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On the Busemann Area in Minkowski Spaces

Rolf Schneider

Mathematisches Institut, Albert-Ludwigs-Universität
Eckerstr. 1, 79104 Freiburg i.Br., Germany

Abstract. Among the different notions of area in a Minkowski space, those due to Busemann and to Holmes and Thompson, respectively, have found particular attention. In recent papers it was shown that the Holmes-Thompson area is integral-geometric, in the sense that certain integral-geometric formulas of Crofton-type, well known for the area in Euclidean space, can be carried over to Minkowski spaces and the Holmes-Thompson area. In the present paper, the Busemann area is investigated from this point of view.

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1. Introduction and results

A Minkowski space (a finite-dimensional real Banach space) carries a natural metric and hence admits a canonical notion of curve length. The metric gives also rise to Hausdorff measures of any dimension. For a positive integer k less than the dimension of the space, the k-dimensional Hausdorff measure can serve as a notion of surface area for k-dimensional surfaces. There are, however, other reasonable and essentially different ways of introducing a notion of area in a Minkowski space. This is explained in detail in the book of Thompson [11]. The few natural requirements for such a notion of area (see [11], Chapter 5, or the brief summary in [7]) can be satisfied in many different ways. Two particularly well studied notions of area in Minkowski spaces are the Busemann area and the Holmes-Thompson area. As soon as there are different notions of area, the question arises whether there are viewpoints under which one of them might seem preferable. In earlier papers ([9], [7], [8]), an attempt was made to extend certain integral-geometric results for areas from Euclidean spaces to Minkowski spaces. It was found that the Holmes-Thompson area is suitable for that purpose. A similar conclusion can be drawn from some recent results on integral geometry in

Finsler spaces (Álvarez & Fernandes [1], [2]). What we intend here is a closer inspection of the Busemann area from this point of view. For a k-rectifiable Borel set M, the Busemann k-area of M coincides with the k-dimensional Hausdorff measure of M (a proof can be found, e.g., in [9], Section 5). For that reason, the Busemann area might appear as a first choice for a notion of area in Minkowski spaces. Our results will show, in particular, that this is no longer true from an integral-geometric point of view. We restrict our consideration to areas in codimension one, briefly called *areas*.

We assume $n \geq 3$ and represent an n-dimensional Minkowski space in the form $X = (\mathbb{R}^n, \|\cdot\|_B)$, where $\|\cdot\|_B$ is a norm on $\mathbb{R}^n$, with unit ball $B = \{x \in \mathbb{R}^n : \|x\|_B \leq 1\}$. A Minkowskian $(n-1)$-area α_{n-1} (satisfying the requirements of [11], Chapter 5) will be called *integral-geometric for* X, if there exists a translation invariant (locally finite) Borel measure μ on the space $\mathcal{E}_1^n$ of lines in $\mathbb{R}^n$ such that, for every $(n-1)$-dimensional compact convex set $K \subset \mathbb{R}^n$, the area of K is given by

$$\alpha_{n-1}(K) = \mu\left(\{L \in \mathcal{E}_1^n : L \cap K \neq \emptyset\}\right). \tag{1}$$

Equation (1) is the simplest version of an integral-geometric formula for the area, and if it holds, then more general versions also hold. In Euclidean space, (1) is true for the Euclidean $(n-1)$-area, if μ is the suitably normalized rigid motion invariant measure on $\mathcal{E}_1^n$.

The Holmes-Thompson area is integral-geometric for every Minkowski space. In [7] it was proved that for the spaces ℓ_∞^n and ℓ_1^n, among all Minkowskian areas only the multiples of the Holmes-Thompson area are integral-geometric. In the following, we investigate more closely how far the Busemann area deviates from being integral-geometric.

Since we are dealing with properties of isometry classes of Minkowski spaces, we formulate the results in terms of the Minkowski (or Banach-Mazur) compactum $\mathcal{M}_n$. This is the space of all isometry classes of n-dimensional Minkowski spaces, metrized by the logarithm of the Banach-Mazur distance. However, in order to simplify the formulations, we often identify a Minkowski space with its isometry class.

We conjecture that the Busemann area is generically not integral-geometric. The set of Minkowski spaces for which the Busemann area is not integral-geometric is open in $\mathcal{M}_n$, but we do not know whether it is dense. We have only been able to prove the following.

