Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 42 (2001), No. 1, 179-184.

Locally Compact Topologically Nil and Monocompact PI-rings

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1. Introduction

In this note we shall investigate a topological version of the problem of Kurosh: "Is any algebraic algebra locally finite?"

Kaplansky's theorem concerning the local nilpotence of nil PI-algebras is well-known. We will prove a generalization of Kaplansky's theorem to the class of locally compact rings. We use in the proof a theorem of A. I. Shirshov [8] concerning the height of a finitely generated PI-algebra. We will use also the locally projectively nilpotent radical of a locally compact ring constructed in [5].

For a discrete Φ-algebra R the locally nilpotent radical in the class of Φ-algebras coincides with the locally nilpotent radical of the ring R (considered as a $\mathbf{Z}$-algebra). We give an example which shows that for locally compact Φ-algebras the locally projectively nilpotent radical does not always exist.

K. I. Beidar posed the following question: Let R be a simple nil ring. Does R admit a non-discrete locally compact ring topology?

We proved in [7] that if R is a simple nil ring then R doesn't admit a locally compact ring topology relative to which it can be represented as a union of a familiy of cardinality $< c$ compact subsets. In particular, there are no second countable locally compact ring topologies on R. We will give in this paper other partial answers to the question of K. I. Beidar. In this context, let us mention that the longstanding problem of the existence of a simple nil ring has been solved recently affirmatively by A. Smoktunowicz [3].

Notation and definitions. Fix a discrete associative commutative ring Φ with identity. A locally compact Φ-algebra R is said to be *projectively nilpotent* provided for each neighborhood V of zero there exists a natural number n such that $R^n \subseteq V$.

All topological rings are assumed to be Hausdorff and associative.

A locally compact Φ-algebra R is said to be *topologically nil* provided it is a union of its projectively nilpotent subalgebras. A locally compact Φ-algebra R is said to be *locally projectively nilpotent* provided each finite subset of it is contained in a projectively nilpotent subalgebra.

We will say that an element x of a topological Φ-algebra R is *compact* provided it is contained in a compact subalgebra.

This concept is analogous to the notion of a compact element of a topological group [2]. A topological ring considered as a $\mathbb{Z}$-algebra all whose elements are compact was called in [4] *monocompact*.

Recall that a topological ring $(R, \mathfrak{T})$ is called *minimal* provided there is no ring topology $\mathfrak{T}' \leq \mathfrak{T}$, $\mathfrak{T}' \neq \mathfrak{T}$.

The reader may consult all necessary algebraic notions from [8]. We shall assume below that R is a locally compact ring considered as an associative $\mathbb{Z}$-algebra over the ring $\mathbb{Z}$ of integers satisfying an admissible identity. We will say for the simplicity that A is a PI-ring. The additive group of a ring R will be denoted by R^+. The closure of a subset A of a topological space X will be denoted by $\overline{A}$. The unit circle group $\mathbb{R}/\mathbb{Z}$ will be denoted by $\mathbb{T}$. Recall that a locally compact abelian group A is called *self-dual* provided it is topologically isomorphic to its dual A^*.

2. Monocompact and topologically nil locally compact rings

Lemma 1. *A monocompact locally compact ring R whose additive group R^+ is compactly generated is compact.*

Proof. Indeed, by the well-known result from the theory of LCA-groups R^+ is topologically isomorphic to a topological product of a compact group, a finite number of copies of the discrete group $\mathbb{Z}$ and a finite number of copies of $\mathbb{R}$. Since each element of R^+ is contained in a compact group, R is compact. □

Lemma 2. *Any quasiregular locally compact ring R for which R^+ is compactly generated is projectively nilpotent.*

Proof. We will reduce the proof to the case when R is a discrete ring. Since R does not contain non-zero idempotents, the component C of zero of R is nilpotent, hence by [5] we may assume that R is a locally compact totally disconnected ring. We claim that R is a bounded ring. Indeed, let K be a compact subset that algebraically generates the additive group R^+ of R. If V is an open subgroup of R then choose an open subgroup U of R such that $UK \subseteq V$ and $KU \subseteq V$. Then, obviously, $UR \subseteq V$ and $RU \subseteq V$, i.e. R is a bounded ring.

