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# Moufang Buildings and Twin Buildings

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## 1. Introduction

The "Moufang Condition" for spherical buildings was introduced by J. Tits in the appendix of [9], as a tool to give more structure to the classification of spherical buildings of rank at least three (which are automatically "Moufang"). More recently, also the spherical buildings of rank 2 satisfying the Moufang Condition are classified [13]. Hence one could say that, on the geometric level, spherical buildings axiomatize the situation of a simple group of Lie type, while on the group-theoretic level, the Moufang Condition characterizes the groups themselves as automorphism groups. A few years ago, a similar phenomenon occured after the discovery of Kac-Moody algebras and Kac-Moody groups. In [11] Tits gave a group theoretical definition of a "Moufang Condition" intrinsically generalizing the notion of "Moufang spherical building". This definition was formally translated into geometrical language by Ronan in his book [7]. The main motivation was an attempt to characterize the Kac-Moody groups as automorphism groups of certain buildings (namely, the Moufang buildings). However, this equivalence is not yet established. On the geometric level, Ronan and Tits introduced the so-called "twin buildings" (see [12]), and these axiomatize the situation of a Kac-Moody group geometrically. Again, the equivalence of (simple) Kac-Moody groups and twin buildings (possibly under some additional hypotheses) has not been established yet. However, the work of Tits (see [11], [12], [9]), Mühlherr (see [4], [5], [6]) and Ronan points in the direction that a classification of 2-spherical twin buildings (i.e., twin buildings with a diagram containing no edges labelled $\infty$) is feasible. Moreover, Mühlherr and Ronan show in [3] that, under some mild restrictions, both combinatorial buildings of a 2-spherical twin building satisfy the Moufang Condition. Hence, in view of the analog for the spherical case, and in view of a possible classification of Moufang buildings (still under some additional,

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natural, conditions), the question of whether every Moufang building corresponds to a twin building is very interesting. In fact, Proposition 4 of [12] says that this is indeed the case. For a proof of that proposition, Tits refers the reader to the paper [11] without any additional specification or hint. The paper [11] indeed contains a rough outline of a possible proof of the proposition, and with some effort, one can reconstruct in detail the arguments and (non-trivial technical) computations. However, in the present paper, we want to present a proof of Tits' proposition partly following his ideas, but mainly using alternative geometric arguments. In my opinion this gives better insight into the geometry of twin buildings and the relation with groups, as a generalization of the connection between geometries and groups in the theory of (semi-simple) algebraic groups.

The paper is organized as follows. In Section 2 we give some definitions and facts to introduce the setting for the Moufang building $\Delta$. In Section 3 we construct a chamber system $\mathcal{C}^-$ using the groups which come from the Moufang structure on $\Delta$. At the end of this section a covering $\kappa$ between $\mathcal{C}^-$ and $\Delta$ is defined. Using universal properties of $\Delta$ this implies that $\kappa$ is an isomorphism. In the last section we give a proof that $\Delta$ is the half of a twin building.

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## 2. Preliminaries

By a Coxeter matrix $M$ over the finite index set $I$ we will mean a symmetric matrix $M = (m_{ij})_{i,j\in I}$ with entries in $\mathbb{N} \cup \infty$ such that $m_{ii} \geq 1$ and $m_{ij} \geq 2$ if $i \neq j$. With every such Coxeter matrix one can associate a group $W$ with presentation

$$W = \langle s_i | (s_i s_j)^{m_{ij}} \rangle.$$

This is the Coxeter group or Weyl group of type $M$. We sometimes write $(W, S)$ instead of $W$ where $S = \{s_i | i \in I\}$. This couple is called a Coxeter system of type $M$. This notation is used when we have a particular set $S$ of generators of $W$ in mind. The length of an element $w \in W$ with respect to this generating set is denoted by $l(w)$. An expression $w = s_{i_1} \dots s_{i_m}$ with $l(w) = m$ is called *reduced* or *minimal*.

**Definition 1.** *Let $M$ be a Coxeter matrix and $(W, S)$ a Coxeter system of type $M$. A building of type $M$ is a quadruple $(\Delta, W, S, d)$ with $\Delta$ a set whose elements are called chambers, $(W, S)$ the given Coxeter system and $d$ a distance function going from $\Delta \times \Delta$ to $W$ satisfying* 3 *axioms:*

**Bu1** $w = 1$ *if and only if* $x = y$.

**Bu2** *If $z \in \Delta$ is a chamber such that $d(y, z) = s$ with $s \in S$ then $d(x, z) = w$ or $ws$. Moreover if $l(ws) > l(w)$ then $d(x, z) = ws$.*

**Bu3** *If $s \in S$ then there exists a chamber $z$ of $\Delta$ such that $d(x, z) = ws$.*

This definition was taken from [12]. When there is no confusion possible, or when $(W, S)$ is not important a building $(\Delta, W, S, d)$ will also briefly be written as $(\Delta, d)$ or as $\Delta$.

An obvious example of a building of type $M$ is given by the Coxeter group itself. Chambers are elements of $W$ and distance between two chambers $x$ and $y$ is defined as $x^{-1}y$. In the sequel we sometimes view the Coxeter group either as group or as a building. Which approach is used will be clear from the context unless stated otherwise.

If the associated Coxeter group is finite, then the building $\Delta$ is called a spherical building. For more information about spherical buildings we refer to [9], where they are classified in case $|S| \geq 3$. As a generalisation of the idea of spherical building the concept of twin buildings was introduced by M.Ronan and J.Tits. The paper [12] can be seen as a standard reference on this subject. We give the definition of [12] of a twin building.

**Definition 2.** *For a certain Coxeter matrix and associated Weyl group* $(W, S)$ *a twinned pair of buildings or a twin building of type* $M$ *is a pair of buildings* $(\Delta_+, W, S, d_+)$ *and* $(\Delta_-, W, S, d_-)$ *with a codistance function* $d^*$ *going from* $\Delta_+ \times \Delta_- \sqcup \Delta_- \times \Delta_+$ *to* $W$ *satisfying* ($\epsilon \in \{-1, 1\}$, $x \in \Delta_\epsilon, y \in \Delta_{-\epsilon}$ *and* $d^*(x, y) = w$)*:*

**Tw1** $d^*(y, x) = w^{-1}$.

**Tw2** *If* $z \in \Delta_{-\epsilon}$ *is such that* $d_{-\epsilon}(y, z) = s \in S$ *and* $l(ws) < l(w)$ *then* $d^*(x, z) = ws$.

**Tw3** *For every* $s \in S$ *there exists at least one chamber* $z \in \Delta_{-\epsilon}$ *with* $d^*(x, z) = ws$.

Apart from the geometrical description of buildings there is also a nice group theoretical counterpart. This is the notion of a $BN$-pair or Tits system.

**Definition 3.** *Let* $G$ *be a group with two subgroups* $B$ *and* $N$. *Then* $(G, B, N, S)$ *forms a* $BN$*-pair or Tits system if the following axioms are satisfied:*

**BN0** $\langle B, N \rangle = G$.

**BN1** $H = B \cap N \trianglelefteq N$ *and* $N/H$ *is a Coxeter group with generating set*

$$S = \{s_i | i \in I\}.$$

**BN2** $BsBwB \subset BswB \cup BwB$ *whenever* $w \in W$ *and* $s \in S$.

**BN3** $sBs \neq B$ *for* $s \in S$.

More information about $BN$-pairs can be found in [9].

Remains to define what a Moufang building is. To do this we follow the approach in Paragraph 4 of Chapter 6 in [7]. First we need some additional notions like panel, apartment and root. Let $(\Delta, W, S, d)$ be a building then two chambers $x$ and $y$ are called *$s$-adjacent* for $s \in S$ whenever $d(x, y) = s$ or $x = y$. An *$s$-panel* $P$ is a maximal set of mutually $s$-adjacent chambers. When we don't have a specific $s$ in mind an $s$-panel will also be called a *panel*.