Theorem 1. *In $\mathcal{M}_n$, every neighbourhood of the Euclidean space ℓ_2^n contains Minkowski spaces for which the Busemann area is not integral-geometric, as well as spaces* (*different from ℓ_2^n*) *for which the Busemann area is integral-geometric.*

In the neighbourhood of other spaces, the situation can be even worse:

Theorem 2. *If $n = 3$ or n is sufficiently large, then in $\mathcal{M}_n$ a full neighbourhood of ℓ_∞^n consists of Minkowski spaces for which the Busemann area is not integral-geometric.*

The dimensional restriction in Theorem 2 is probably unnecessary.

2. Preliminaries

For convenience, we equip $\mathbb{R}^n$ with an auxiliary Euclidean structure, given by a scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $|\cdot|$. For notions and results from the theory of convex bodies that are used without explanation, we refer to [6].

First we recall the definition of Minkowski $(n-1)$-areas. Let $\mathcal{C}^{n-1}$ denote the set of all $(n-1)$-dimensional convex bodies in $\mathbb{R}^n$ which have the origin as centre of symmetry. By an *area generating function* we understand a function $\alpha : \mathcal{C}^{n-1} \to \mathbb{R}^+$ which is invariant under non-degenerate linear transformations of $\mathbb{R}^n$, continuous (with respect to the Hausdorff metric) and normalized by $\alpha(C) = \kappa_{n-1}$ (the volume of the $(n-1)$-dimensional Euclidean unit ball) if C is an $(n-1)$-dimensional ellipsoid. If such a function α and a Minkowski space $X = (\mathbb{R}^n, \|\cdot\|_B)$ are given, the induced Minkowski area of a compact C^1-hypersurface M in X is defined by

$$\alpha_{n-1}^B(M) := \int_M \frac{\alpha(B \cap T_x M)}{\lambda_{n-1}(B \cap T_x M)}\, d\lambda_{n-1}(x),$$

where $T_x M$ denotes the tangent space of M at x, considered as a linear subspace of $\mathbb{R}^n$, and λ_{n-1} is the $(n-1)$-dimensional Lebesgue area measure induced by the Euclidean metric. The Minkowski area $\alpha_{n-1}^B(M)$ does not depend on the choice of this metric. We consider only area generating functions α for which the *scaling function* defined by

$$\sigma_{\alpha,B}(u) := |u| \frac{\alpha(B \cap u^\perp)}{\lambda_{n-1}(B \cap u^\perp)} \qquad \text{for } u \in \mathbb{R}^n \setminus \{0\} \tag{2}$$

is convex. (The scaling function depends on the Euclidean structure, but not its convexity property.) Under this assumption, $\sigma_{\alpha,B}$ is the support function of a convex body $\mathbf{I}_{\alpha,B}$, which is called the *isoperimetrix* of the pair (α, B) (see [11] for the motivation and for further discussion).

Lemma 1. *The Minkowski area α_{n-1} is integral-geometric for $(\mathbb{R}^n, \|\cdot\|_B)$ if and only if the isoperimetrix $\mathbf{I}_{\alpha,B}$ is a zonoid.*

Essentially, this is a special case of Theorem 3.1 in [9]. For the reader's convenience, we give the short proof. If α_{n-1} is integral-geometric, there is a translation invariant, locally finite Borel measure μ on the space $\mathcal{E}_1^n$ of lines such that (1) holds whenever $K \subset u^\perp$, $u \in S^{n-1}$. Since μ is translation invariant, there is a finite, even measure φ on the sphere S^{n-1} such that

$$\int_{\mathcal{E}_1^n} f\, d\mu = \int_{S^{n-1}} \int_{v^\perp} f(t + \operatorname{lin}\{v\})\, d\lambda_{n-1}(t)\, d\varphi(v)$$

for every nonnegative measurable function f on $\mathcal{E}_1^n$ (a proof may be found, e.g., in [10], Section 4.1). This gives

$$\alpha_{n-1}^B(K) = \lambda_{n-1}(K) \int_{S^{n-1}} |\langle u, v\rangle|\, d\varphi(v),$$

and since $\alpha_{n-1}^B(K) = \sigma_{\alpha,B}(u)\lambda_{n-1}(K)$, we obtain

$$\sigma_{\alpha,B}(u) = \int_{S^{n-1}} |\langle u, v\rangle|\, d\varphi(v) \qquad \text{for } u \in \mathbb{R}^n \setminus \{0\}. \tag{3}$$

Since $\sigma_{\alpha,B}$ is the support function of $\mathbf{I}_{\alpha,B}$, this body is a zonoid. The argument can be reversed.

