Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 42 (2001), No. 1, 71-87.

The Multiple Conical Surfaces

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Abstract. We give a classification of the surfaces that can be generated by a moving conic in more than one way. It turns out that these surfaces belong to classes which have been thoroughly studied in other contexts (ruled surfaces, Veronese surfaces, del Pezzo surfaces).

1. Introduction

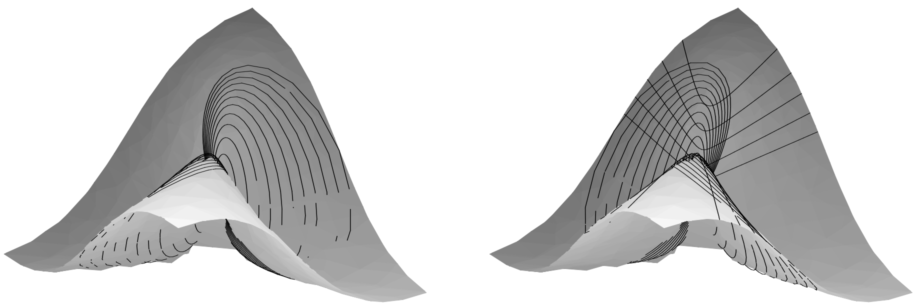

Figure 1: A multiple conical surface

A pencil of conics is a one-parameter family of conics. A conical surface is a surface which is generated as the union of conics in a pencil. A multiple conical surface is a surface which is a conical surface in more than one way. For instance, the surface in Figure 1 is generated by two families of circles passing through a fixed point. The horizontal sections form a third generating family, consisting of hyperbolas (see Example 5 below).

The class of multiple conical surfaces includes several classical examples, such as the quadric surfaces, the torus, the cyclides of Dupin (see [14]), the surfaces of Darboux [6], the double surfaces of Blutel [9], and a class of surfaces that arises in kinematics and has been investigated in [15, 16]. They also arise in the surface parametrization problem (see [22, 21]); there, the problem is to locate one generating pencil, and the fact that there is no unique one is responsible for the difficulty of this location task (for instance, one needs to introduce algebraic numbers).

Traditionally, the conical surfaces have been investigated using methods of projective duality and projective differential geometry (see [1, 25, 7, 8, 2, 19, 18]). Here, we use intersection theory for divisors on surfaces (see [24, 28]).

The classification results obtained this way are quite satisfying: any such surface is algebraic of degree at most 8, and they belong to classes of surfaces which have been thoroughly investigated from a different point of view: ruled surfaces, Veronese surfaces and del Pezzo surfaces. We list all possibilities for the exact number of pencils of conics and give other results of geometrical relevance.

The most abundant subclass of multiple conical surfaces is the one consisting of del Pezzo surfaces. The del Pezzo surfaces of degree 3 have already been classified in [23, 4] (see also [3]). For del Pezzo surfaces of higher degree, we use the classification in [5].

This work was supported by the Austrian Science Fund (FWF) in the frame of the "Adjoints" project and of the project SFB F013. I also want to thank M. Husty for a fruitful discussion which started this investigation, and for drawing my attention to many classical examples.

2. General results

We consider complex analytic surfaces in 3-dimensional space. Our first result establishes that all multiple conical surfaces are algebraic. Subsequently, we show that these surfaces are rational, and that the generic plane section has genus zero or one. Finally, we introduce the concept of linear normalization which will be used in subsequent sections.

If we assume that S is a closed analytic subvariety of projective space, than S would automatically be algebraic by Chow's theorem (see [12]). But the algebraic nature of multiple conical surfaces can be shown under much weaker assumptions, which include also the affine case.

Let U be an open subset of complex 3-dimensional projective space (for instance, 3-dimensional affine space). Let S be a closed analytic subvariety of U of dimension 2. Let Π be the algebraic variety of dimension 8 parametrizing all conics in projective 3-space. A *family of conics* on S is a set of conics parametrized by an analytic subvariety of Π, where the generic conic is irreducible, such that the intersection of each conic with U lies in S. If the parametrizing variety is a curve, then we speak of a *pencil* of conics. We say that the surface is generated by a pencil of conics if it is the union of all intersections of the conics in the pencil with U. A surface which is generated by a pencil of conics is called a *conical surface*. A *multiple conical surface* is a surface that is generated by more than one pencil of conics.

A degenerate case is the plane, which has of course infinitely many pencils of conics generating it. From now on, we assume that S is not the plane.

Theorem 1. *Any multiple conical surface is algebraic.*

Proof. In the following, we identify subvarieties of Π with their corresponding families of conics.

Let Γ_1, Γ_2 two (maybe transcendental) pencils in Π generating S. The generic conic in Γ_1 intersects the generic conic in Γ_2 (maybe outside U), hence all conics in Γ_1 intersect all conics in Γ_2.

Let Λ be the family of all conics intersecting each conic in Γ_2, maybe outside U. Then Λ is an algebraic subvariety of Π, because it is the solution set of infinitely many algebraic conditions. Moreover Λ contains Γ_1.

Suppose, indirectly, that S is not algebraic. Since the union V of all conics in Λ is an algebraic set containing S, it must be 3-dimensional. Clearly, there exists an algebraic subvariety of Λ of dimension 2 such that the union is still 3-dimensional and still contains S. Let Δ be such a subvariety.

Let p be a generic point on S. Let C_0 be a conic in Γ_2 passing through p. For each point q in a neighborhood of p, there is only a finite set of conics in Δ passing through q, because otherwise we would have $\dim(\Delta) > 2$. Especially, this holds if $q \in C_0$. If we take the union of all conics in Δ passing through q, where q ranges over C_0 in a small neighborhood of p, then we get a two-dimensional set. On the other hand, any conic in Δ intersects C_0, and therefore the union must be three-dimensional. This is a contradiction. □

Theorem 2. *Let S be a multiple conical surface. Then S has several algebraic pencils of conics.*

Proof. Let Γ_1, Γ_2 two (maybe transcendental) pencils in Π generating S. Let Λ_1 be the algebraic family of all conics intersecting each conic in Γ_2. Similarly, let Λ_2 be the algebraic family of all conics intersecting each conic in Γ_1. We distinguish two cases.