To relate buildings one needs a notion of morphisms.

**Definition 4.** *Given two buildings* $(\Delta, W, S, d)$ *and* $(\Delta', W, S, d')$ *of the same type a morphism from* $(\Delta, W, S, d)$ *to* $(\Delta', W, S, d')$ *is a mapping* $\theta$ *going from* $\Delta$ *to* $\Delta'$ *such that if* $x$ *and* $y$ *are* $s$*-adjacent then* $\theta(x)$ *and* $\theta(y)$ *are also* $s$*-adjacent where* $x, y \in \Delta$. *An isomorphism is also called an isometry.*

A nice example of an isometry of the Coxeter group onto itself is given by left multiplication with a fixed element.

**Definition 5.** *Given a Coxeter group viewed as a building then for an element $s \in S$ the fundamental root defined by $s$ is the set $\alpha_s = \{w \in W | l(sw) > l(w)\}$. All other roots in $W$ are subsets of the form $w(\alpha_s)$ for some $w \in W$. The opposite root of a root $\alpha$ is the root $-\alpha$ having an empty intersection with $\alpha$. For every root $\alpha$ the boundary of $\alpha$ denoted by $\partial\alpha$, is the set of panels that have non-empty intersection with both $\alpha$ and $-\alpha$. Roots are called positive or negative according whether they contain 1 or not. If a root $\alpha$ is positive, this is denoted by $\alpha > 0$. Similarly $\alpha < 0$ means that the root $\alpha$ is a negative root.*
*The set of all roots in $W$ is denoted by $\Psi$.*

We fix some notation. If $s_i, i \in I$ denotes a fundamental reflection then the root associated with $s_i$ is written as $\alpha_i$.

**Definition 6.** *An apartment $\Sigma$ in a building $\Delta$ is an isometric copy of the Coxeter group $W$ viewed as a building in $\Delta$. A root in a building $\Delta$ is defined as an isometric copy of a root $\alpha$ in $W$. The boundary of a root in $\Delta$, and the notion of positive and negative roots are defined in a similar way.*

It can be proven that apartments always exist in buildings and that they characterize the geometry (cf. Theorem 3.11 of [7]).

**Definition 7.** *Two roots $\alpha$ and $\beta$ in $W$ are called prenilpotent if and only if $\alpha \cap \beta \neq \varnothing$ and $(-\alpha) \cap (-\beta) \neq \varnothing$. If two roots $\alpha$ and $\beta$ in $W$ are prenilpotent then the interval $[\alpha, \beta]$ is defined as the set*

$$\{\gamma \in \Psi \,|\, \alpha \cap \beta \subset \gamma \text{ and } (-\alpha) \cap (-\beta) \subset (-\gamma)\}.$$

*The notation $(\alpha, \beta)$, with $\alpha, \beta$ a prenilpotent pair of roots, stands for $[\alpha, \beta] \setminus \{\alpha, \beta\}$.*

Starting with a building $(\Delta, W, S, d)$ of a certain type $M$ and the set $\Phi$ of all roots in a fixed apartment $\Sigma_0$ (called the standard apartment) of $\Delta$ we call the building Moufang if there exists a set of automorphism groups $(U_\alpha)_{\alpha\in\Phi}$ also called root groups such that:

**Mo1** *Every element $u \in U_\alpha$ fixes all chambers of $\alpha$. If $\pi$ is a panel on $\partial\alpha$ and $c$ is the chamber of $\pi$ lying in $\alpha$ then $U_\alpha$ fixes $c$ and acts regularly on all the chambers of $\pi \setminus \{c\}$.*

**Mo2** *If $\{\alpha, \beta\}$ is a pair of prenilpotent distinct roots then*

$$[U_\alpha, U_\beta] \subset U_{(\alpha,\beta)}.$$

**Mo3** *For each $u_\alpha \in U_\alpha \setminus \{1\}$ there exists an element $m(u_\alpha) \in U_{-\alpha} u_\alpha U_{-\alpha}$ stabilizing $\Sigma$.*

**Mo4** *If $n = m(u_\alpha)$ then for every root $\beta$ we have $nU_\beta n^{-1} = U_{s_\alpha(\beta)}$.*

Given a Moufang building $\Delta$ with root groups $(U_\alpha)_{\alpha\in\Phi}$, we define the group $G = \langle U_\alpha \rangle_{\alpha\in\Phi}$, $N$ the group generated by all $m(u_\gamma)$ with $u_\gamma \in U_\gamma$ for a root $\gamma$ in $\Phi$. The standard chamber $c_+$ is defined as the image of 1 under the isometry going from the Coxeter group $W$ to the standard apartment $\Sigma_0$. It follows from the construction that $c_+$ is the intersection of all positive roots in $\Sigma_0$. The subgroup of elements of $N$ that fix $\Sigma_0$ is denoted by $H$, the torus in the classical sense. It is easy to check that $H \subset N_G(U_\alpha)$ for all root groups $U_\alpha$. The group $B_+$ stands for $\langle H, U_\alpha \rangle_{\alpha>0}$, $B_- = \langle H, U_\alpha \rangle_{\alpha<0}$ and $B_\alpha = \langle H, U_\alpha \rangle$ for every $\alpha \in \Phi$. The group $B_+$ also has a geometrical meaning: it is the full stabilizer in $G$ of the standard chamber

$c_+$ in $\Delta$ and $N$ is the stabilizer of the apartment $\Sigma_0$ in $G$. The first fact is not obvious to show. It follows mainly from Lemma 4 in Section 5 of [11]. In fact this lemma yields that $G = \cup(B_+wB_+)_{w\in W}$. With this setup we thus get a system $(G,(U_\alpha)_{\alpha\in\Phi})$ of groups. In the paper [11] J.Tits considers similar systems satisfying 5 axioms, namely (RD1) till (RD5). It is not hard to check that $(G,(U_\alpha)_{\alpha\in\Phi})$ satisfies (RD1), (RD2), (RD3) and (RD4). However axiom (RD5) cannot be proved by elementary techniques. It states that $\forall i \in I$ and $\alpha_i \in \Phi$ with $\alpha_i > 0$, $B_{\alpha_i} \not\subset B_-$ and $B_{-\alpha_i} \not\subset B_+$. That $B_{-\alpha_i} \not\subset B_+$ is easy to check. This follows essentially from the equality $Stab_G(c_+) = B_+$. Namely if $B_{-\alpha_i} \subset B_+$ then every $u_{-\alpha_i} \in U_{-\alpha_i}$ would fix $c_+$ contradicting the regular action of $U_{-\alpha_i}$. To exclude that $B_{\alpha_i}$ is contained in $B_-$ one cannot use the same argument as for the other case. The difference here is that $B_-$ has no interpretation in terms of the building geometry. For this we will have to look deeper into the structure of $\Delta$. Moreover the following results are true. For proofs we refer to [11].

**Theorem 1.** *Given a Moufang building $(\Delta, W, S, d)$ of type $M$ (with notations as above) then there is a unique homorphism $\nu : N \mapsto W$ such that for $n \in N$ and $\alpha \in \Phi$*

$$nB_\alpha n^{-1} = B_{\nu(n)(\alpha)}.$$

*The kernel of $\nu$ is $H$. This implies that $N/H \cong W$ and $N/H$ is generated by a set $\tilde{s}_iH$ where $\{\tilde{s}_i\}$ is a set of $m(u_{\alpha_i})$ with $u_{\alpha_i} \neq 1$ and $\{\alpha_i \mid i \in I\}$ a fundamental root system in $\Phi$.*

*Proof.* This is a restatement of Lemma 3(i),(iii) in Paragraph 5 of [11]. The only thing one has to check are axioms (RD2), (RD3) and (RD4) of loc. cit. for the system $(G,(B_\alpha)_{\alpha\in\Phi})$. □

In what follows we will also consider elements of $w$ as they were in the group $G$ and write for example $wB_\alpha w^{-1} = B_{w(\alpha)}$ if there is no confusion. If we consider $w$ as being an element of $G$ we have a representative of $w$ in $N/H$ in mind.