As a first consequence of Lemma 1, we see that the set $\mathcal{I}_\alpha$ of (isometry classes of) Minkowski spaces for which a given Minkowski area α_{n-1} is integral-geometric, is a closed subset of $\mathcal{M}_n$. In fact, let $(m_i)_{i\in\mathbb{N}}$ be a sequence in $\mathcal{I}_\alpha$ converging to $m \in \mathcal{M}_n$. We can choose representatives of m_i, m with unit balls B_i, B so that $B_i \to B$ in the Hausdorff metric. From (2) and the continuity of the area generating function α it follows that $\sigma_{\alpha,B_i} \to \sigma_{\alpha,B}$ pointwise, and this implies $\mathbf{I}_{\alpha_i,B_i} \to \mathbf{I}_{\alpha,B}$ in the Hausdorff metric ([6], Theorems 1.8.12 and 1.8.11). Each I_{α,B_i} is a zonoid, and the set of zonoids is closed in the space of convex bodies. Hence $\mathbf{I}_{\alpha,B}$ is a zonoid, which means that α_{n-1} is integral-geometric for $(\mathbb{R}^n, \|\cdot\|_B)$ and thus $m \in \mathcal{I}_\alpha$.

The Busemann area β_{n-1} is defined by the constant area generating function, $\beta(C) = \kappa_{n-1}$ for $C \in \mathcal{C}^{n-1}$. Hence, its scaling function is given by

$$\sigma_{\beta,B}(u) = |u| \frac{\kappa_{n-1}}{\lambda_{n-1}(B \cap u^\perp)} \qquad \text{for } u \in \mathbb{R}^n \setminus \{0\}. \tag{4}$$

Here

$$\lambda_{n-1}(B \cap u^\perp) = \frac{1}{n-1} \int_{s_u} \rho(B,v)^{n-1}\, d\sigma(v), \tag{5}$$

where $\rho(B,\cdot)$ denotes the radial function of B,

$$s_u := \{v \in S^{n-1} : \langle u, v\rangle = 0\}$$

is the great subsphere $S^{n-1} \cap u^\perp$, and σ is the $(n-2)$-dimensional spherical Lebesgue measure on s_u. The *intersection body* $\mathrm{I}B$ of B is defined by its radial function

$$\rho(\mathrm{I}B, u) = \frac{1}{|u|} \lambda_{n-1}(B \cap u^\perp) \qquad \text{for } u \in \mathbb{R}^n \setminus \{0\}, \tag{6}$$

hence

$$\mathbf{I}_{\beta,B} = \kappa_{n-1} \mathrm{I}^o B,$$

where $\mathrm{I}^o B := (\mathrm{I}B)^o$ denotes the polar body of $\mathrm{I}B$.

3. Proof of Theorem 1

The isoperimetrix of the Busemann area for the Minkowski space $(\mathbb{R}^n, \|\cdot\|_B)$ will now be denoted by $\mathbf{I}_B$. The proof of the first part of Theorem 1 requires the construction of unit balls B for which $\mathbf{I}_B$ is not a zonoid. Let B be given. We write

$$g(v) := \frac{1}{(n-1)\kappa_{n-1}} \rho(B,v)^{n-1}, \qquad v \in S^{n-1},$$

and

$$G(u) := \int_{s_u} g(v)\, d\sigma(v) \qquad \text{for } u \in \mathbb{R}^n \setminus \{0\},$$

so that G is homogeneous of degree zero. We extend also g to $\mathbb{R}^n \setminus \{0\}$ by homogeneity of degree zero. By (4), (5), the support function of the isoperimetrix $\mathbf{I}_B$ is given by

$$h(\mathbf{I}_B, u) = \frac{|u|}{G(u)} \qquad \text{for } u \in \mathbb{R}^n \setminus \{0\}. \tag{7}$$

We compute the directional derivatives of G. Let $u \in S^{n-1}$ and $w \in S^{n-1}$ with $w \perp u$ be given, let $0 < \epsilon < 1$. Let $\vartheta \in SO_n$ be the rotation with

$$\vartheta u = \frac{u + \epsilon w}{|u + \epsilon w|}$$

and $\vartheta x = x$ for $x \in \mathrm{lin}\{u, w\}^\perp$. Then

$$\vartheta w = \frac{w - \epsilon u}{|w - \epsilon u|}.$$

Let $v \in s_u \setminus \{\pm w\}$ and write

$$v = \alpha w + \sqrt{1-\alpha^2}\, \overline{v} \qquad \text{with } \overline{v} \in s_u \cap w^\perp.$$