If $\Lambda_1 \neq \Lambda_2$, then one can choose algebraic pencils $\Gamma'_1 \subset \Lambda_1$, $\Gamma'_2 \subset \Lambda_2$, each generating S.

If $\Lambda_1 = \Lambda_2$, then Λ_1 cannot be a pencil because it contains two distinct pencils. Hence it is a family of dimension at least two, and one can find two distinct algebraic generating pencils within it. □

The results above show that we may restrict our attention to the algebraic situation. Consequently, we may assume that $U = \mathbf{P}^3$ from now on.

We adopt the terminology (see [13]) that a birationally ruled surface is a surface that has a rational map to a curve, such that the generic fiber has genus zero (and is therefore birationally equivalent to a line). By a theorem of Enriques (see [13]), the birationally ruled surfaces can be characterized as the surfaces with Kodaira dimension $-\infty$, i.e. all plurigeni equal to 0.

Theorem 3. *Any multiple conical surface is rational.*

Proof. Let $\mathcal{F}_1$ be an algebraic family of conics generating S, with parameter space T_1, an algebraic curve. Then the algebraic set

$$S' := \{(x,t) \mid x \in C(t)\} \subset S \times T_1$$

is a birationally ruled surface, therefore all its plurigenera are 0. The second projection is a rational map from S' to S, hence all plurigenera of S are zero (otherwise, the pullback of a pluricanonical section would give rise to a pluricanonical section on S' which is impossible). Consequently, S is also a birationally ruled surface.

Suppose, indirectly, that S is irrational. Then any rational curve on S is contained in a fiber of the ruling. Therefore, there is precisely one pencil of rational curves on S, and S cannot be multiple conical. □

Recall that an algebraic family is called *linear* iff it consists of the inverse images of the hyperplanes under a rational map to a projective space (see [13]). Especially, a pencil is linear iff it consists of the fibers of a rational map to the projective line.

In order to apply intersection theory, we consider a desingularization $d : \tilde{S} \to S$ of the multiple conical surface S. We consider *divisors* (i.e. composite curves where the components are counted with multiplicity) on the desingularization $\tilde{S}$. Two divisors are called linear equivalent iff they are contained in a common linear family. This relation is transitiv (see [13]). The set of all divisors linear equivalent to D forms a linear family, which is denoted by $|D|$ (the complete linear family defined by D). The set of equivalence classes of divisors modulo linear equivalence form a group, called the *class group*. The intersection product is well-defined on classes.

Theorem 4. *Let S be a multiple conical surface. Then S has several linear pencils of conics.*

Proof. If all pencils of S are linear, then there is nothing to prove.

The pullbacks of any two curves in an arbitrary pencil are numerically equivalent divisors on $\tilde{S}$, by continuity. On a nonsingular rational surface, numerical equivalence implies linear equivalence (see [28]). Therefore, the pullback of a nonlinear pencil on S is contained in a linear family of divisors on $\tilde{S}$ of dimension at least 2. It is clear that such a linear family contains an infinite number of linear pencils. By mapping them down to S, we get infinitely many pencils of conics on S. □

Theorem 5. *Let S be a multiple conical surface. Then the generic plane section of S has genus zero or one.*

Proof. The pullback H of a generic section is a nonsingular curve on $\tilde{S}$, by Bertini's theorem (see [12]). Any linear family on $\tilde{S}$ cuts out a linear family on H. The pullback of a linear pencil on S cuts out a linear family of degree 2 (because the generic plane section intersects a generic conic in the pencil in two points) and dimension 1. We distinguish two cases.

If one of these linear families on H is incomplete, then its completion must be of dimension 2. But the existence of a complete linear family of dimension 2 and degree 2 (a g_2^2 in the terminology of the theory of algebraic curves) implies that H has genus zero (see [26]).

Now, suppose that all these linear families are complete (i.e. g_2^1's). We claim that the linear families are distinct.

Proof of the claim: choose first a generic point p on S, and then a generic plane E through p. (Note that we do not give away the generic-ness of the plane by this choice!) Let C be the plane section. Let P_1 be the conic in the linear pencil $\mathcal{F}_1$ passing through p, and let P_2 be conic in the linear pencil $\mathcal{F}_2$ passing through p. Suppose, indirectly, that the two linear families cut out the same g_2^1. Then P_1 and P_2 intersect the plane E in the same point q. Because the plane was chosen generic, and the coincidence condition is closed, P_1 and P_2 intersect any plane through p in the same point. This implies $P_1 = P_2$. Now recall that also the choice of p was generic, hence $\mathcal{F}_1 = \mathcal{F}_2$. This proves the claim.

Now, it is well known (see [26]) that any curve that has two different g_2^1's has genus one. Therefore, H has genus one. $\square$

For the rest of the paper, let us fix the following notation. The pullback of a generic plane section is denoted by H. For any family $\mathcal{F}_i$ of conics, let P_i be the pullback of a generic conic.

We have already shown that we have $P_i{\cdot}H = 2$ for all i. Similarly, one can show that $H^2 = \deg S$.

Let $i_H : \tilde{S} \to \mathbf{P}^{\dim|H|}$ be the rational map defining the linear structure of $|H|$. The image variety $\overline{S} \subset \mathbf{P}^{\dim|H|}$ is called the *linear normalization* of S. A surface is called *linearly normal* if is isomorphic to its linear normalization.

Algebraically, the graded coordinate ring of the linear normalization can be obtained as the subalgebra of the graded integral closure of the given coordinate ring which is generated in degree 1.

The following theorem is well-known. We include the proof because we could not find a self-contained proof in the literature. Moreover, it gives geometrical insight to the concept of linear normalization.

