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Hereditariness, Strongness and Relationship between Brown-McCoy and Behrens Radicals

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Abstract. In this paper we explore the properties of being hereditary and being strong among the radicals of associative rings, and prove certain results such as a relationship between Brown-McCoy and Behrens radicals.

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I.

In this paper rings are all associative, but not necessarily with a unit element. As usual, $I \triangleleft A$ and $L \triangleleft_l A$ ($R \triangleleft_r A$) denote that I is an ideal and L is a left ideal (R is a right ideal) in A, respectively. A^o will stand for the ring on the additive group $(A, +)$ with multiplication $xy = 0$, for all $x, y \in A$.

Let us recall that a (Kurosh-Amitsur) *radical* γ is a class of rings which is closed under homomorphisms, extensions (I and A/I in γ imply A in γ), and has the inductive property (if $I_1 \subseteq \cdots \subseteq I_\lambda \subseteq \ldots$ is a chain of ideals, $A = \cup I_\lambda$, and each I_λ is in γ, then A is in γ).

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The unique largest γ-ideal $\gamma(A)$ of A is then the γ-radical of A. A hereditary radical containing all nilpotent rings is called *a supernilpotent radical.* Let $\mathcal{M}$ be a class of rings. Put

$$\overline{\mathcal{M}} = \{A \mid \text{every ideal of } A \text{ is in } \mathcal{M}\}.$$

A radical γ is said to be *principally left (right) hereditary* if $a \in A \in \gamma$ implies $Aa \in \gamma$ ($aA \in \gamma$, respectively). A radical γ is said to be *left (right) strong* if $L \triangleleft_l A$ ($R \triangleleft_r A$) and $L \in \gamma$ ($R \in \gamma$) imply $L \subseteq \gamma(A)$ ($R \subseteq \gamma(A)$, respectively). A radical γ is *normal* if γ is left strong and principally left hereditary. We shall make use of the following condition a left ideal L of a ring A may satisfy with respect to a class $\mathcal{M}$ of rings:

($*$) $L \triangleleft_l A$ *and* $Lz \in \mathcal{M}$ *for all* $z \in L \cup \{1\}$.

A radical γ is said to be *principally left strong* if $L \subseteq \gamma(A)$ whenever the left ideal L of a ring A satisfies condition ($*$) with respect to the class $\gamma(= \mathcal{M})$. Principally right strongness is defined analogously.

We will focus on two conditions that a class $\mathcal{M}$ can satisfy.

(H) *If* $A^o \in \mathcal{M}$ *then* $S \in \mathcal{M}$ *for every subring* $S \subseteq A^o$.

(Z) *If* $A \in \mathcal{M}$ *then* $A^o \in \mathcal{M}$.

A class $\mathcal{M}$ of rings is said to be *regular* if every nonzero ideal of a ring in $\mathcal{M}$ has a nonzero homomorphic image in $\mathcal{M}$. Starting from a regular (in particular, hereditary) class $\mathcal{M}$ of rings the *upper radical operator* $\mathcal{U}$ yields a radical class

$$\mathcal{U}\mathcal{M} = \{A \mid A \text{ has no nonzero homomorphic image in } \mathcal{M}\}.$$

Recall that the *Baer radical* β is the upper radical determined by all prime rings, the *Brown-McCoy radical* $\mathcal{G}$ is the upper radical determined by all simple rings with unity element, and the Behrens radical $\mathcal{B}$ is the upper radical of all subdirectly irreducible rings having a nonzero idempotent in their hearts.

The lower principally left strong radical construction $\mathcal{L}_{ps}(\mathcal{M})$ is similar to the lower (left) strong radical construction $\mathcal{L}_s(\mathcal{M})$ (see [1]).