**Theorem 2.** *Given a Moufang building $(\Delta, W, S, d)$ of type $M$ (with notations as above) then $G$ acts transitively on $\Delta$ and the system $(G, B_+, N, S)$ forms a $BN$-pair where $S$ is a set of generators of the group $N/H$.*

*Proof.* One checks that axiom (RD1) of RD-systems (cf. Paragraph 5 in [11]) is satisfied for $(G,(B_\alpha)_{\alpha\in\Phi})$ and that $B_{-\alpha} \subseteq B_+$ for all $\alpha > 0$. Then the proof of Lemma 4 in Paragraph 5 of loc. cit. is still valid. Following the strategy of Proposition 4(i) of loc. cit. one deduces that $(G, B_+, N, S)$ is a $BN$-pair. □

We also mention the following property which we will use later on.

**Lemma 1.** *Given a Moufang building $(\Delta, W, S, d)$ as above then for every $i \in I$ the group $B_{\alpha_i}$ with $\alpha_i > 0$ has two double cosets in the group it generates with $B_{-\alpha_i}$, these are $B_{\alpha_i}$ and $B_{\alpha_i}s_iB_{\alpha_i}$. Hence we can write*

$$\langle B_{\alpha_i}, B_{-\alpha_i}\rangle = B_{\alpha_i} \cup B_{\alpha_i}s_iB_{\alpha_i}.$$

*Proof.* If $u_{-\alpha_i} \in B_{-\alpha_i}$ and $u_{-\alpha_i} \notin H$ then there exist elements $u_{\alpha_i}$ and $u'_{\alpha_i}$ in $U_{\alpha_i}$ such that $m(u_{-\alpha_i}) = u_{\alpha_i}u_{-\alpha_i}u'_{\alpha_i}$. From the properties of the Moufang building $\Delta$ it follows that $u_{\alpha_i}u_{-\alpha_i}u'_{\alpha_i}s_i^{-1} \in H$, hence $u_{-\alpha_i} \in B_{\alpha_i}s_iB_{\alpha_i}$. This proves the claim. □

## 3. The chamber system $\mathcal{C}^-$

In this paragraph we construct a chamber system $\mathcal{C}^-$ using the groups. First we need some lemmas.

**Lemma 2.** *Given a negative root $\alpha_i$ with $i \in I$ then*

$$B_\alpha B_{\alpha_i} \tilde{s}_i B_- \subset B_{\alpha_i} \tilde{s}_i B_-$$

*for every negative root $\alpha \in \Phi$.*

*Proof.* For the proof we refer to Lemma 4 in Section 5 of [11]. One replaces all positive roots by negative roots. □

**Lemma 3.** *Let $w \in W$ (with $(W, S)$ a Weyl group) , and $s_{i_1} \dots s_{i_m}$ a reduced expression of $w$. Set for $j \in \{1, \dots, m\}$ $w_j = s_{i_1} \dots s_{i_j}$, $w_0 = 1$ and $\beta_j = w_{j-1}(\alpha_j)$ then $\{\beta_1, \dots, \beta_m\}$ is the set of all positive roots sent by $w^{-1}$ to a negative root.*

*Proof.* This lemma is a restatement of Proposition 3($i$) of [11], Section 5. The proof can be found there. □

**Lemma 4.** *Given any $w \in W$ and a reduced expression $s_{i_1} \dots s_{i_m}$ of $w$ then the set*

$$U_{-\beta_1} \dots U_{-\beta_m}$$

*is a group $U_{-w}$ only depending on $w$. The group $B_{-w}$ satsifies $B_{-w} w B_- = B_- w B_-$.*

*Proof.* The statement of this lemma is analogous to the statement of Proposition 3(ii), (iv) in Section 5 of [11]. The only difference is that the groups here are parametrized by negative roots. One can easily check that the proof given in loc. cit. remains partially valid if positive roots are replaced by negative roots. □

Using the groups $U_{-w}$ we construct the following chamber system $\mathcal{C}^-$. Let $U_- = \langle U_{-\alpha} \rangle_{\alpha > 0}$. For a given $w \in W$ the group $U_{-w}$ defines a coset structure on $U_-$. We define $\mathcal{C}^-_w$ as the set of all right cosets of $U_{-w}$ in $U_-$. The set of chambers of $\mathcal{C}^-$ is the disjoint union $\sqcup \mathcal{C}^-_w$. As we want the chamber system $\mathcal{C}^-$ to be defined over the set $I$ we have to define an $i$-adjacency relation for every $i \in I$. To do this we first fix some terminology which is introduced in [11] in Section 5.11.

Given $J \subseteq I$ such that $W_J = \langle s_i | i \in J \rangle$ is finite and an element $w \in W$, then $w$ is called *right $J$-anti-reduced* if $l(w) = max\{l(u) | u \in wW_J\}$. For $w \in W$ and $i \in I$, $w^i$ stands for the unique right $\{i\}$-anti-reduced element in the $i$-panel in $W$ containing $w$. For adjacency we state the following rule:

Two chambers $xU_{-w}$ and $yU_{-v}$ are $i$-adjacent if and only if

(1) $w^i = v^i$,

(2) $xU_{-w^i} = yU_{-w^i}$.

It is easily checked that $\mathcal{C}^-$ equipped with this adjacency relation is indeed a chamber system over $I$ in the sense of [7], Chapter 1.

We also remark that the group $U_-$ acts on the chamber system $\mathcal{C}_-$ by left multiplication. It is easily checked that under this action $i$-adjacent chambers are sent to $i$-adjacent chambers. This means that the group $U_-$ acts as a group of type preserving automorphisms of the chamber system $\mathcal{C}^-$.

The next step is to construct a chamber systems morphism between $\mathcal{C}^-$ and $(\Delta, W, S, d)$.

**Lemma 5.** *The map $\kappa$ between $\mathcal{C}^-$ and $(\Delta, W, S, d)$ that sends $xU_w$ to $xw(c_+)$ is a type preserving morphism between the chamber systems $\mathcal{C}^-$ and $(\Delta, W, S, d)$ (i.e. it sends $i$-adjacent chambers to $i$-adjacent chambers).*

*Proof.* We have to check that $\kappa$ is well defined and that if $xU_{-w}$ and $yU_{-v}$ are $i$-adjacent, then also $\kappa(xU_{-w})$ and $\kappa(yU_{-v})$ are $i$-adjacent. To see this we rely on the following property:

$$U_{-w} \subset Stab_G(w(c_+)). \tag{$*$}$$

Let's first check this property. By Theorem 1 and Lemma 2 the group $w^{-1}U_{-w}w$ is contained in $B_+$. As $Stab_G(c_+) = B_+$ formula $(*)$ is clear. Because of property $(*)$ the map $\kappa$ is well defined, i.e. if $xU_{-w} = x'U_{-w}$ then $x(w(c_+)) = x'(w(c_+))$.