Theorem 6. *Any surface is a projection from its linear normalization. The projection is birational, and it preserves the degree of the surface and the degree of any curve not contained in the singular locus.*

Proof. Since the desingularization map $d : \tilde{S} \to S$ defines the linear structure of a subfamily of $|H|$, it follows that i_H is regular and d is i_H composed with a projection p to $\mathbf{P}^3$. The inverse of the projection is $i_H \circ d^{-1}$ (this argument actually shows that the inverse is defined on all nonsingular points of S). The subfamily of $|H|$ does not have based points, hence the center of projection is disjoint from the surface $\overline{S}$.

If C is a point not contained in the singular locus, its pre-image $\overline{C}$ on $\overline{S}$ is birationally equivalent to it because the inverse map is defined almost everywhere. The map p is a projection from a subspace disjoint from $\overline{C}$, hence $\deg(\overline{C}) = \deg(C)$. Taking for C a generic plane section, we obtain $\deg(\overline{S}) = \deg(S)$. $\square$

Let S be a surface in $\mathbf{P}^3$, and let $\overline{S}$ be its normalization. By Theorem 6 above, there is a one-to-one correspondence between pencils of conics on $\overline{S}$ and pencils of conics on S. Therefore, it makes sense to classify multiple conical surfaces up to linear normalization.

3. Ruled surfaces

In this section, we recall the definitions of the Veronese surface and of the ruled surfaces. It is well-known that the surfaces with generic plane section of genus zero are the projections of ruled surfaces or of the Veronese surface. We give a complete classification of the multiple conical surfaces of this type.

The Veronese surface $V\subset\mathbf{P}^5$ is given by the parametrization

$$(1 : s : t : s^2 : st : t^2)$$

(up to a change of projective coordinates in $\mathbf{P}^5$). The class group is generated by L, the divisor $t = 0$. We have $H\sim 2L$ and $L^2 = 1$. Thus, the degree of V is $H^2 = 4$.

For any $m, n\geq 0$ such that $2m + n\geq 2$, the ruled surface $R_{n,m}\subset\mathbf{P}^{2m+n+1}$ is given by the parametrization $(1 : t : \ldots : t^{m+n} : s : st : \ldots : st^m)$. The ruling is formed by the lines where t is constant. The class group is generated by the ruling P and the cross section $B : s = 0$. (The ruled surface $R_{0,1}$ (a nonsingular quadric) has a second ruling, formed by the lines where s is constant.) We have $H = B + (m + n)P$, $B\cdot P = 1$, $P^2 = 0$, $B^2 = -n$. Thus, the degree of $R_{n,m}$ is $H^2 = 2(m + n) - n = 2m + n$.

We will use the following well-known theorem.

Theorem 7. *Let S be a surface with generic plane section of genus zero. Then* $\dim|H| = \deg(S) + 1$, *and the linear normalization of S is either a ruled surface $R_{n,m}$ or the Veronese surface (up to projective isomorphism).*

Proof. Well-known (see [12]). □

Theorem 8. *Let S be a multiple conical surface with generic plane section of genus zero. Then we have one of the following cases.*

- *S is a quadric surface; any pencil of conics is contained in the 3-dimensional linear family of plane sections.*
- *S is a cubic surface; the linear normalization of S is the ruled surface $R_{1,1}$; any pencil of conics is contained in the 2-dimensional linear family corresponding to $|B + P|$.*
- *S is a quartic surface; the linear normalization of S is the Veronese surface; any pencil of conics is contained in the 2-dimensional linear family corresponding to $|L|$.*

Proof. Let $\overline{S}$ be the linear normalization of S. By Theorem 7, we know that $\overline{S}$ is either a ruled surface $R_{n,m}$ or the Veronese surface. In the second case, the equation $P_i\cdot H = 2$ implies that $P_i\sim L$. Hence any pencil of conics is contained in the two-dimensional linear family $|L|$.

Now, suppose that $\overline{S} = R_{n,m}$. Since the class group is generated by B, P, we may write $P_i\sim a_iB + b_iP$. Because P_i is in a moving linear family, we have $P\cdot P_i = a_i\geq 0$ and $B\cdot P_i = b_i - na_i\geq 0$. We also have

$$P_i \cdot H = (a_iB + b_iP)\cdot(B + (m + n)P) = b_i + a_im = 2,$$

and the restriction $2m + n\geq 2$ above. Thus, we are left with the following possibilities.

1. $a_i = 0$, $b_i = 2$: Then $P_i \sim 2P$, and $P_i \cdot P = 0$, hence all curves in $|P_i|$ decompose into two curves in $|P|$. This contradicts our assumption that the generic conic of the pencil $\mathcal{F}_i$ is irreducible. Hence this case is impossible.
2. $a_i = 1$, $b_i = 0$, $m = 2$, $n = 0$: Then $\overline{S} = R_{0,2}$. Since the complete linear family $|P_i| = |B|$ has dimension 1, the pencil $\mathcal{F}_i$ is equal to $|B|$.
3. $a_i = 1$, $b_i = 1$, $m = 1$, $n = 0$: Then $P_i \sim H$, and $H^2 = 2$. Hence $\overline{S}$ is a quadric surface, and $\mathcal{F}_i$ is a pencil of plane sections. In this case, $S = \overline{S}$ because a quadric surface is linearly normal.
4. $a_i = 1$, $b_i = 1$, $m = 1$, $n = 1$: Then $\overline{S} = R_{1,1}$, and the pencil $\mathcal{F}_i$ is contained in $|B + P|$.
5. $a_i = 1$, $b_i = 2$, $m = 0$, $n = 2$: Again, $P_i \sim H$, and $H^2 = 2$. Hence we have another quadric surface and a pencil of plane sections (the difference to case 3 is that here we have a singular quadric).
6. $a_i = 2$, $b_i = 0$, $m = 1$, $n = 0$: Then $P_i \sim 2B$, and $P_i \cdot B = 0$, hence we obtain the same contradiction as in case 1. So this case is impossible.