We shall construct the lower principally left strong radical (see also [7]) in the following way. Let $\mathcal{M}$ be a homomorphically closed class of rings and define $\mathcal{M} = \mathcal{M}_1$,

$$\mathcal{M}_{\alpha+1} = \left\{ A \;\middle|\; \begin{array}{l} \text{every nonzero homomorphic image of } A \text{ has a} \\ \text{nonzero left ideal with } (*) \text{ in } \mathcal{M}_\alpha \text{ or a nonzero} \\ \text{ideal } I \in \mathcal{M}_\alpha \end{array} \right\}$$

for ordinals $\alpha \geq 1$ and $\mathcal{M}_\lambda = \bigcup\limits_{\alpha<\lambda} \mathcal{M}_\alpha$ for limit ordinals λ. In particular,

$$\mathcal{M}_2 = \left\{ A \;\middle|\; \begin{array}{l} \text{every nonzero homomorphic image of } A \text{ has a} \\ \text{nonzero left ideal with } (*) \text{ in } \mathcal{M} \text{ or a nonzero ideal} \\ I \in \mathcal{M} \end{array} \right\}.$$

The class $\mathcal{L}_{ps}(\mathcal{M}) = \bigcup\limits_{\alpha} \mathcal{M}_\alpha$ is called the lower principally left strong radical class. As shown in [6] $\mathcal{L}_{ps}(\mathcal{M})$ is the smallest principally left strong radical containing $\mathcal{M}$ and

$$\mathcal{M} \subseteq \mathcal{L}(\mathcal{M}) \subseteq \mathcal{L}_{ps}(\mathcal{M}) \subseteq \mathcal{L}_s(\mathcal{M}).$$

For any class $\mathcal{M}$ let us define $\mathcal{M}^{\text{o}} = \{A \mid A^{\text{o}} \in \mathcal{M}\}$. It is easy to see that if $\mathcal{M}$ is a radical then so is $\mathcal{M}^{\text{o}}$. Let

$$\gamma_l = \{A \in \gamma \mid \text{every left ideal of } A \text{ is in } \gamma\}$$

and

$$\gamma_r = \{A \in \gamma \mid \text{every right ideal of } A \text{ is in } \gamma\}.$$

Next, we recall some results which will be used later on.

Proposition 1. [2, Lemma 1] *Let γ be a radical. If S is a subring of a ring A such that $S^{\text{o}} \in \gamma$, then also $(S^*)^{\text{o}} \in \gamma$ where S^* denotes the ideal of A generated by S.*

Proposition 2. [5, Lemma 2.4] *Let γ be a radical. If $(\beta(A))^{\text{o}} \in \gamma$, then $\beta(A) \in \gamma$.*

Proposition 3. [2, Corollary 1] *If $\mathcal{M} \subseteq \mathcal{M}^{\text{o}}$ then $\mathcal{L}(\mathcal{M}) \subseteq (\mathcal{L}(\mathcal{M}))^{\text{o}}$ and $\mathcal{L}_s(\mathcal{M}) \subseteq (\mathcal{L}_s(\mathcal{M}))^{\text{o}}$.*

Proposition 4. [4, Theorem 4] *If a radical γ is left strong and principally left hereditary, then γ is normal.*

Proposition 5. [2, Lemma 2] *For any element a of a ring A, $I = r(a)a$, where $r(a) = \{x \in A \mid ax = 0\}$ is an ideal of Aa and $I^2 = 0$. In addition Aa/I is a homomorphic image of aA.*

Proposition 6. [5, Corollary 4.2] *A radical γ is hereditary and normal if and only if γ is principally left strong, principally left hereditary and satisfies condition* (H).

Proposition 7. [7, Theorem 6] *A radical γ is normal if and only if γ is principally left or right hereditary and principally left or right strong.*

Proposition 8. [6, Theorem 3.3] *Let $\mathcal{M}$ be a homomorphically closed class of rings satisfying:*

1) *$\mathcal{M}$ contains all zero rings;*
2) *$\mathcal{M}$ is hereditary;*
3) *if $I \triangleleft A$, $I^2 = 0$ and $A/I \in \mathcal{M}$ then $A \in \mathcal{M}$.*

Then $\mathcal{L}_{ps}(\mathcal{M}) = \mathcal{M}_2$.

Proposition 9. [5, Theorem 5.1] *The Behrens radical class $\mathcal{B}$ is the largest principally left hereditary subclass of the Brown-McCoy radical class $\mathcal{G}$, in fact*

$$\mathcal{B} = \mathcal{MG},$$

where

$$\mathcal{MG} = \{A \mid Aa \in \mathcal{G} \text{ for all } a \in A\}.$$

A ring A is said to be *(right) strongly prime* if every non-zero ideal I of A contains a finite subset F such that $r_A(F) = 0$, where $r_A(F) = \{x \in A \mid Fx = 0\}$.