Suppose that $xU_{-w}$ and $yU_{-v}$ are $i$-adjacent, i.e. $w^i = v^i$ and $xU_{-w^i} = yU_{-w^i}$. From $w^i = v^i$ it follows that $w(c_+)$ and $v(c_+)$ are $i$-adjacent and belong to the $i$-panel containing $w^i$. From $y^{-1}x \in U_{-w^i}$ we deduce that $y^{-1}x$ stabilizes $w^i(c_+)$, hence also stabilizes the $i$-panel through $w^i(c_+)$. This means that $y^{-1}x(w(c_+))$ and $v(c_+)$ are $i$-adjacent, hence also $x(w(c_+))$ and $y(v(c_+))$ are $i$-adjacent. This completes the proof of the lemma. □

## 4. Properties of $\kappa$

In this paragraph we show that $\kappa$ is a 2-covering from $\mathcal{C}^-$ onto $(\Delta, W, S, d)$. We start by showing that $\kappa$ is surjective. For this we need an additional property of Moufang buildings.

**Proposition 1.** *Given a Moufang building $(\Delta, W, S, d)$ with standard apartment $\Sigma_0$, then the orbit of $\Sigma_0$ (as a set of chambers) under $B_-$ is the full building $\Delta$, i.e. $B_-(\Sigma_0) = \Delta$.*

*Proof.* The proposition follows from the decomposition $G = B_-WB_+$ regarded the fact that $\{w(c_+)|w \in W\} = \Sigma_0$. First we show that $G = \cup(B_-wB_-)_{w\in W}$.

Using Lemma 4 we write:

$$B_-s_iB_-wB_- = B_-s_iB_{-w}wB_-.$$

Two cases occur:

(1) $l(s_iw) > l(w)$.
Then $s_iB_{-w}s_i \subset B_-$ and

$$B_-s_iB_-wB_- = B_-(s_iB_{-w}s_i)s_iwB_- = B_-s_iwB_-.$$

(2) $l(s_iw) < l(w)$.
Hence

$$\begin{aligned} B_-s_iB_-wB_- &= B_-s_iB_-s_is_iwB_- \\ &\subset \{B_-s_iB_-, B_-\}s_iwB_- \\ &\subset B_-wB_- \cup B_-s_iwB_-. \end{aligned}$$

From this one deduces that $\cup(B_-wB_-)_{w\in W} = G$.

Then we show that for every $w \in W$ and $s_i \in S$

$$B_-s_iB_-wB_+ \subseteq B_-swB_+ \cup B_-wB_+.$$

As above again two cases can occur:
(1) $l(s_iw) < l(w)$.
This means that the root $w^{-1}(\alpha_i)$ is negative, hence

$$\begin{aligned} B_-s_iB_-wB_+ &= B_-s_iB_{-\alpha_i}wB_+ \\ &= B_-s_iw(w^{-1}B_{-\alpha_i}w)B_+ \\ &= B_-s_iwB_+. \end{aligned}$$

(2) $l(s_iw) > l(w)$.
Then we use the above equation and calculate:

$$\begin{aligned} B_-s_iB_-wB_+ &= B_-s_iB_-s_is_iwB_+ \\ &\subset B_-\{1, s_i\}B_-s_iwB_+ \\ &= B_-s_iwB_+ \cup B_-wB_+. \end{aligned}$$

It follows that $B_-WB_+ = G$. □

**Corollary 1.** *The morphism $\kappa$ is surjective.*

*Proof.* Consider an arbitrary chamber $a$ in $\Delta$. Then by Proposition 1 we have $a = b_-v(c_+)$ for some $b_- \in B_-$ and $v \in W$. As for every root $\alpha$, $H \subset Stab_G(U_\alpha)$ we can write $b_-$ as $u_-h$ for $u_- \in U_-$ and $h \in H$. Because $H$ fixes every chamber of $\Sigma_0$ we can write $a = u_-v(c_+)$. If we consider the element $u_-U_{-v}$ of $\mathcal{C}^-$ then clearly $\kappa(u_-U_{-v}) = a$. □

The only problem that remains to prove is that $\kappa$ is a 2-covering.

**Theorem 3.** *The map $\kappa$ is 2-covering from $\mathcal{C}^-$ to $\Delta$, i.e. it sends spherical rank 2 residues isomorphically onto spherical rank 2 residues.*

*Proof.* To prove this we remark that the action of $U_-$ on $\mathcal{C}_-$ and $\Delta$ is compatible with $\kappa$, i.e. for all $xU_{-w} \in \mathcal{C}_-$ and $u_- \in U_-$ we have $\kappa(u_-xU_{-w}) = u_-\kappa(xU_{-w})$. In order to prove that $\kappa$ is a 2-covering, it will then be enough to show that $\kappa$ induces an isomorphism between every $\{i,j\}$ residue containing a chamber $U_{-w}$, with $w$ an $\{i,j\}$-anti-reduced element in $W$, and its image in $\Delta$. To see this we remark that every rank 2 residue in $\mathcal{C}_-$ always contains a chamber $xU_{-w}$ where $w$ is $\{i,j\}$-anti-reduced and $x \in U_{-}$. The morphism determined by $x^{-1}$ will then send the given rank 2 residue to another rank 2 residue that contains $U_{-w}$.

Fix a certain rank 2 residue in $\mathcal{C}_{-}$ of spherical type $\{i,j\}$ (hence $m_{ij} < \infty$). Call this residue $R_{-}^{ij}$. Suppose that $R_{-}^{ij}$ contains a chamber $U_{-w}$ with $w$ $\{i,j\}$-anti-reduced. As $U_{-w} \in R_{-}^{ij}$, we see that $w(c_{+}) \in \kappa(R_{-}^{ij})$. If we denote by $R^{ij}$ the $\{i,j\}$-residue in $\Delta$ which contains $w(c_{+})$ then we have to show that $\kappa$ induces an isomorphism between $R_{-}^{ij}$ and $R^{ij}$.

(1) The map $\kappa$ induces a surjection between $R_{-}^{ij}$ and $R^{ij}$.
This will follow from the fact that $\kappa$ induces a surjection between rank 1 residues. Consider a fixed $i \in I$ and a chamber $a$ in $\Delta$. Using Proposition 1 and the action of $U_{-}$ on $\Delta$ we can assume that $a = v(c_{+})$, $v \in W$. Then every chamber of the $i$-residue containing $a$ can be written under the form $v(u_{\alpha_i}s_i(c_{+}))$ with $u_{\alpha_i} \in U_{\alpha_i}$ and $\alpha_i > 0$.

Two cases occur:

(i) $l(vs_i) < l(v)$.
Then $vu_{\alpha_i} = vu_{\alpha_i}v^{-1}v$ with $vu_{\alpha_i}v^{-1} \in U_{v(\alpha_i)}$. Granted the condition on $v$, one has $U_{v(\alpha_i)} \subset U_{-v^i}$. If we consider in $\mathcal{C}^{-}$ the chamber $vu_{\alpha_i}v^{-1}U_{-vs_i}$, then this chamber is $i$-adjacent to $U_{-v}$ and $\kappa(vu_{\alpha_i}v^{-1}U_{-v}) = a$.

(ii) $l(vs_i) > l(v)$.
Using Lemma 1, one starts by rewriting $u_{\alpha_i}$ as $u_{-\alpha_i}s_ib_{-\alpha_i}$ with $u_{-\alpha_i} \in U_{-\alpha_i}$ and $b_{-\alpha_i} \in B_{-\alpha_i}$. As we also know that $s_ib_{-\alpha_i}s_i \subset B_{\alpha_i}$ the chamber $a$ coincides with $vu_{-\alpha_i}(c_{+})$. Because of the condition on $v$ we have that $vu_{-\alpha_i}v^{-1} \in U_{-v(\alpha_i)} \subset U_{-v^i}$. Hence the chamber $vu_{-\alpha_i}v^{-1}U_{-vs_i}$ is $i$-adjacent to $U_{-v}$ and $\kappa(vu_{-\alpha_i}v^{-1}U_{-vs_i}) = a$.
This completes the proof that $\kappa$ induces a surjection between rank 1 residues in $\mathcal{C}^{-}$ and $\Delta$. Because rank 2 residues are connected it is clear that $\kappa$ induces a surjection of $R_{-}^{ij}$ onto $R^{ij}$.