Since S is a multiple conical surface, case 2 is also impossible because the ruled surface $R_{0,2}$ has only one pencil of conics (namely $|B|$). $\square$

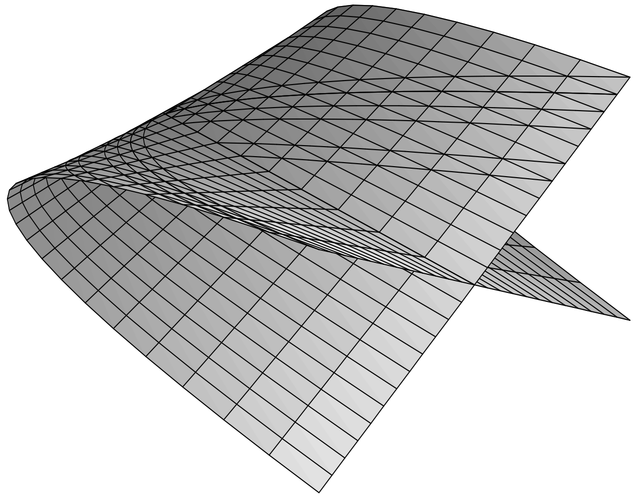

Figure 2: The ruled surface $x^2y - z^2 = 0$ and two pencils of conics on it

Example 1. Here is an example of the second case. The ruled cubic surface S with equation $x^2y - z^2w = 0$ has the linear normalization

$$u^2 - yw = ux - zw = uz - xy = 0.$$

A parametrization is

$$(x : y : z : w : u) = (s : t^2 : st : 1 : t),$$

hence we have indeed the ruled surface $R_{1,1}$. The parametrization of S is obtained by omitting the last coordinate.

For two parameters λ_1, λ_2, we get a conic by substituting $s = \lambda_1 t + \lambda_2$ in the parametrization. Obviously, these conics form a two-dimensional family of conics. Figure 2 shows the pencil of parabolas $s =$ constant, the ruling $t =$ constant, and some elements of the pencil of parabolas $s + t =$ constant.

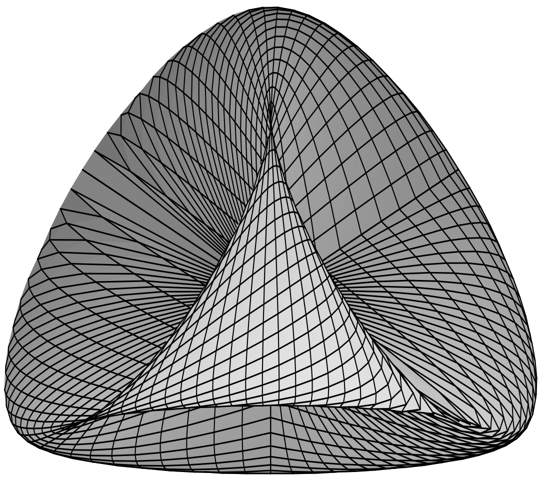

Figure 3: The Steiner surface and two pencils of conics on it

Example 2. Here is an example of the third case. The Steiner surface with equation

$$x^2y^2 + y^2z^2 + z^2x^2 + xyzw = 0$$

has the linear normalization

$$\begin{aligned} uz - xy = vx - yz = uv - y^2 = u^2 + uv + uw + x^2 = \\ = v^2 + uv + vw + z^2 = uy + vy + wy + xz = 0. \end{aligned}$$

A parametrization is

$$(x : y : z : w : u : v) = (s : st : t : -s^2 - t^2 - 1 : s^2 : t^2),$$

which shows that linear normalization is a Veronese surface.

As in the previous example, we get a pencil of conics by substituting $s = \lambda_1 t + \lambda_2$ in the parametrization. Figure 3 shows the two pencils $s =$ constant. and $t =$ constant.

In the proof above, we also recovered Brauner's classification [2] of the conical surfaces which are also ruled surfaces (these are the quadric surfaces and the surfaces $R_{0,2}$ and $R_{1,1}$). We refer to [2] for a more detailed discussion of these surfaces.

Remark 1. If $\overline{S}$ is the ruled surface $R_{1,1}$ or the Veronese surface, then all pencils are contained in a single linear family of dimension 2. Because any two plane algebraic curves intersect, we have the following interesting corollary of Theorem 8: if S is a multiple conical surface of degree greater than 2 with generic plane section of genus zero, then any two algebraic pencils have a common conic. It can also be seen in Figure 3, where it is projected to a segment of the singular line $x = z = 0$.

4. Del Pezzo surfaces

We give a complete classification of the multiple conical surfaces with generic plane section of genus one, using the theory of del Pezzo surfaces.

A *del Pezzo surface* is a linearly normal rational surface with generic hyperplane section of genus one. Equivalently, del Pezzo surfaces can be defined as the linearly normal rational surfaces of degree d in $\mathbf{P}^d$. They have been investigated by numerous authors [10, 11, 27, 5].

Our motivation for studying del Pezzo surfaces is the following fact.

Proposition 1. *Let S be a multiple conical surface with generic plane section of genus one. Then the linear normalization $\overline{S}$ of S is a del Pezzo surface.*

Vice versa, let S be a surface such that its linear normalization $\overline{S}$ is a del Pezzo surface. If $\overline{S}$ has several pencils of conics, then S is a multiple conical surface.

Proof. This is an immediate corollary of the properties of the linear normailzation described in Section 2. □

Here is a summary of the relevant facts on del Pezzo surfaces. For the proofs, we refer to [11].