The (right) strongly prime radical S is defined as the upper radical determined by the class of all strongly prime rings, i.e. for any ring A,

$$S(A) = \cap\{I \triangleleft A \mid A/I \text{ is strongly prime}\}.$$

It is known that the radical S is special: so, in particular, S is hereditary and contains the prime radical β.

Proposition 10. [3, Corollary 1] *The (right) strongly prime radical S is right strong.*

II.

Proposition 11. *Let γ be a principally left strong radical satisfying the conditions* (H) *and* (Z)*. Then the largest hereditary subclass $\overline{\gamma}$ of γ will be principally left strong.*

Proof. Let $L \triangleleft_l A$ be such that $L \in \overline{\gamma}$ and $Lz \in \overline{\gamma}$ for every $z \in L$. Let L^* be the ideal in A generated by L, $L^* = L + LA$ and suppose $I \triangleleft L^*$. Then $IL \triangleleft L$, $IL \triangleleft_l I$ and $ILz \triangleleft Lz \in \overline{\gamma}$ for all $z \in L$. Since γ satisfies condition (H), $\overline{\gamma}$ is hereditary, and so $ILz \in \overline{\gamma}$ for all $z \in IL$. Since γ is principally left strong $IL \subseteq \gamma(I)$. We have

$$I(L^*)^2 = I(L + LA)L^* = (IL + ILA)L^* \subseteq ILL^* \subseteq \gamma(I)L^* \subseteq \gamma(I).$$

So $I^3 \subseteq I(L^*)^2 \subseteq \gamma(I)$ and therefore $I/\gamma(I)$ is nilpotent, implying $I/\gamma(I) \in \beta$. We claim that $I^{\mathrm{o}} \in \gamma$. Since $L \in \overline{\gamma} \subseteq \gamma$, by (Z) we conclude that $L^{\mathrm{o}} \in \gamma$. Now Proposition 1 implies that $(L^*)^{\mathrm{o}} \in \gamma$ and so by (H) it follows $I^{\mathrm{o}} \in \gamma$. Hence $(I/\gamma(I))^{\mathrm{o}} \in \gamma \cap \beta$ and applying Proposition 2 and taking into consideration that $I/\gamma(I)$ is nilpotent, we get

$$I/\beta(I) = \beta(I/\gamma(A)) \in \gamma.$$

Thus $I \in \gamma$ and so $\overline{\gamma}$ is principally left strong. □

Corollary 12. *If a class $\mathcal{M}$ is hereditary and satisfies* (Z) *then $\mathcal{L}_{ps}(\mathcal{M})$ is hereditary.*

Proof. By Proposition 3, we have $\mathcal{L}_{ps}(\mathcal{M}) \subseteq \mathcal{L}_s(\mathcal{M}) \subseteq (\mathcal{L}_s(\mathcal{M}))^{\mathrm{o}}$. Let $A \in \mathcal{L}_{ps}(\mathcal{M})$ then we get $\overline{A^{\mathrm{o}} \in \mathcal{L}_s(\mathcal{M})}$ and so $A^{\mathrm{o}} \in \overline{\mathcal{L}(\mathcal{M})}$. Since $\mathcal{L}(\mathcal{M})$ is hereditary, we conclude that $A^{\mathrm{o}} \in \overline{\mathcal{L}(\mathcal{M})}$ and so $A^{\mathrm{o}} \in \overline{\mathcal{L}_{ps}(\mathcal{M})}$. This means that $\overline{\mathcal{L}_{ps}(\mathcal{M})}$ satisfies the conditions (Z) and (H). By Proposition 11, $\overline{\mathcal{L}_{ps}(\mathcal{M})}$ is principally left strong and $\mathcal{M} \subseteq \overline{\mathcal{L}_{ps}(\mathcal{M})} \subseteq \mathcal{L}_{ps}(\mathcal{M})$ and this implies $\overline{\mathcal{L}_{ps}(\mathcal{M})} = \mathcal{L}_{ps}(\mathcal{M})$. □