(2) The morphism $\kappa$ induces an injection of $R_{-}^{i,j}$ into $R^{ij}$.
Suppose that we have two chambers $u'_{-}U_{-w'}$ and $u''_{-}U_{-w''}$ in $R_{-}^{ij}$ such that $\kappa(u'_{-}U_{-w'}) = \kappa(u''_{-}U_{-w''})$. This means that $u'_{-}w'(c_{+}) = u''_{-}w''(c_{+})$ and both $w'$ and $w''$ belong to the $\{i,j\}$-residue in $W$ determined by $w$. Because of the conditions on $w$ it is easy to check that both $u'_{-}$ and $u''_{-}$ belong to $U_{-w}$. We rewrite the above equality as

$$(w^{-1}u'_{-}w)w^{-1}w'(c_{+}) \;=\; (w^{-1}u''_{-}w)w^{-1}w''(c_{+}).$$

As both $u'_{-}$ and $u''_{-}$ belong to $U_{-w}$ the elements $w^{-1}u'_{-}w$ and $w^{-1}u''_{-}w$ belong to $B_{+}$. Call the first one $b'_{+}$ and the second one $b''_{+}$, then we find

$$b'_{+}w^{-1}w'(c_{+}) \;=\; b''_{+}w^{-1}w''(c_{+}).$$

But this implies by the Bruhat decomposition of the group $G$ (as we have a $BN$-pair in $G$) that $w^{-1}w' = w^{-1}w''$, yielding $w' = w''$.

There remains to show that $u'_{-}U_{-w'} = u''_{-}U_{-w''}$.
From the equality $u'_{-}w'(c_{+}) = u''_{-}w'(c_{+})$ one deduces that $w'^{-1}u_{-}''^{-1}u'_{-}w' \in B_{+}$. The element $u_{-}''^{-1}u'_{-}$ is contained in $U_{-w}$ and we call it $u_{-w}$. Then $u_{-w}$ satisfies $w'^{-1}u_{-w}w' \in B_{+}$. Consider the set of positive roots sent by $w^{-1}$ into negative roots, namely $\{\gamma_1,\ldots,\gamma_n\}$. Because of the properties of $w$ we can divide this set into two subsets (after possibly reordering) $\{\gamma_1,\ldots,\gamma_{l-1}\} \sqcup \{\gamma_l,\ldots,\gamma_n\}$. Here $\{\gamma_1,\ldots,\gamma_{l-1}\}$ is the set of positive roots sent by $w'$ to a

negative root and $\{\gamma_l, \dots, \gamma_n\}$ is the set of remaining roots. With this notation in mind we write $u_{-w}$ as $u_{-w'}u_{-r}$ with $u_{-w'} \in U_{-w'}$ and $u_{-r} = u_{-\gamma_l} \dots u_{-\gamma_n}$. We rewrite the formula $w'^{-1}u_{-w}w' \in B_+$ as

$$w'^{-1}u_{-r}w' \;=\; w'^{-1}u^{-1}_{-w'}w'\tilde{b}_+$$

for a $\tilde{b}_+ \in B_+$. The element $w'^{-1}u_{-r}w'$ apparently belongs to $B_+$. Suppose that $w = w' \underbrace{s_j s_i \dots s_j}_{m \text{ terms}}$ with $l(w) = l(w') + m$. Then

$$w'^{-1}\{\gamma_l, \dots, \gamma_n\} \;=\; \{\alpha_j, s_j(\alpha_i), \dots, s_j s_i \dots s_i(\alpha_j)\}.$$

Hence we can write $w'^{-1}u_{-r}w'$ as $u_{-\alpha_j}u_{-s_j(\alpha_i)} \dots u_{-s_j s_i \dots s_i(\alpha_j)}$ yielding that $w^{-1}u_{-r}w' \in U_- \cap B_+$. Now we look at the rank 2 building $\Gamma_{ij}$ determined by $B_{\alpha_i}, B_{-\alpha_i}, B_{\alpha_j}$ and $B_{-\alpha_j}$ (i.e. the rank 2 building we get by considering the group $\langle B_{\alpha_i}, B_{\alpha_j}, B_{-\alpha_i}, B_{-\alpha_j}\rangle$ and the induced $BN$-pair in it). It follows that $w'^{-1}u_{-r}w'$ is inside the group generated by these four groups. But $w'^{-1}u_{-r}w'$ fixes the fundamental chamber $c_+^{ij}$ in this polygon. Hence this element is inside $U_-^{ij} \cap B_+^{ij}$ where the groups $B_+^{ij}$ and $B_-^{ij}$ are similarly as above. The proof that $\kappa$ is a 2-covering will be done if we show the following lemma.

**Lemma 6.** *If we are given a spherical rank* 2 *building with Weylgroup* $\langle s_1, s_2 | (s_1 s_2)^{m_{12}}\rangle$ *then*

$$B_+ \cap B_- = H.$$

*Proof.* If we consider a spherical rank 2 Moufang building, the groups $B_+$ and $B_-$ both have a geometric meaning. Indeed, in the standard apartment $\Sigma$ there will be two chambers $c_+$ and $c_-$ such that the $l(d(c_+, c_-))$ is maximal in the Weylgroup. The group $B_+$ will then be the stabilizer of $c_+$ in $G$, $B_-$ will be the stabilizer of $c_-$ and $H$ will be the stabilizer of the standard apartment in $G$. This implies in particular that $B_- \cap B_+ \subseteq H$ and as $H \subseteq B_+ \cap B_-$ we have

$$H = B_+ \cap B_-. \qquad \square$$

This lemma implies that $w'u_{-r}w'^{-1}$ lies in $H$. Moreover by properties of spherical Moufang buildings explained in [7] on pages 75 and 76 it follows that $u_r^{-1} = 1$. This yields $u_- \in U_{-w'}$ or $u''^{-1}_- u'_- \in U_{-w'}$, hence $u''_- U_{-w'} = u'_- U_{-w'}$ what we wanted to show. This completes the proof of Theorem 3. $\square$

As already mentioned the group $U_-$ acts on both $\mathcal{C}^-$ and $\Delta$ in a way compatible with $\kappa$. This implies that $Stab_{U_-}(a) = Stab_{U_-}(\kappa(a))$ with $a \in \mathcal{C}^-$. If we do this for $c_+$ then $\kappa(1) = c_+$ and $Stab_{U_-}(1) = 1$ and $Stab_{U_-}(c_+) = U_- \cap B_+$. This gives us $U_- \cap B_+ = \{1\}$, which is a very strong condition. Consider $B_- \cap B_+$. Every element in this intersection can be written as $hu_-$ for $h \in H$ and $u_- \in U_-$. But then $u_- = 1$ and the element is contained in $H$. As also $H \subseteq B_- \cap B_+$ we find

$$B_- \cap B_+ \;=\; H.$$

Using the universal properties of buildings we get the following corollary.

**Corollary 2.** *The chamber system* $\mathcal{C}^-$ *is a building of type* $M$ *isomorphic to* $(\Delta, W, S, d)$ *under* $\kappa$.