Therefore R possesses a local base consisting of two-sided ideals. We reduced the proof of the lemma to the discrete case. The set P of periodic elements of the group R^+ is a finite nilpotent ideal. Therefore we may assume that R^+ is a finitely generated torsion free abelian group. Without loss of generality we may assume that $R^+ = \mathbb{Z}^n$ for some natural number n. Fix a prime number p. Then the factor ring R/pR is a quasiregular algebra of dimension n over the field $\mathbb{Z}/p\mathbb{Z}$. It follows immediately that $R^{n+1} \subseteq pR$. Then $R^{n+1} \subseteq \cap\{pR : p$ runs all prime numbers$\} = \{0\}$. □

Theorem 1. *Let R be a locally compact compactly generated PI-ring. Then:*

i) *if R is monocompact then R is compact;*

ii) *if R is topologically nil then it is projectively nilpotent.*

Proof. i) We may assume that R is a topologically finitely generated ring. Without loss of generality we may assume that R is totally disconnected topologically finitely generated monocompact ring which satisfies the conditions of the theorem. We claim that R^+ is a compactly generated group.

Denote by $\{a_1, \ldots, a_k\}$ a set of topological generators of R and by n the degree of an admissible identity true on R. Denote by Y the set of elements of R which can written as products of $< n$ elements from $\{a_1, \ldots, a_k\}$. Then by a theorem of Shirshov [8, Chapter 5, §2] R has a bounded height q relative to Y.

Denote by $T_1, \ldots, T_s$ the subrings of R generated by the elements of the set Y. We affirm that $<a_1, \ldots, a_k>^+$ is algebraically generated by a compact subset. By the theorem of Shirshov [8, Chapter 5, §2] $<a_1, \ldots, a_k>^+$ is generated by the union of subsets $T_{i_1} \ldots T_{i_r}$, where $r < q$. Therefore $<a_1, \ldots, a_k>^+$ is contained in the subgroup of R generated by the union of subsets $\overline{T_{i_1}} \ldots \overline{T_{i_r}}$, where $r < q$. The group R^+ is the closure of a compactly generated subgroup, hence is it compactly generated. By Lemma 1 R is compact.

ii) Keep the notations of i). Denote by V a compact neighborhood of zero of R.

There exists a natural number m such that $b^{m+j} \in V$ for every non-negative integer j and for each $b \in Y$. Denote by Y' the set consisting of elements of the form b^t, where $b \in Y$ and $t \leq m$.

It follows from the theorem about the height of Shirshov [8, Chapter 5, §2] that $<a_1, \ldots, a_k>^+$ is generated by the union of the sets $A_{i_1} \ldots A_{i_r}$, where $r < q$ and $A_{i_j} = V \cup Y'$. By Lemma 2 R is a projectively nilpotent ring. □

Lemma 3. *If Φ is an infinite discrete field and A a compact (not necessarily associative) Φ-algebra, then $A^2 = \{0\}$.*

Proof. The neighborhood $\overline{O_{1/3}}=\{x : x \in \mathbb{T}, |x| < 1/3\}$ does not contain a non-zero subgroup. Fix any character ξ of A. There exists a neighborhood V of zero of A such that $\xi(V) \subseteq \overline{O_{1/3}}$. There exists a neighborhood U of zero of A such that $A \cdot U = \{au : a \in A, u \in U\} \subseteq V$. Then $\xi(A \cdot U) \subseteq \xi(V) \subseteq \overline{O_{1/3}}$, hence $\xi(A \cdot U) = 0$.