1. The degree of a del Pezzo surface is less than or equal to 9.
2. There is only one del Pezzo surface of degree 9, up to projective isomorphism. It is embedded in $\mathbf{P}^9$ and has the parametrization
$$(1 : s : s^2 : s^3 : t : st : s^2t : t^2 : st^2 : t^3).$$
The class group is generated by $L : s = 0$, and we have $L^2 = 1$ and $H{\sim}3L$.
3. There are three del Pezzo surfaces of degree 8, up to projective isomorphism. They are embedded in $\mathbf{P}^8$, and their parametrizations are
$$(1 : s : s^2 : t : st : s^2t : t^2 : st^2 : s^2t^2),$$
$$(1 : s : s^2 : t : st : s^2t : t^2 : st^2 : t^3),$$
$$(1 : t : t^2 : t^3 : t^4 : s : st : st^2 : s^2).$$
The class group is generated by $P : t = 0$ and $B : s = 0$. We have $P^2 = 0$, $P{\cdot}B = 1$, $B^2 = -n$, and $H{\sim}2B + (n + 2)P$, where $n = 0, 1, 2$, respespectively.
4. For $3{\le}d{\le}7$, any del Pezzo surface of degree d is embedded in $\mathbf{P}^d$ and has a parametrization by polynomials of total degree at most 3. The class group is a free abelian group of rank $10-d$, generated by divisors $L, E_1, \cdots, E_{9-d}$. We have $L^2 = 1$, $E_i^2 = -1$, $L{\cdot}E_i = 0$, $E_i{\cdot}E_j = 0$ for $i{\ne}j$, and $H{\sim}3L - E_1 - \ldots - E_{9-d}$.

5. The embedding divisor H is anticanonical (in fact, this property characterizes del Pezzo surfaces).
6. A del Pezzo surface has only rational singularities. The preimage of the singularities are unions of curves F such that $F^2 = -2$ and $F{\cdot}K = 0$, where K is the canonical class (these curves are called -2-curves). Their intersection graphs are classified by the Dynkin diagrams $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{A}_4$, $\mathbf{A}_5$, $\mathbf{D}_4$, $\mathbf{D}_5$, $\mathbf{E}_6$.
7. A del Pezzo surface has only finitely many lines. They correspond to curves E such that $E^2 = E{\cdot}K = -1$ (so-called -1-curves) on the desingularization.

All references to the class group refer to the minimal desingularization of S.

Remark 2. In the literature, one finds also del Pezzo surfaces of "degree" 1 and 2 (the notion of degree is abused here). We do not need to consider these here.

Theorem 9. *On a del Pezzo surfaces, any pencil of conics is the image of a pencil* $|P|$*, where* P *is a divisor such that* $P^2 = 0$ *and* $P{\cdot}K = -2$*. All these divisors satisfy the conditions* $P{\cdot}F{\geq}0$*, for all -2-curves* F*.*

Proof. Let $\mathcal{F}$ be a pencil, and let P be the pullback of a generic conic in $\mathcal{F}$. Then $P{\cdot}K = -P{\cdot}H = -2$. By the genus formula, $P^2 + P{\cdot}K = -2$, hence $P^2 = 0$. Any curve in $\mathcal{F}$ must be numerically equivalent to P, hence linearly equivalent, hence contained in $|P|$. Because $P^2 = 0$, there is at most one curve in $|P|$ passing through a fixed point. It follows that $|P|$ is a pencil and $\mathcal{F} = |P|$.

Because P is an irreducible curve distinct from any -2-curve F, the intersection product $P{\cdot}F$ cannot be negative. □

Theorem 10. *A del Pezzo surface of degree 9 does not have any pencil of conics.*
For $d = 8$*, the pencils of conics are* $|P|$*, and* $|B|$ *if* $n = 0$*.*
For $3{\leq}d{\leq}7$*, the pencils of linear conics are among the linear families* $|L - E_i|$*,* $|2L - E_{i_1} - E_{i_2} - E_{i_3} - E_{i_4}|$ *where* i_1, i_2, i_2, i_4 *are distinct (only for* $d{\leq}5$*), and* $|3L - 2E_{i_1} - E_{i_2} - \ldots - E_{i_6}|$ *where* $i_1, \ldots, i_6$ *are distinct (only for* $d = 3$*).*

Proof. If $d = 9$, then $P{\cdot}H$ is a multiple of 3 for each curve P, hence there are no conics at all.

If $d = 8$, set $P_i{\sim}xB + yP$. By the conditions $P_i^2 = 0$ and $P_i{\cdot}H = 2$, we get the equations

$$2xy - nx^2 = 0, \quad -nx + 2y + 2x = 2,$$

with integral solutions $(x, y) = (0, 1)$ and $(x, y) = (1, \frac{n}{2})$ (only if $n = 0$ or $n = 2$). If $n = 2$, then the second solution $B + P$ violates the additional condition in Theorem 9, because $(B + P){\cdot}B = -1$. For the remaining solutions, $|xB + yP|$ is indeed a pencil of conics, as it can easily be checked using the explicit parametrizations above.

If $3{\leq}d{\leq}7$, then we set $P_i{\sim}xL - x_1E_1 - \ldots - x_{9-d}E_{9-d}$. The conditions $P_i^2 = 0$ and $P_i{\cdot}H = 2$ yield the equations

$$x^2 = x_1^2 + \ldots + x_{9-d}^2, \quad 3x - 2 = x_1 + \ldots + x_{9-d}. \tag{1}$$

We assume that $x_1 \geq \ldots \geq x_{9-d}$ (otherwise one has to permute the x_i). By the Cauchy-Schwarz inequality, we have

$$(3x-2)^2 = (x_1 + \ldots + x_{9-d})^2 \leq (9-d)(x_1^2 + \ldots + x_{9-d}^2) \leq 6x^2.$$

The only integers satisfying $(3x-2)^2 \leq 6x^2$ are $x = 1, 2, 3$.

If $x = 1$, then the equations (1) give $x_1 = 1$, $x_2 = \ldots = x_{9-d} = 0$. If $x = 2$, then the equations (1) give $d \leq 5$, $x_1 = x_2 = x_3 = x_4 = 1$, $x_5 = \ldots = x_{9-d} = 0$. If $x = 3$, then the equations (1) give $d = 6$, $x_1 = 2$, $x_2 = \ldots = x_6 = 1$. □

The theorem has the following converse.

