtained again. This is seen at once if we consider the union of the tetrahedron $V'_0V'_1V'_2V'_3$ with three of its copies. These are as follows. Its mirror image with respect to the wall $V'_0V'_1V'_2$, its mirror image with respect to the wall $V'_1V'_2V'_3$ and its transform by a half-turn about the edge $V'_1V'_2$, respectively. These four copies are united to a larger tetrahedron which can be identified with the tetrahedron $V_0V_1V_2V_3$ we started from in our former construction. Then $A_{(0)}$ is the same as above and we have the identifications $V'_0=V_0$ and $V'_3=V_3$. This correspondence shows the following isomorphism: $[\,p,2,p\,]^+ \rtimes D_{2d} \cong [[\,2p,2^+,2p\,]]$. (Here the double brackets in the right-hand term indicate that $[\,2p,2^+,2p\,]$ is extended by the half-turn about the join of the midpoints of edges $V'_0V'_3$ and $V'_1V'_2$.) Thus we may say that the symmetry group of a p-gonal antiprismatic polytope is isomorphic to $[[\,2p,2^+,2p\,]]$.

Finally, from simple considerations we obtain the f-vector as follows: $f(P)=(2p^2+4p, 8p^2+4p, 8p^2, 2p^2)$.

We summarize some of our more important findings in

Proposition 3.14. *The p-gonal antiprismatic polytopes ($p\geq 3$) constructed in this section are Wythoffian prime perfect polytopes which form an infinite series of counter-examples to Rostami's conjecture. Furthermore, neither of them can be regarded a Wythoffian perfect polytope of the second kind, thus they form a third class of Wythoffian perfect polytopes. Their symmetry group is isomorphic to $[[\,2p,2^+,2p\,]]$. They have the f-vector $f(P^*)=(2p^2,8p^2,8p^2+4p,2p^2+4p)$, and orbit vector $\theta(P^*)=(1,2,2,2)$.*

4. Conclusion and problems

In Section 3.1 and 3.2 we briefly described the prime perfect 4-polytopes which are already known. These are the Wythoffian perfect 4-polytopes of the first kind and their polars, the Kepler polytopes of dimension 4. The former are known as a special class of uniform polytopes, thus their properties can be studied in terms of the comprehensive theory of uniform polytopes due to Coxeter [4–8]. The latter were described first by the author [13].

These are the only prime perfect 4-polytopes which are allowed by Rostami's conjecture. However, in Sections 3.3 and 3.5 constructions are given for 2 new classes of perfect 4-polytopes, of which Rostami's conjecture cannot give an account. Hence the existence of these polytopes disproves the conjecture:

Theorem 4.1. *Rostami's conjecture is false.*

At the same time, Madden's classification theorem [19], which is an (erroneous) affirmation of the conjecture, is also disproved (the errors we found are mentioned in Section 3.3). We remark here as well, that the classification of perfect 4-polytopes cannot be regarded complete as yet. In fact, work is in progress in this direction, and our hope is that in subsequent publications an account on that will be given. In particular, here we just indicate that we have a construction even for a class of *non-Wythoffian* perfect 4-polytopes as well.

In order to understand the geometry of perfect polytopes, there may be a number of intermediate problems to be solved. To mention some of them, first we formulate the following:

Problem 4.2. *Does there exist a chiral perfect polytope?*

(Here by chirality of a polytope is meant that its symmetry group is generated by pure rotations.)

Or, is it true that a perfect polytope P or its polar P^* is necessarily nodal? Let us call P *seminodal* in case both P and P^* have vertices which are not nodal. Then we have:

Problem 4.3. *Does there exist a seminodal perfect polytope?*

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References

[1] Ashley, J.; Grünbaum, B.; Shephard, G. C.; Stromquist, W.: *Self-duality groups and ranks of self-dualities.* DIMACS Ser. Discrete Math. Theor. Comp. Sci. **4** (1991), 11–50. Zbl 0752.52003

[2] Brøndsted, A.: *An Introduction to Convex Polytopes.* Graduate Texts in Math. **90**, Springer, New York 1983. Zbl 0509.52001

[3] Buerger, M. J.: *Introduction to Crystal Geometry.* McGraw-Hill Book Co., New York 1971.

[4] Coxeter, H. S. M.: *Wythoff's construction for uniform polytopes.* Proc. London. Math. Soc. **38** (1934), 327–339. Zbl 0010.27503

[5] Coxeter, H. S. M.: *Regular and semi-regular polytopes. I.* Math. Z. **46** (1940), 380–407. Zbl 0022.38305

[6] Coxeter, H. S. M.: *Regular Polytopes.* Methuen, London 1948. Zbl 0031.06502

[7] Coxeter, H. S. M.: *Regular and semi-regular polytopes. II.* Math. Z. **188** (1985), 559–591. Zbl 0553.52007

[8] Coxeter, H. S. M.: *Regular and semi-regular polytopes. III.* Math. Z. **200** (1988), 3–45. Zbl 0633.52006

[9] Coxeter, H. S. M.; Moser, W. O. J.: *Generators and Relations for Discrete Groups.* Springer-Verlag, Berlin 1972. Zbl 0239.20040

[10] Engel, P.: *Geometric Crystallography.* Reidel, Dordrecht 1986. Zbl 0659.51001

[11] Farran, H. R.; Robertson, S. A.: *Regular convex bodies.* J. London Math. Soc. (2) **49** (1994), 371–384. Zbl 0801.52007

[12] Gévay, G.: *Icosahedral morphology.* In: Hargittai, I. (Ed.): *Fivefold Symmetry.* World Scientific, Singapore 1992, 177–203.

[13] Gévay, G.: *Kepler hypersolids.* In: Böröczky, K.; Fejes Tóth, G. (Eds.): *Intuitive Geometry.* Proc. 3rd Int. Conf. Szeged, Hungary, Sept. 2–7, 1991, North-Holland, Amsterdam. Colloq. Mat. Soc. János Bolyai **63** (1994), 119–129. Zbl 0809.00022

[14] Gévay, G.: *Changes of shape and symmetry in the construction of perfect polytopes.* In: Extended Abstract of the International Katachi ∪ Symmetry Symposium, University of Tsukuba, Japan, 1994. Reprinted in: Hyperspace **9**(1) (2000), 77–80.

[15] Gévay, G.: *Chiral facet-transitive 4-polytopes related to regular polytopes.* Hyperspace **9**(3) (2000), 9–21.

[16] Grünbaum, B.: *Convex Polytopes.* Wiley, London 1967. Zbl 0163.16603

[17] Hahn, T. (Ed.): *International Tables for X-ray Crystallography.* Vol A., Reidel, Dordrecht 1983.

[18] Kleber, W.: *An Introduction to Crystallography.* VEB Technik Verlag, Berlin 1970.

[19] Madden, T. M.: *A classification of perfect 4-solids.* Beiträge Algebra Geom. **36** (1995), 261–279. Zbl 0838.52016

[20] Madden, T. M.; Robertson, S. A.: *The classification of regular solids.* Bull. London Math. Soc. **27** (1995) 363–370. Zbl 0852.52002

[21] Robertson, S. A.: *Polytopes and Symmetry.* London Math. Soc. Lecture Notes **90**, Cambridge University Press, Cambridge 1984.

[22] Schulte, E.: *Symmetry of polytopes and and polyhedra.* In: Goodman, J. E. and O'Rourke, J. O. (Eds.): *Handbook of Discrete and Computational Geometry.* CRC Press, Boca Raton 1997, pp. 311-330. Zbl 0916.52004

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