Proposition 13. *Let γ be a principally left strong radical satisfying the conditions* (H) *and* (Z)*. Then γ_r is left strong.*

Proof. Let $L \triangleleft_l A$ and $L \in \gamma_r$ and let K be a left ideal of $L^* = L + LA$. Since $L \in \gamma_r$, $kL \in \gamma$ for every $k \in K$. Let $R \triangleleft_r kL$. Then it is easy to see that $RkL \in \gamma$, and by conditions (Z) and (H), $R/RkL \in \gamma$ and so $R \in \gamma$. Hence $kL \in \gamma_r$ for every $k \in K$. An argument similar to the proof of Proposition 5 will show that $(Lk + r(k)k)/r(k)k$ is a homomorphic image of kL, where $r(k) = \{x \in L^*/kx = 0\}$. Hence $(Lk + r(k)k)/r(k)k \in \gamma$. By (H) and (Z) we have $r(k)k \in \gamma$ and so $Lk \in \gamma$ for every $k \in K$. Therefore $Lk \subseteq \gamma(K)$ and $LK \subseteq \gamma(K)$. Clearly

$$K^3 \subseteq (L^*K)K \subseteq (LA^1K)K \subseteq LL^*K \subseteq LK \subseteq \gamma(K)$$

hence $K \in \gamma$ by Proposition 2. □

The next result is a generalization of [2, Corollary 4].

Corollary 14. *If $\mathcal{M}$ is a right hereditary class with* (Z)*, then $\mathcal{L}_{ps}(\mathcal{M})$ is one-sided hereditary and $\mathcal{L}_{ps}(\mathcal{M}) = \mathcal{L}_s(\mathcal{M})$ (i.e. $\mathcal{L}_{ps}(\mathcal{M})$ is left and right hereditary).*

Proof. By Corollary 12, $\mathcal{L}_{ps}(\mathcal{M})$ satisfies condition (H). Let $A \in \mathcal{L}_{ps}(\mathcal{M})$. Then it is easy to see that $A^o \in \mathcal{L}_{ps}(\mathcal{M})$. Hence $\mathcal{L}_{ps}(\mathcal{M})$ satisfies condition (Z). Hence $\mathcal{L}_{ps}(\mathcal{M})_r$ is a radical. By Proposition 13, $\mathcal{L}_{ps}(\mathcal{M})_r$ is left strong. Since $\mathcal{M} \subseteq \mathcal{L}_{ps}(\mathcal{M})_r$ we get $\mathcal{M} \subseteq \mathcal{L}_{ps}(\mathcal{M})_r \subseteq \mathcal{L}_{ps}(\mathcal{M}) \subseteq \mathcal{L}_s(\mathcal{M})$ and $\mathcal{L}_{ps}(\mathcal{M})_r = \mathcal{L}_s(\mathcal{M})$. Hence $\mathcal{L}_{ps}(\mathcal{M}) = \mathcal{L}_s(\mathcal{M})$. Since $\mathcal{L}_{ps}(\mathcal{M})_r$ is right hereditary and left strong, we have that $\mathcal{L}_{ps}(\mathcal{M})$ is one-sided hereditary. □

Theorem 15. *Let $\gamma \neq 0$ be a principally left strong radical with* (Z) *and* (H). *Then γ_r is contained in γ as a largest nonzero hereditary and normal subradical. Furthermore, $\overline{\gamma}$ is contained in γ as a largest non-zero hereditary principally left strong subradical.*

Proof. Let $0 \neq A \in \gamma$. By (Z), $A^o \in \gamma$ and by (H), $A^o \in \gamma_r$. All zero-rings of γ are in γ_r and so $\gamma_r \neq 0$. Hence γ_r satisfies conditions (Z) and (H). By Propositions 13, 6 and 4, γ is normal and hereditary.