Lemma 1. *Let P be a divisor such that $P^2 = 0$, $P{\cdot}H = 2$, and $P{\cdot}F \geq 0$ for each -2-curve F. Then $|P|$ is a pencil of conics on S.*

Proof. Let $R := P + H$. We claim that $R{\cdot}C \geq 0$ for any curve C. Suppose, indirectly, that $R{\cdot}C < 0$ for an irreducible curve C. By the Riemann-Roch theorem,

$$\dim|R| \geq \frac{R{\cdot}(R+H)}{2} - 1 = d - 2 \geq 0,$$

hence C cannot move in a linear family. On a del Pezzo surface, this implies that C is a -1-curve or a -2-curve (see [11]). Using the explicit description of the possible classes of -1-curves in [17], one checks easily that $P{\cdot}E \geq 0$ for all these classes. Moreover, $P{\cdot}F \geq 0$ for any -2-curve F. This is a contradiction, and our claim is proven.

Now, we apply Reider's theorem [20] and obtain that the linear family $|P|$ does not have base points. By Bertini's theorem (see [12]), the generic element in $|P|$ is nonsingular. Because its arithmetic genus is zero, and its degree is 2, it must be an irreducible conic. □

We continue the investigation by case distinction on the degree, following the classification in [5].

Degree 8. The only multiple conical surface is the one with $n = 0$. The two pencils of conics are formed by the curves $s =$ constant and $t =$ constant in the parametrization above.

Degree 7. If S is nonsingular, then we have a multiple conical surface with two pencils $|L - E_1|$, $|L - E_2|$. If S is singular, then $E_1 - E_2$ is a -2-curve (maybe after renaming), and then $(L - E_2){\cdot}(E_1 - E_2) < 0$, hence we have only one pencil.

Example 3. The surface S with parametrization

$$(x : y : z : w) = (s^2t + t + 1 : st^2 + s + 1 : s^2 + 1 : t^2 + 1)$$

has an implicit equation of degree 7 (which is quite complicated). Its linear normalization has the parametrization

$$\begin{aligned}(x : y : z : w : u_1 : u_2 : u_3 : u_4) &= \\ = (s^2t + t + 1 : st^2 + s + 1 : s^2 + 1 &: t^2 + 1 : 1 : s : t : st).\end{aligned}$$

This is a nonsingular del Pezzo surface of degree 7. The two pencils of conics are given by $s =$ constant and $t =$ constant. Figure 4 shows the image of these two pencils on the projection S.

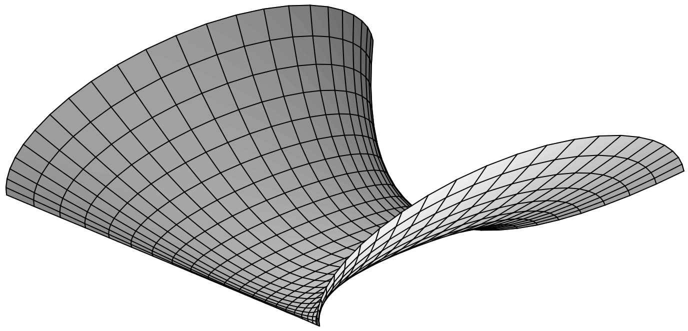

Figure 4: A surface of degree 7 with two pencils of conics

Degree 6. Up to the choice of the orthogonal basis of the class group, the set D of -2-curves is one of the following.

1. $D = \emptyset$ (S is nonsingular). Then we have three pencils of conics: $|L - E_1|$, $|L - E_2|$, $|L - E_3|$.
2. $D = \{E_1 - E_2\}$ (S has one double point of type $\mathbf{A}_1$ and 4 lines). Then we have two pencils of conics: $|L - E_1|$, $|L - E_3|$. The divisor $L - E_2$ fails to satisfy the additional condition in Theorem 9.
3. $D = \{L - E_1 - E_2 - E_3\}$ (S has one double point of type $\mathbf{A}_1$ and 3 lines). Then we have three pencils of conics: $|L - E_1|$, $|L - E_2|$, $|L - E_3|$.
4. $D = \{L - E_1 - E_2 - E_3, E_1 - E_2\}$ (S has two double points of type $\mathbf{A}_1$). Then we have two pencils of conics: $|L - E_1|$, $|L - E_3|$.
5. $D = \{E_1 - E_2, E_2 - E_3\}$ (S has one double point of type $\mathbf{A}_2$). Then we have one pencil of conics: $|L - E_1|$.
6. $D = \{L - E_1 - E_2 - E_3, E_1 - E_2, E_2 - E_3\}$ (S has one double point of type $\mathbf{A}_2$ and one double point of type $\mathbf{A}_1$). Then we have one pencil of conics: $|L - E_1|$.

Degree 5. Up to the choice of the orthogonal basis of the class group, the set D of -2-curves is one of the following.

1. $D = \emptyset$ (S is nonsingular). Then we have 5 pencils of conics: $|L - E_1|$, $|L - E_2|$, $|L - E_3|$, $|L - E_4|$, $|2L - E_1 - E_2 - E_3 - E_4|$.
2. $D = \{E_1 - E_2\}$ (S has one double point of type $\mathbf{A}_1$). Then we have 4 pencils of conics: $|L - E_1|$, $|L - E_3|$, $|L - E_4|$, $|2L - E_1 - E_2 - E_3 - E_4|$.
3. $D = \{E_1 - E_2, E_3 - E_4\}$ (S has two double points of type $\mathbf{A}_1$). Then we have 3 pencils of conics: $|L - E_1|$, $|L - E_3|$, $|2L - E_1 - E_2 - E_3 - E_4|$.
4. $D = \{E_1 - E_2, E_2 - E_3\}$ (S has one double point of type $\mathbf{A}_2$). Then we have 3 pencils of conics: $|L - E_1|$, $|L - E_4|$, $|2L - E_1 - E_2 - E_3 - E_4|$.
5. $D = \{L - E_1 - E_2 - E_3, E_1 - E_2, E_2 - E_3\}$ (S has one double point of type $\mathbf{A}_2$ and one double point of type $\mathbf{A}_1$). Then we have 2 pencils of conics: $|L - E_1|$, $|L - E_4|$.