*Proof.* This follows from the results in [10]. It is shown in this paper that every building is a universal object with respect to 2-coverings. This means that if we cover a building by a chamber system, then this covering is necessarily an isomorphism. □

**Corollary 3.** *The pair $(G, B_-, N, S)$ is a $BN$-pair.*

*Proof.* The proof is similar to the proof of Theorem 2 as $B_\alpha \not\subset B_-$, $\forall \alpha > 0$. □

## 5. The relation $\mathcal{O}$

We start from a Moufang building $(\Delta, W, S, d)$. The set of all roots in $W$ is given by $\Phi = \{\alpha\}$. The rootgroups are denoted by $U_\alpha$. We use notation as before. Then we know that there are two $BN$-pairs involved, $(G, B_+, N, S)$ and $(G, B_-, N, S)$. The first $BN$-pair yields a building $(\Delta_+, W, S, d_+)$ isomorphic to $(\Delta, W, S, d)$. From the second the building $(\Delta_-, W, S, d_-)$ is constructed. As the chambers of $\Delta_+$ and $\Delta_-$ correspond to cosets of $B_+$ respectively $B_-$ the group $G$ acts in a natural way on both buildings. Let $c_+$ and $c_-$ be the chambers of $\Delta_+$ and $\Delta_-$ corresponding to $B_+$ and $B_-$. We define the relation $\mathcal{O} \subset \Delta_+ \times \Delta_- \cup \Delta_- \times \Delta_+$ by the following rules:

( $(x_+, y_-) \in \Delta_+ \times \Delta_-$, $(y_-, x_+) \in \Delta_- \times \Delta_+$)

$$\begin{gathered}(x_+, y_-) \in \mathcal{O} \\ \Updownarrow \\ \exists g \in G \text{ such that } (g(x_+), g(y_-)) = (c_+, c_-)\end{gathered}$$

$$\begin{gathered}(y_-, x_+) \in \mathcal{O} \\ \Updownarrow \\ (x_+, y_-) \in \mathcal{O}\end{gathered}$$

We describe the relation $\mathcal{O}$ for rank 2 Moufang buildings.

**Theorem 4.** *Suppose that $(\Delta, W, S, d)$ is a rank 2 Moufang building of spherical type then the relation $\mathcal{O}$ defines a twinning between $\Delta_+$ and $\Delta_-$.*

*Proof.* As the building $\Delta$ is of spherical type there exists a unique element $w_0$ in $W$ such that $l(w_0) > l(w)\ \forall w \in W$. We make the following construction. Set $(\Delta_1, W, S, d_1) = (\Delta, W, S, d)$, $(\Delta_2, W, S, d_2) = (\Delta, W, S, w_0 d w_0)$. Define a codistance function $d^*$ between $\Delta_1$ and $\Delta_2$ by:

( $(x_1, x_2) \in \Delta_1 \times \Delta_2$, $(x_2, x_1) \in \Delta_2 \times \Delta_1$)

$$\begin{aligned} d^*(x_1, x_2) &= w_0 d(x_1, x_2) \\ d^*(x_2, x_1) &= d(x_1, x_2) w_0. \end{aligned}$$

It follows from Proposition 1 in [12] that the couple $((\Delta_1, W, S, d_1), (\Delta_2, W, S, d_2))$ with the codistance function $d^*$ is a twin building. It can be shown that this is the only possible twinning on $\Delta$.

We know that two $BN$-pairs $(G, B_+, N, S)$ and $(G, B_-, N, S)$ can be constructed. Each of these $BN$-pairs has an associated building. Denote them by $(\Delta_+, W, S, d_+)$ and $(\Delta_-, W, S, d_-)$.

We give a short description of $(\Delta_+, W, S, d_+)$. The set of chambers $\Delta_+$ is given by the set $\{gB_+ \mid g \in G\}$. Let $s \in S$ then $g_1B_+$ is $s$-adjacent to $g_2B_+$ if and only if $B_+g_1^{-1}g_2B_+ = B_+sB_+$. To define the distance between two chambers one uses the Bruhat decomposition of the group $G$. This means that the group $G$ has a decomposition

$$G = \sqcup(B_+wB_+)_{w\in W}.$$

Moreover if $B_+w'B_+ = B_+w''B_+$ then it follows that $w' = w''$. For two chambers $g_1B_+$ and $g_2B_+$ of $\Delta_+$ the distance $d(g_1B_+, g_2B_+)$ is defined as the unique element $v \in W$ such that

$$B_+g_1^{-1}g_2B_+ = B_+vB_+.$$

Using standard arguments it follows that $(\Delta_+, W, S, d_+)$ is a building. The same can be done for $(G, B_-, N, S)$. This gives the building $(\Delta_-, W, S, d_-)$. From the construction of $(\Delta_+, W, S, d_+)$ it can be proved that it is isomorphic to $(\Delta, W, S, d)$. The isomorphism is given by

$$\begin{array}{rcl} \varphi_1 : \Delta_+ & \to & \Delta \\ \varphi_1(gB_+) & \mapsto & g(c_+). \end{array}$$

In a similar way $(\Delta_-, W, S, d_-)$ is isomorphic to $(\Delta_2, W, S, d_2)$.
Consider the group $B_-$. As we work in a spherical building it follows that $B_- = \langle U_{-w_0}, H\rangle$. Hence $w_0B_+w_0 = B_-$. By this we can map every chamber of $\Delta_-$ to a chamber of $\Delta_+$. Namely every $hB_-$ can be written as $hw_0B_+w_0$. If we send every $hB_-$ to $hw_0B_+$ this is well defined. Call this map $Opp_{w_0}$. The composition $\varphi_2 = Opp_{w_0} \circ \varphi_1$ defines a bijection of $\Delta_-$ to $\Delta$. Moreover $\varphi_2$ sends $s$-adjacent chambers to $w_0sw_0$-adjacent chambers. This implies that $(\Delta_-, W, S, d_-)$ is isomorphic to $(\Delta_2, W, S, w_0dw_0)$ under $\varphi_2$. The explicit formula for $\varphi_2$ is given by

$$\begin{array}{rcl} \varphi : \Delta_- & \to & \Delta_2 \\ hB_- & \mapsto & hw_0(c_+). \end{array}$$

To finish the proof we show the following equivalence:
$((x_+, y_-) \in \Delta_+ \times \Delta_-)$

$$(x_+, y_-) \in \mathcal{O} \Leftrightarrow d^*(\varphi_1(x_+), \varphi_2(y_-)) = 1.$$

(1) If $(x_+, y_-) \in \mathcal{O}$ then $x_+ = gB_+$ and $y_- = gB_-$, with $g \in G$. Hence $\varphi_1(x_+) = g(c_+)$ and $\varphi_2(y_-) = gw_0(c_+)$. We calculate

$$\begin{array}{rcl} d(g(c_+), gw_0(c_+)) & = & d(c_+, w_0(c_+)) \\ & = & d(c_+, w_0c_+) \\ & = & w_0. \end{array}$$

This implies that $d^*(\varphi_1(x_+), \varphi_2(y_-)) = 1$.

(2) Suppose $gB_+$ and $hB_-$ are such that $d^*(\varphi_1(gB_+), \varphi_2(hB_-)) = 1$. This means that $d(g(c_+), hw_0(c_+)) = w_0$. Using the isomorphism $\varphi_1$ and the Bruhat decomposition of $G$ if follows that

$$hb_- = gb_+$$

for appropriate $b_- \in B_-$ and $b_+ \in B_+$. This means that $(gB_+, hB_-) \in \mathcal{O}$. □

Remains to prove the same result for non-spherical rank 2 Moufang buildings. Let $(\Gamma, W, S, d)$ be such a building. We consider a graph whose vertex set $V$ is the set of all residues in $\Gamma$. Two vertices are joined by an edge if and only if they lie in a chamber. In this way we get a bipartite graph $(V, E)$, which turns out to be a tree. It can also be easily checked that every isomorphism of $\Gamma$ as building induces an isomorphism of the tree $(V, E)$. For more information about non-spherical rank 2 Moufang buildings we refer to [8]. The result we will prove is:

**Theorem 5.** *Given a non-spherical rank 2 Moufang building $(\Gamma, W, S, d)$ then the relation $\mathcal{O}$ defines the opposition relation of a twinning between $\Delta_+$ and $\Delta_-$.*