The second part of the theorem follows from Proposition 11. □

Corollary 16. *The largest left hereditary subclass S_l of strongly prime radical S is the largest normal radical contained in S.*

Theorem 17. *The following statements are equivalent for a radical γ.*

1) *γ is hereditary and normal.*
2) *γ is left or right principally hereditary, principally left or right strong and satisfies condition* (H).
3) *There exists a principally left (right, respectively) strong radical δ such that $\delta_r = \gamma$ ($\delta_l = \gamma$, respectively) and satisfies conditions* (Z) *and* (H).
4) *There exists a right (left, respectively) hereditary class $\mathcal{M}$ of rings satisfying* (Z) *such that $\gamma = \mathcal{L}_{ps}(\mathcal{M})$ ($\gamma = \mathcal{L}'_{ps}(\mathcal{M})$, respectively), where $\mathcal{L}'_{ps}(\mathcal{M})$ is principally right strong radical generated by $\mathcal{M}$.*

Proof. 2) $\Longrightarrow$ 1): By Proposition 7, γ is normal and by Proposition 6, γ is hereditary.

1) $\Longrightarrow$ 3): We claim that γ is one-sided hereditary. So let $L \triangleleft_l A \in \gamma$. Since γ is normal, γ is principally left hereditary, so $Aa \in \gamma$, for all $a \in L$. Therefore $Aa \cdot z \in \gamma$ for every $z \in Aa$. Hence $Aa \subseteq \gamma(L)$ for all $a \in L$, and this gives $L^2 \subseteq \gamma(L)$. Again, since γ is normal and satisfies condition (Z), $A^o \in \gamma$ and by hereditariness $L^o \in \gamma$. Therefore $L \in \gamma$. Right hereditariness is proved analogously. Now we choose δ to be γ, $\delta = \gamma$ and we have $\gamma = \delta = \delta_l = \delta_r$.

3) $\Longrightarrow$ 4): We choose $\mathcal{M} = \delta_r$ ($\mathcal{M} = \delta_l$, respectively). Then $\delta_r = \mathcal{L}_{ps}(\delta_r) = \mathcal{L}_{ps}(\mathcal{M})$ ($\delta_l = \mathcal{L}'_{ps}(\delta_l) = \mathcal{L}'_{ps}(\mathcal{M})$, respectively) by Proposition 13 and clearly δ_r satisfies (Z).

4) $\Longrightarrow$ 2): By Corollary 14, $\gamma = \mathcal{L}_{ps}(\mathcal{M})$ ($\gamma = \mathcal{L}'_{ps}(\mathcal{M})$) is one-sided hereditary and left strong. Hence by Proposition 4 it is normal. It is easy to see that γ satisfies 2). □

Proposition 18. *Let γ be a supernilpotent radical and let us assume that $\gamma_l = \gamma_r$ is the largest principally left hereditary subclass of γ which we will denote by δ. Then*

$$\mathcal{L}_{ps}(\gamma) = \mathcal{L}_{ps}(\delta) \vee \gamma$$

where $\vee$ denotes the union in the lattice of all radicals (i.e. the lower radical determined by the union of the components).

Proof. Clearly $\mathcal{L}_{ps}(\delta) \vee \gamma \subseteq \mathcal{L}_{ps}(\gamma)$. Conversely, let $A \in \mathcal{L}_{ps}(\gamma)$. Under our hypothesis, we can apply Proposition 8 and so $\mathcal{L}_{ps}(\gamma) = \gamma_2$. Thus any non-zero homomorphic image A' of A has a non-zero γ-ideal or a nonzero left ideal L such that $La \in \gamma$ for all $a \in L \cup \{1\}$. Using our hypothesis again, we conclude that $L \in \delta$ and therefore the $\mathcal{L}_{ps}(\delta)$-radical of A' is nonzero. Hence A' has a nonzero ideal in $\mathcal{L}_{ps}(\delta) \cup \gamma$ and so $A \in \mathcal{L}_{ps}(\delta) \vee \gamma$. □

Corollary 19. *$\mathcal{L}_{ps}(\mathcal{G}) = \mathcal{L}_{ps}(\mathcal{B}) \vee \mathcal{G}$ and $\mathcal{G}_2 = \mathcal{B}_2 \vee \mathcal{G}$.*

Proof. By Proposition 9, the Brown-McCoy radical satisfies the assumption of Proposition 18, in fact, $\mathcal{M}\mathcal{G} = \mathcal{G}_l = \mathcal{G}_r = \mathcal{B}$. □

Remark. This corollary can also be obtained as an application of Proposition 8 to the radicals $\mathcal{G}$ and $\mathcal{B}$.

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