6. $D = \{E_1 - E_2, E_2 - E_3, E_3 - E_4\}$ (S has two double points of type $\mathbf{A}_3$). Then we have 2 pencils of conics: $|L - E_1|$, $|2L - E_1 - E_2 - E_3 - E_4|$.
7. $D = \{L - E_1 - E_2 - E_3, E_1 - E_2, E_2 - E_3, E_3 - E_4\}$ (S has two double points of type $\mathbf{A}_4$). Then we have one pencil of conics: $|L - E_1|$.

Degree 4. Here, one can observe the phenomenon of *complementary pencils*: if a hyperplane contains a conic, then it contains also a second one because the degree of any hyperplane section is 4.

Lemma 2. *The pencil complementary to $|P|$ can be constructed by the following procedure: Initialize $P' := H - P$; while there exists a -2-curve F such that $F{\cdot}P' < 0$, replace P' by $P' - F$.*

Proof. Any curve that has negative intersection number with $H - P$ must be a fixed component of $|H - P|$. Because the number of fixed components is finite, the procedure above terminates.

If $e := -F{\cdot}(H-P) > 1$, then $P{\cdot}(2F+eH) = 0$ and $(2F+eH)^2 > 0$. By the Hodge index theorem, it follows that P is numerically zero, which is a contradiction. Hence $F{\cdot}(H-P) < 0$ implies $F{\cdot}(H - P) = -1$. Thus, $(H - P - F)^2 = 0$. By induction over the while loop, we see that $(P')^2 = 0$. By Lemma 1, $|P'|$ is a pencil of lines. Moreover, there is a hyperplane containing $P + P'$, hence $|P'|$ is complementary to $|P|$. □

Using the tables of possible sets of -2-curves (up to choice of the orthogonal basis) in [5] and the theorems 9 and 10, we can compute all possible configurations of pencils of conics. The result is displayed in Figure 4. Complementary pencils are indicated by a dash between them. The pencils in the table without attaching dash are self-complementary.

Example 4. The torus, with homogeneous equation

$$(x^2 + y^2 + z^2 + (1 - r^2)w^2)^2 - 4x^2 - 4y^2 = 0,$$

where r is a parameter in the open interval $(0, 1)$, has the linear normalization

$$x^2 + y^2 + z^2 - wu = (u + (1 - r^2)w)^2 - 4x^2 - 4y^2 = 0$$

in $\mathbf{P}^4$. This is a del Pezzo surface with four double points of type A^1, namely $(1 : \pm i : 0 : 0 : 0), (0 : 0 : \pm i(1 - r) : 1 : -(1 - r)^2)$. According to Figure 5, the surface has four pencils of conics, two of them being self-complementary. The two pencils are shown in Figure 6. Indeed, the two formed by the rotating circle and by the orbits under this rotation are self-complementary. The pair of complementary pencils is cut out by the tangent planes through the origin.

Degree 3. (Cubic surfaces with at most isolated double points) For any line on S, the planes through the lines cut out a pencil of conics. Vice versa, let P be an irreducible conic. Then the plane carrying P intersects S in P and a line L, and P is contained in the pencil

-2-curves	singularities	pencils of conics
none	none	$\|L-E_1\|$ - $\|2L-E_2-E_3-E_4-E_5\|$ $\dots$ $\|L-E_5\|$ - $\|2L-E_1-E_2-E_3-E_4\|$
E_4-E_5	1 $\mathbf{A}_1$	$\|L-E_1\|$ - $\|2L-E_2-E_3-E_4-E_5\|$ $\|L-E_2\|$ - $\|2L-E_1-E_3-E_4-E_5\|$ $\|L-E_3\|$ - $\|2L-E_1-E_2-E_4-E_5\|$ $\|L-E_4\|$ - $\|2L-E_1-E_2-E_3-E_4\|$
$L-E_1-E_2-E_3$, E_4-E_5	2 $\mathbf{A}_1$	$\|L-E_1\|$ - $\|2L-E_2-E_3-E_4-E_5\|$, $\|L-E_2\|$ - $\|2L-E_1-E_3-E_4-E_5\|$, $\|L-E_3\|$ - $\|2L-E_1-E_2-E_4-E_5\|$, $\|L-E_4\|$
E_3-E_4, E_4-E_5	1 $\mathbf{A}_2$	$\|L-E_1\|$ - $\|2L-E_2-E_3-E_4-E_5\|$, $\|L-E_2\|$ - $\|2L-E_1-E_3-E_4-E_5\|$, $\|L-E_3\|$ - $\|2L-E_1-E_2-E_3-E_4\|$
$L-E_1-E_2-E_3$, E_2-E_3, E_4-E_5	3 $\mathbf{A}_1$	$\|L-E_1\|$ - $\|2L-E_2-E_3-E_4-E_5\|$, $\|L-E_2\|$ - $\|2L-E_1-E_2-E_4-E_5\|$, $\|L-E_4\|$
E_1-E_2, E_3-E_4, E_4-E_5	1 $\mathbf{A}_1$, 1 $\mathbf{A}_2$	$\|L-E_1\|$ - $\|2L-E_1-E_3-E_4-E_5\|$, $\|L-E_3\|$ - $\|2L-E_1-E_2-E_3-E_4\|$
E_2-E_3, E_3-E_4, E_4-E_5	1 $\mathbf{A}_3$	$\|L-E_1\|$ - $\|2L-E_2-E_3-E_4-E_5\|$, $\|L-E_2\|$ - $\|2L-E_1-E_2-E_3-E_4\|$
$L-E_1-E_2-E_3$, E_3-E_4, E_4-E_5	1 $\mathbf{A}_3$	$\|L-E_1\|$ - $\|2L-E_2-E_3-E_4-E_5\|$, $\|L-E_2\|$ - $\|2L-E_1-E_3-E_4-E_5\|$, $\|L-E_3\|$
$L-E_1-E_2-E_3$, $L-E_3-E_4-E_5$, E_1-E_2, E_4-E_5	4 $\mathbf{A}_1$	$\|L-E_3\|$ - $\|2L-E_1-E_2-E_4-E_5\|$, $\|L-E_1\|$, $\|L-E_4\|$
$L-E_1-E_2-E_3$, E_1-E_2, E_2-E_3, E_4-E_5	2 $\mathbf{A}_1$, 1 $\mathbf{A}_2$	$\|L-E_1\|$ - $\|2L-E_1-E_2-E_4-E_5\|$, $\|L-E_4\|$
$L-E_1-E_2-E_3$, E_1-E_2, E_3-E_4, E_4-E_5	1 $\mathbf{A}_1$, 1 $\mathbf{A}_3$	$\|L-E_1\|$ - $\|2L-E_1-E_3-E_4-E_5\|$, $\|L-E_3\|$
E_1-E_2, E_2-E_3, E_3-E_4, E_4-E_5	1 $\mathbf{A}_4$	$\|L-E_1\|$ - $\|2L-E_1-E_2-E_3-E_4\|$
$L-E_1-E_2-E_3$, E_2-E_3, E_3-E_4, E_4-E_5	1 $\mathbf{D}_4$	$\|L-E_1\|$ - $\|2L-E_2-E_3-E_4-E_5\|$, $\|L-E_2\|$
$L-E_1-E_2-E_3$, $L-E_3-E_4-E_5$, E_1-E_2, E_3-E_4, E_4-E_5	2 $\mathbf{A}_1$, 1 $\mathbf{A}_3$	$\|L-E_1\|$, $\|L-E_3\|$
$L-E_1-E_2-E_3$, E_1-E_2, E_2-E_3, E_3-E_4, E_4-E_5	1 $\mathbf{D}_5$	$\|L-E_1\|$