*Proof.* First we fix some notations and terminology.
Denote $W = \{s, t\}$. The chambers of $\Gamma$ will be considered as doubletons $\{x, x'\}$, where $x$ and $x'$ stand for the simplices in the chamber $\{x, x'\}$. We assume that the standard chamber is given by $c_0 = \{x_0, x_1\}$ and the standard apartment $\Sigma_0$ is the sequence $\ldots c_{-2} \stackrel{t}{\sim} c_{-1} \stackrel{s}{\sim} c_0 \stackrel{t}{\sim} c_1 \stackrel{s}{\sim} c_2 \ldots$. Write $c_i = \{x_i, x_{i+1}\}$, $\forall i$. Then the standard apartment $\Sigma_0$ corresponds to a sequence $\ldots x_{-2} \sim x_{-1} \sim x_0 \sim x_1 \sim x_2 \ldots$ in the tree $(V, E)$.
As to the Moufang structure on $\Gamma$ we keep the notations from above.
Let $\alpha_i^+$ be the positive root of $\Sigma_0$ such that $x_i$ lies on its boundary. Similarly $\alpha_i^-$ is the negative root of $\Sigma_0$ such that $x_i$ lies on $\partial\alpha_i^-$. By calculations already made there are two $BN$-pairs involved: $(G, B_+, N, S)$ and $(G, B_-, N, S)$. They give rise to two buildings $(\Delta_+, W, S, d_+)$ and $(\Delta_-, W, S, d_-)$. To prove that $\mathcal{O}$ is the opposition relation of a twinning between $\Delta_+$ and $\Delta_-$ we refer to Proposition 5.4. of [2]. In order to use this proposition we show the following:
(i) The relation $\mathcal{O}$ defines a 1-twinning between $\Delta_+$ and $\Delta_-$.
(ii) For any four chambers $y_-$, $c_-^1$ and $c_-^2$ in $\Delta_-$ and $e_+ \in \Delta_+$ such that $(e_+, c_-^1) \in \mathcal{O}$, $(e_+, c_-^2) \in \mathcal{O}$ and

$$\begin{aligned} l(d_-(c_-^1, y_-)) &= l(d_-(c_-^2, y_-)) \\ &= min\{l(d_-(a_-, y_-)) | (e_+, a_-) \in \mathcal{O}\} \end{aligned}$$

we have $d_-(c_-^1, y_-) = d_-(c_-^2, y_-)$.
(iii) For any four chambers $y_- \in \Delta_-$, $y_+^1, y_+^2, e_+ \in \Delta_+$ such that $(y_+^1, y_-) \in \mathcal{O}$, $(y_+^2, y_-) \in \mathcal{O}$ and

$$\begin{aligned} l(d_+(e_+, y_+^1)) &= l(d_+(e_+, y_+^2)) \\ &= min\{l(d_+(a_+, c_+)) | (a_+, y_-) \in \mathcal{O}\} \end{aligned}$$

we have $d_+(y_+^1, e_+) = d_+(y_+^2, e_+)$.

(iv) Given chambers $y_-$, $a_- \in \Delta_-$, $e_+$ and $b_+ \in \Delta$ such that $a_-$ is as in (ii), $l(d(a_-, y_-))$ is minimal, $b_+$ is as in (iii) and $l(d(d_+, b_+))$ is minimal then

$$d_+(e_+, b_+) = d(a_-, y_-).$$

If (i), (ii), (iii) and (iv) are satisfied we define for every $x \in \Delta_\epsilon$ ($\epsilon \in \{1, -1\}$) a codistance function $d_x : \Delta_- \mapsto W$. For every $z \in \Delta_{-\epsilon}$, $d_x(z)$ equals $d_{-\epsilon}(x_{-\epsilon}, z)$ with $(x, x_{-\epsilon}) \in \mathcal{O}$ such that $l(d(x_{-\epsilon}, z)$ is mimimal as in (ii) or (iii). One easily checks this defines a codistance funtion for every $x$.
Remains to check these 4 properties:

(1) Because of the definition of $\mathcal{O}$ it suffices to check that $\mathcal{O}$ defines a 1-twinning between the $s$-residue $R^s_+$ in $\Delta_+$ containing $c_+$ and the $s$-residue $R^s_-$ in $\Delta_-$ containing $c_-$. We check that for all the chambers $x_-$ of $R^s_-$ satisfy $(x_-, c_+) \in \mathcal{O}$ except $s(c_-)$.
As the stabilizer of $R^s_+$ and $R^s_-$ acts transitively on the chambers of these residues this will be enough to ensure that $\mathcal{O}$ defines a 1-twinning between $R^s_+$ and $R^s_-$. Every element of $R^s_-$ has the form $u_{-\alpha_s}s(c_-)$ for $u_{-\alpha_s} \in U_{-\alpha_s}$. Suppose that $u_{-\alpha_s} \neq 1$. Granted the properties of the $BN$-pair $(G, B_-, N, S)$ we can write $u_{-\alpha_s}sc_- = u_{\alpha_s}su'_{\alpha_s}sc_-$ for appropriate $u_{\alpha_s}$ and $u'_{\alpha_s} \in U_{\alpha_s}$. But then $u_{-\alpha_s}s(c_-) = u_{\alpha_s}(c_-)$. And $(c_+, u_{-\alpha_s}(c_-)) = (u_{\alpha_s}(c_+), u_{\alpha_s}(c_-))$. Hence $(c_+, u_{-\alpha_s}(c_-)) \in \mathcal{O}$.
Consider the chamber $s(c_-)$. If $(c_+, s(c_-)) \in \mathcal{O}$ then there would exist a $g \in G$ such that $g(c_+) = c_+$ and $g(s(c_-)) = c_-$. But then $g \in B_+$ and $gs \in B_-$ or $s = b_+b_-$ for $b_+ \in B_+$ and $b_- \in B_-$. This contradicts the fact that $s$ stabilizes the standard apartment $\Sigma_0$.
Hence $(s(c_-), c_+) \notin \mathcal{O}$.
Granted the action of $G$ on $\Delta_+$ and $\Delta_-$ we may assume that $d_+ = c_+$ in $(2), (3)$ and $(4)$.

(2) Suppose that $y_-$, $c^1_-$ and $c^2_-$ are chambers as in (ii) with $(c_+, y_-) \notin \mathcal{O}$. Then $y_- = gB_-$, $c^1_- = b^1_+B_-$ and $c^2_- = b^2_+B_-$ for $g \in G$, $b^i_+ \in B_+$. Let $d_-(c^1_-, y_-) = w_1$ and $d_-(c^2_-, y_-) = w_2$. It follows from the assumptions that $l(w_1) = l(w_2)$.
Assume $w_1 \neq w_2$.
Because we work in a non-spherical Coxeter group two possibilities occur. Namely $w_1^2 = w_2^2 = 1$ or $w_1^2 \neq 1$ and $w_2^2 \neq 1$.
Expressing that the distances from $c^1_-$ and $c^2_-$ to $y_-$ are $w_1$ and $w_2$ gives:

$$\begin{aligned} gB_- &= b^1_+b^1_-w_1B_- \\ &= b^2_+b^2_-w_2B_- \end{aligned}$$

for $b^i_- \in B_-$.
Hence

$$b^1_+b^1_-w_1 = b^2_+b^2_-w_2b_-$$

for $b_- \in B_-$.
But then

$$(b^2_+)^{-1}b^1_+ = b^2_-w_2b_-w_1^{-1}(b^1_-)^{-1}.$$

If $w_1^2 = w_2^2 = 1$ then

$$b^2_-w_2b_-w_1^{-1}(b^1_-)^{-1} = b'_-w_2w_1b''_-$$

for $b'_-, b''_- \in B_-$.
If $w_1^2 \neq 1$ and $w_2^2 \neq 1$ then $w_1 w_2 = 1$ and

$$b_-^2 w_2 b_- w_1^{-1} (b_-^1)^{-1} = b'_- w_2^2 b''_-$$

for $b'_-, b''_- \in B_-$.
In all cases we find that if $w_1 \neq w_2$ then for a $v \neq 1$, $b'_-$ and $b''_- \in B_-$

$$b'_- v b''_- \in B_+,$$

with $l(v) = 0 \ mod\ 2$. This means that $b'_- v b''_-$ has to fix the chamber $c_+$. Write $b'_- v b''_- = u'_- v u''_- h$ for $h \in H$. Then $u'_- v u''_-$ has to fix $c_+$.
Two cases occur:

(a) $u''_- = 1$.
Then we have $u'_-(v(x_0) = x_0$ and $u'_-(v(x_1)) = x_1$. This is only possible if $v = 1$ and $u'_- = 1$.