Put $H = \{x : x \in A, \xi(Ax) = 0\}$. Obviously, H is an open subgroup of A. If $\alpha \in \Phi$, $h \in H$, then $\xi(A(\alpha h)) = \xi((\alpha A)h) \subseteq \xi(Ah) = 0$, hence $\alpha h \in H$. We proved that H is an open Φ-subspace of A. Then A/H is a compact discrete topological vector Φ-space, hence $A/H = 0$ or $A = H$. We get that $\xi(A^2) = 0$ which implies that $A^2 = 0$. □

Corollary 1. *For an infinite discrete field Φ and a locally compact Φ-algebra A the following conditions are equivalent:*

1) *A is projectively nilpotent;*
2) *there exists a natural number n such that $\overline{A^n}$ is compact;*
3) *A is nilpotent.*

Corollary 2. *For an infinite discrete field Φ a locally compact Φ-algebra A is locally projectively nilpotent if and only if it is a locally nilpotent algebra.*

We will say that a locally compact Φ-algebra R over a discrete ring Φ possesses the *locally projectively nilpotent radical* provided it has a closed ideal $\mathfrak{L}(R)$ such that the factor algebra $R/\mathfrak{L}(R)$ does not contain non-zero locally projectively nilpotent ideals.

The existence of the locally projectively nilpotent radical for each locally compact ring was proved in [5]. The following example shows that this result cannot be extended to the class of locally compact Φ-algebras.

Example. Let k be an infinite field which is a union of its finite subfields: $k = \cup_{i\geq 0} k_i, k_0 \subset k_1 \subset \cdots \subset k_n \subset k_{n+1} \subset \cdots$. Consider for each natural number n the nilpotent k-algebra

$$A_n = \begin{bmatrix} 0 & k & k & \dots & k \\ 0 & 0 & k & \dots & k \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & k \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

and its finite subring

$$B_n = \begin{bmatrix} 0 & k_n & k_n & \dots & k_n \\ 0 & 0 & k_n & \dots & k_n \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & k_n \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Consider the local direct product $R = \prod(A_n : B_n)$ of discrete rings A_n relative to subrings B_n [6]. Remind that R consists of elements $\{x_n\}$ from the cartesian product $\prod A_n$ with the property that $x_n \in B_n$ for almost all n. The family of all neighborhoods of zero of the topological product $\prod B_n$ defines a ring topology on R. It is easy to see that R becomes a locally compact k-algebra. Obviously, R is not locally nilpotent (in the algebraic sense).

The algebra R contains a dense locally nilpotent two-sided ideal $\bigoplus A_n$. Therefore R does not possess the locally projectively nilpotent radical.

3. Locally compact simple nil rings

Theorem 2. *There are no non-discrete locally compact simple nil rings R satisfying one of the following conditions:*

a) *R is minimal;*
b) *$charR = p > 0$ and R^+ is self-dual.*

Proof. We shall repeat partially for the sake of completeness the essential arguments used in the proof of Theorem II.5.6 of [7]. We will prove that if $(R, \mathfrak{T})$ is a locally compact minimal simple nil ring then each compact open subring V is nilpotent.

By the Remark 5.2 on page 120 of [7], $(R, \mathfrak{T})$ is totally disconnected. Let V be any compact open subring of $(R, \mathfrak{T})$. By corollary II.8.14 of [7], V is a nil ring of bounded degree. According to Lemma II.8.9 of [7], V contains an ideal $I \neq 0 . I^2 = 0$. If $i \neq 0$, $i \in I$, then $iVi = 0$.

Put $I_1 = \{x : x \in R$ and there exists a natural number n such that $xV^n i = 0\}$.

If $x, y \in I_1$, then $xV^n i = 0 = yV^m i$ for some $m, n \in \mathbb{N}$. This implies $(x-y)V^{n+m} i = 0$, hence $x - y \in I_1$. Obviously, $RI_1 \subseteq I_1$. If $r \in R$, $x \in I_1$, then there exists a neighborhood W of zero such that $rW \subseteq V$. There exists $m \in \mathbb{N}$ such that $V^m \subseteq W$. Then $rV^m \subseteq rW \subseteq V$ and so $xrV^m V^n i \subseteq xV^{n+1} i \subseteq xV^n i = 0$ follows implying $xr \in I_1$.