Then $\alpha = \langle v, w \rangle$. Determine t so that $v + tu \perp u + \epsilon w$. This condition gives $t = -\epsilon\alpha$. We have

$$\begin{aligned} \vartheta v &= \vartheta\left(\alpha w + \sqrt{1-\alpha^2}\, \overline{v}\right) = \alpha \frac{w - \epsilon u}{|w - \epsilon u|} + \sqrt{1-\alpha^2}\, \overline{v} \\ &= \frac{v + tu}{|v + tu|} + (v + tu)\left(1 - \frac{1}{|v + tu|}\right) + \alpha(w - \epsilon u)\left(\frac{1}{|w - \epsilon u|} - 1\right), \end{aligned}$$

hence, using $t = -\epsilon\alpha$ and $|\alpha| \le 1$,

$$\left| \vartheta v - \frac{v + tu}{|v + tu|} \right| \le 2\epsilon^2.$$

Since the radial function of a convex body with 0 in the interior is a Lipschitz function on S^{n-1} ([6], Lemma 1.8.10 and Remark 1.7.7), we get

$$|g(\vartheta v) - g(v + tu)| \le c\epsilon^2$$

with a constant c depending only on B. We deduce that

$$\begin{aligned} G(u + \epsilon w) - G(u) &= \int_{s_u} [g(\vartheta v) - g(v)]\, d\sigma(v) \\ &= \int_{s_u} [g(v + tu) - g(v)]\, d\sigma(v) + O(\epsilon^2) \\ &= \int_{s_u} [g(v - \epsilon\langle v, w\rangle u) - g(v)]\, d\sigma(v) + O(\epsilon^2). \end{aligned}$$

The radial function of a convex body with interior points has directional derivatives on $\mathbb{R}^n \setminus \{0\}$, hence the same holds for g. It follows that

$$\lim_{\epsilon \to 0+} \frac{1}{\epsilon}[g(v - \epsilon\langle v, w\rangle u) - g(v)] = g'(v; (-\mathrm{sgn}\langle v, w\rangle)u)|\langle v, w\rangle|.$$

Using the bounded convergence theorem, we obtain

$$G'(u; w) = \int_{s_{u,w}} g'(v; -u)|\langle v, w\rangle| \, d\sigma(v) + \int_{s_{u,-w}} g'(v; u)|\langle v, w\rangle| \, d\sigma(v)$$

with

$$s_{u,w} := \{v \in s_u : \langle v, w\rangle \geq 0\}.$$

From (7) we get

$$h'(\mathbf{I}_B, u; w) = \frac{\langle u, w\rangle}{G(u)} - \frac{G'(u; w)}{G(u)^2} \qquad \text{for } u \in S^{n-1},$$

hence

$$\begin{aligned} &h'(\mathbf{I}_B, u; w) + h'(\mathbf{I}_B, u; -w) \\ &\quad = -h(\mathbf{I}_B, u)^2 \int_{s_u} |\langle v, w\rangle| [g'(v; u) + g'(v; -u)] \, d\sigma(v) \end{aligned} \tag{8}$$

for $u \in S^{n-1}$.

We use this to construct the required examples. We start with the Euclidean unit ball B^n and choose orthogonal unit vectors $u, z \in S^{n-1}$ and a number $\epsilon > 0$. Let

$$B_0 := \mathrm{conv}\,(B^n \cup (1 + \epsilon)(B^n \cap u^\perp))$$

and $B := B_0 + \epsilon[-z, z]$, where $[-z, z]$ is the closed segment with endpoints $-z$ and z. For this body B, let g be defined as above. One easily checks that

$$g'(v; u) + g'(v; -u) < 0 \qquad \text{for } v \in s_u. \tag{9}$$

From (8) and (9) it follows that

$$h'(\mathbf{I}_B, u; w) + h'(\mathbf{I}_B, u; -w) > 0 \tag{10}$$

for all $w \in S^{n-1}$ with $w \perp u$. If $F(\mathbf{I}_B, u)$ denotes the support set of the convex body $\mathbf{I}_B$ with outer normal vector u, then

$$h'(\mathbf{I}_B, u; x) = h(F(\mathbf{I}_B, u), x) \qquad \text{for } x \in \mathbb{R}^n$$

(Theorem 1.7.2 in [6]). Therefore, (10) implies that the face $F(\mathbf{I}_B, u)$ is of dimension $n - 1$. Since B is invariant under reflection in the line $\mathrm{lin}\{u\}$, this face is centrally symmetric, hence we get

$$h(F(\mathbf{I}_B, u), w) = \frac{1}{2} h(\mathbf{I}_B, u)^2 \int_{s_u} |\langle v, w\rangle| \, |g'(v; u) + g'(v; -u)| \, d\sigma(v)$$

for $w \in s_u$. In particular, the face $F(\mathbf{I}_B, u)$ is a zonoid, and since $|g'(v;u) + g'(v;-u)|$ has a positive lower bound, this face has a summand K which is an $(n-1)$-dimensional ball.