(b) The element $u''_- \neq 1$.
Suppose that $W = \{s, t\}$, $\partial\alpha_s = x_0$, $\partial\alpha_t = x_1$.
If $u''_- \in U_{-\alpha_s}$ we find that $u'_- v u''_-(x_0) = x_0$. Granted the condition on $u''_-$ this implies that $u'_-(v(x_0)) = x_0$. Again a contradiction.
Hence there exists an index $j$ such that $x_j \sim y_1 \sim y_2 \sim \ldots \sim u''_-(x_0) \sim u''_-(x_1)$ is the gallery in $\Gamma$ from $\Sigma_0$ to $u''_-(x_1)$.
Suppose that $j < 0$ (we already excluded the case where $j = 0$).
Because $l(v) = 0 \ mod\ 2$, $v$ acts as a translation of $\Sigma_0$, i.e.

$$v(x_l) = x_{l+k_0}, \ \forall l$$

for a fixed $k_0 \in \mathbb{Z}$.
Let $v(x_j) = x_m$.
If $m \leq 0$ then $d_+(x_j, u''_-(x_0)) \neq d_+(x_m, x_0)$. One easily checks that there cannot exist a $u'_- \in U_-$ with $u'_-(v u''_-(x_0)) = x_0$.
If $m \geq 1$ then

$$d_+(x_m, v u''_-(x_0)) < d_+(x_m, v u''_-(x_1)).$$

Using this fact one also checks that for no $u'_- \in U_-$ we can have $u'_-(v u''_-(x_0)) = x_0$.
If $j > 0$ one uses similar arguments to deduce a contradiction.

(3) If $(y_+^1, y_-)$ and $(y_+^2, y_-) \in \mathcal{O}$ then

$$\begin{aligned} y_- &= g(c_-) \\ y_+^1 &= g(c_+) \\ y_+^2 &= g b_-(c_+) \end{aligned}$$

for $g \in G$ and $b_- \in B_-$.
A symmetric proof completely analogous to (2) gives $d_+(y_+^1, c_+) = d_+(y_+^2, c_+)$.

(4) Let $y_-$ and $c_-^1$ be chambers of $\Delta_-$ with $(c_+, c_-^1) \in \mathcal{O}$ and $d(c_-^1, y_-)$ being minimal as in (ii). Then we look for a chamber $y_+$ in $\Delta_+$ such that $(y_+, y_-) \in \mathcal{O}$ and $d_+(c_+, y_+) = d_-(c_-^1, y_-)$.

This will imply (iv). Without loss of generality we can assume that $c^1_- = c_-$. Let the minimal gallery in $\Delta$ between $c_-$ and $y_-$ be

$$y^0_- = c_- \overset{s}{\sim} y^1_- \overset{t}{\sim} y^2_- \overset{s}{\sim} \ldots \overset{t}{\sim} y^m_- = y_-$$

If $y^1_- = u_{-\alpha_s} s(c_-)$ let $y^1_+$ be $u_{-\alpha_s} s(c_+)$. If $y^2_- = u_{-\alpha_t} t u_{-\alpha_s} s(c_-)$ let $y^2_+$ be $u_{-\alpha_t} t u_{-\alpha_s} s(c_+)$. If we do this for all $y^i_-$ we get a gallery

$$y^0_+ = c_+ \overset{s}{\sim} y^1_+ \overset{t}{\sim} y^2_+ \overset{s}{\sim} \ldots \overset{t}{\sim} y^m_+$$

from $c_+$ to $y^m_+$. One shows with a proof similar as in (2) that for no $v \in W$ and $b_-, b'_- \in B_-$ we can have that $b_- v b'_- \in B_+$. This ensures us that all the $y^j_+$ are different. The gallery is therefore non-stammering and $d_+(c_+, y^m_+) = d_-(c^1_-, y_-)$. By construction we have $(y_+, y_-) \in \mathcal{O}$.

This completes the proof that (i), (ii), (iii) and (iv) are satisfied for $\mathcal{O}$. Hence $\mathcal{O}$ is the opposition relation of a twinning between $\Delta_+$ and $\Delta_-$. □

## 6. Constructing a 2-twinning

In this paragraph we will show that the building $(\Delta, W, S, d)$ is half of a twin building using a result of B. Mühlherr in [2]. We restate the main result of loc. cit.

**Theorem 6.** *Let $M$ be a Coxeter matrix over $I$, let $(\Delta_+, W, S, d_+)$ and $(\Delta_-, W, S, d_-)$ be two thick buildings of type $M$ and let $\mathcal{O} \subseteq (\Delta_+ \times \Delta_-) \cup (\Delta_- \times \Delta_+)$ be a non-empty symmetric relation. Then $\mathcal{O}$ is the opposition relation of a twinning between $(\Delta_+, W, S, \delta_+)$ and $(\Delta_-, W, S, \delta_-)$ if and only if the following condition is satisfied:*
*If $J \subseteq I$ is of cardinality at most 2 and if $R_+ \subseteq \Delta_+$ and $R_- \subseteq \Delta_-$ are $J$-residues, then either $\mathcal{O} \cap ((R_+ \times R_-) \cup (R_- \times R_+)) = \varnothing$ or $\mathcal{O} \cap ((R_+ \times R_-) \cup (R_- \times R_+))$ is the opposition relation of a twinning between $R_+$ and $R_-$.*

We now have:

**Theorem 7.** *Given a Moufang building $(\Delta, W, S, d)$ with root groups $(U_\alpha)_{\alpha \in \Phi}$ then $\Delta$ is half of a twin building, i.e. there exists a building $(\Delta_-, W, S, d_-)$ and a codistance function $d^*$ such that $((\Delta, W, S, d), (\Delta_-, W, S, d_-), d^*)$ is a twin building.*

*Proof.* By Theorem 2 and Corollary 3 we know that there are two $BN$-pairs involved. Namely $(G, B_+, N, S)$ and $(G, B_-, N, S)$. The building $(\Delta_+, W, S, d_+)$ associated to $(G, B_+, N, S)$ is by construction isomorphic to $\Delta$. We define the symmetric relation $\mathcal{O}$ between $\Delta_+$ and $\Delta_-$ as before. Consider $s_i, s_j \in S$. Let $R^+_{s_i s_j}$ and $R^-_{s_i s_j}$ be the $\{s_i, s_j\}$-residues in $\Delta_+$ and $\Delta_-$ containing $c_+$ and $c_-$ respectively. Then it follows from Theorem 4 and Theorem 5 that $\mathcal{O}$ defines the opposition relation of a twinning between $R^+_{s_i s_j}$ and $R^-_{s_i s_j}$. By construction this implies that $\mathcal{O}$ satisfies the conditions of Theorem 6. Hence $\mathcal{O}$ defines a twinning between $\Delta_+$ and $\Delta_-$. This means that $\Delta \cong \Delta_+$ is half of a twin building. □

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