Figure 5: Del Pezzo Surfaces of degree 4

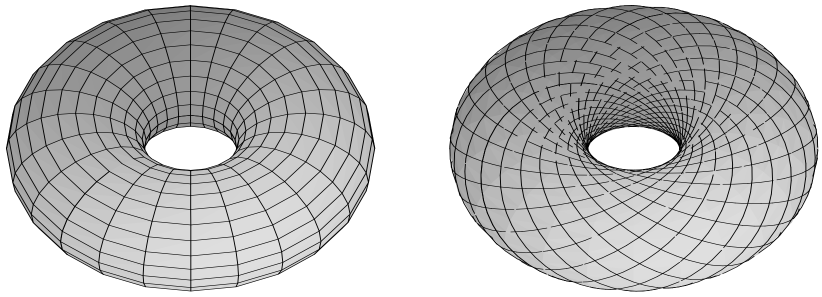

Figure 6: The torus and its four pencils of conics

corresponding to L. Because any irreducible conic is contained in a unique pencil (namely $|P|$), we have a one-to-one correspondence between pencils of conics and lines.

According to the classification in [3], the number of lines can be 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 16, 21, or 27. Thus, we have a multiple conical surface in all these cases except the first.

Example 5. The cubic surface with equation

$$z(x^2 + y^2 + z^2) - xyw = 0$$

(see also Figure 1) has a double point of type $\mathbf{A}^2$ at the point $(0 : 0 : 0 : 1)$. According to the classification in [3], it has 15 lines. Twelve of them are complex. The three real lines are $x = z = 0$, $y = z = 0$, and $w = z = 0$. The pencils of conics corresponding to the two lines through the singular point consist of circles, and they are shown in figure 1. The pencil corresponding to the infinite line $w = z = 0$ is the pencil of horizontal sections. From the implicit equation (considering z as a constant), it is clear that we have hyperbolas for $z \in (-1/2, 1/2)$, and empty sections outside this interval.

The following algebraic characterization of multiple conical surfaces can be proved through the classification above.

Theorem 11. *Any multiple conical surface has a parametrization*

$$(x : y : z : w) = (X(s,t) : Y(s,t) : Z(s,t) : W(s,t)),$$

where the maximum of the degrees in s and in t of X, Y, Z, W is 2.

Proof. For the quadric surface, the claim is obviously true. For the ruled surface $R_{1,1}$ and for the Veronese surface, we have already mentioned parametrizations of the desired type. The same holds for del Pezzo surfaces of degree 8 (recall that we must have $n = 0$ in this case). It remains to show the statement for del Pezzo surfaces of degree less than or equal to 7.

Explicit checking in the classification above shows that there exist always two pencils $|P_1|$ and $|P_2|$ with $P_1 \cdot P_2 = 1$. Their generic conics intersect in a single point. Hence, the product of the two structure maps $i_1 \times i_2 : S \to \mathbf{P}^1 \times \mathbf{P}^1$ is birational. The inverse $\mathbf{P}^1 \times \mathbf{P}^1 \to S$ is a parametrization as desired. Indeed, the two pencils are $s =$ constant and $t =$ constant; because these curves are conics, the degree of the parametrization must be equal 2 in both variables. $\square$

Remark 3. The converse of Theorem 11 does not hold, because the ruled surface $R_{0,2}$ has a degree 2 parametrization but only one pencil of conics.

Remark 4. Over the reals, Theorem 11 does not hold, as one can construct del Pezzo surfaces which do not have two real pencils with intersection product equal to one. An example is the cubic surface $yx^2 + w^3 + wz^2 = 0$. It has a $\mathbf{D}_4$ singularity at $(0 : 1 : 0 : 0)$. According to the classification in [3], there are 6 lines. These lines are given by $x = w = 0$, $y = w = 0$, $x = w \pm iz = 0$, $y = w \pm iz = 0$. Only the first two lines are real and correspond to real pencils. These two pencils have intersection number 2; indeed the generic elements given by $x = sw$ and $y = tw$ intersect in two varying points

$$(x : y : z : w) = (s : t : \sqrt{-s^2t - 1} : 1).$$

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Received November 5, 1999