We proved that I_1 is a two-sided ideal of R. Since $0 \neq i \in I_1$ we get that $I_1 = R$.

Put, for any $n \in \mathbb{N}$, $R_n = \{x : x \in R$ and $xV^n i = 0\}$.

Then R_n is a closed left ideal of R and $\cup R_n = R$. There exists $n_0 \in \mathbb{N}$ such that R_{n_0} is open. There exists $m \in \mathbb{N}$ such that $V^m \subseteq R_{n_0}$, hence $V^{m+n_0} i = 0$.

Put $I_2 = \{x : x \in R$ & there exists $k \in \mathbb{N}$ such that $V^k x = 0\}$. Then $i \in I_2$ and I_2 is a right ideal of R. If $r \in R$, $x \in I_2$ then there exists $n \in \mathbb{N}$ such that $V^n x = 0$. There exists a neighborhood W of zero of R so that $Wr \subseteq V$. There exists $m \in \mathbb{N}$ so that $V^m \subseteq W$, therefore $V^{n+m} rx = V^n V^m rx \subseteq V^{n+1} x \subseteq V^n x = 0$ which gives $rx \in I_2$. We obtain that $I_2 = R$.

Put for each $l \in \mathbb{N}$ $S_l = \{x : x \in R, V^l x = 0\}$. Then S_l is a closed right ideal of R and $\cup S_l = R$. There exists $l_0 \in \mathbb{N}$ so that S_{l_0} is open. There exists $k_0 \in \mathbb{N}$ so that $V^{k_0} \subseteq S_{l_0} \Rightarrow V^{l_0 + k_0} = 0$. We proved that V is a nilpotent ring. We proved that any compact open subring of a simple locally compact nil ring is nilpotent. Since in a locally compact ring with a compact component of zero every compact subring is contained in a compact open subring, we obtained that every compact subring of R is nilpotent.

Now assume that $(R, \mathfrak{T})$ be a non-discrete simple nil ring.

a) Let $\mathfrak{B} = \{W\}$ be a fundamental system of neighborhoods of zero consisting of open subrings. Consider the family $\mathfrak{C} = \{W + RW : W \in \mathfrak{B}\}$ of open left ideals of R. We claim that $\mathfrak{C}$ gives a Hausdorff ring topology $\mathfrak{T}'$ on R coarser than $\mathfrak{T}$. Let $W \in \mathfrak{B}$ and $x \in R$. There exists $W_1 \in \mathfrak{B}$ such that $W_1 x \subseteq W$ which implies that $(W_1 + RW_1)x \subseteq W + RW$. The others axioms are obviously fulfilled. Assume that $V + RV = R$. Denote by n the index of nilpotence of V. Then $0 \neq V^{n-1}$ and $RV^{n-1} = (V + RV)V^{n-1} = 0$, a contradiction, since the right annihilator of any simple ring is zero. This implies that $(R, \mathfrak{T}')$ is a Hausdorff topological ring. Obviously, $\mathfrak{T}' \leq \mathfrak{T}$ and by the condition $\mathfrak{T}' = \mathfrak{T}$.

By a well-known theorem (see, for example, [1]) $(R, \mathfrak{T})$ is a Baire space. By Theorem II.8.12 of [7], $(R, \mathfrak{T})$ has an open left ideal which is a nil ring of bounded degree, a contradiction.

b) The self-duality in this case means that $R^+ \cong_{top} (\mathbb{Z}/(p))^{\mathfrak{m}} \bigoplus (\bigoplus_{\mathfrak{m}} \mathbb{Z}/(p))$ for some cardinal number $\mathfrak{m}$. It is obvious that R contains a dense subgroup S of cardinality $\mathfrak{m} <$ card R. We get a contradiction as in Theorem II.5.6 of [7]. □

Acknowledgement. I'd like to express my gratitude to professors V. Arnautov and R. Wiegandt for their attention to this paper.

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Received March 13, 2000