The body B has a cylindrical part, namely $Z := (B \cap z^\perp) + \epsilon[-z, z]$. There is a neighbourhood U of the vector z so that $B \cap y^\perp = Z \cap y^\perp$ for all $y \in U \cap S^{n-1}$. For these vectors y, we have

$$\lambda_{n-1}(B \cap y^\perp) = \frac{\lambda_{n-1}(B \cap z^\perp)}{\langle y, z \rangle}$$

and hence

$$h(\mathbf{I}_B, y) = h(\mathbf{I}_B, z)\langle y, z \rangle.$$

This means that the point $z_0 := h(\mathbf{I}_B, z)z$ is a vertex of the isoperimetrix $\mathbf{I}_B$ (that is, a point with n-dimensional normal cone).

If we now assume that $\mathbf{I}_B$ is a zonoid, then the face $F(\mathbf{I}_B, u)$ is a summand of $\mathbf{I}_B$ (Corollary 3.5.6 in [6]). In particular, $\mathbf{I}_B$ has a summand K which is an $(n-1)$-dimensional ball. There is a translate K' of K such that $z_0 \in K' \subset \mathbf{I}_B$ (Theorem 3.2.2 in [6]). But this is not possible, since z_0 is a vertex of $\mathbf{I}_B$. Thus $\mathbf{I}_B$ cannot be a zonoid.

If a neighbourhood (with respect to the Hausdorff metric) of the unit ball B^n is given, the number ϵ can be chosen so small that B is contained in that neighbourhood. It follows that every neighbourhood of ℓ_2^n in $\mathcal{M}_n$ contains Minkowski spaces for which the isoperimetrix of the Busemann area is not a zonoid. By Lemma 1, this completes the proof of the first part of Theorem 1.

Remark. The definition of 'integral-geometric' may be relaxed, by requiring only the existence of a signed measure instead of a positive measure (such signed measures, given by densities, appear in the Crofton formulas treated in [1]). Then Lemma 1 remains true if 'zonoid' is replaced by 'generalized zonoid', and also the first part of Theorem 1 with its proof given above remains valid.

Now we prove the second part of Theorem 1. Let $f : S^{n-1} \to \mathbb{R}$ be an even function of class C^∞. For sufficiently small $\epsilon > 0$, the function $\rho(B(\epsilon), \cdot)$ defined by

$$\rho(B(\epsilon), u) := (1 + \epsilon f(u))^{\frac{1}{n-1}}$$

for $u \in S^{n-1}$ and extended to $\mathbb{R}^n \setminus \{0\}$ by positive homogeneity of degree -1, is the radial function of a centrally symmetric convex body $B(\epsilon)$. (In fact, $1/\rho(B(\epsilon), \cdot)$ is convex for sufficiently small $\epsilon > 0$, as follows from the uniform convergence, for $\epsilon \to 0$, of the second derivatives of this function, together with Theorem 1.5.10 in [6].) We choose for f a spherical harmonic of even degree $m \geq 2$; then

$$\int_{s_u} f(v)\, d\sigma(v) = (n-1)\kappa_{n-1} a_m f(u) \qquad \text{for } u \in S^{n-1}$$

with a constant $a_m \neq 0$ (see, e.g., [4]). It follows that

$$h(\mathbf{I}_{B(\epsilon)}, u) = |u| \frac{\kappa_{n-1}}{\rho(\mathrm{I}B(\epsilon), u)} = |u| \frac{1}{1 + \epsilon a_m f(u/|u|)}$$

for $u \in \mathbb{R}^n \setminus \{0\}$. The function $h(\mathbf{I}_{B(\epsilon)}, \cdot)$ is of class C^∞. For $\epsilon \to \infty$, the partial derivatives of this function converge, uniformly on S^{n-1}, to the corresponding partial derivatives of $h(\mathbf{I}_{B(0)}, \cdot) = h(B^n, \cdot)$. Since $h(\mathbf{I}_{B(\epsilon)}, \cdot)$ is of class C^∞, the integral equation

$$h(\mathbf{I}_{B(\epsilon)}, u) = \int\limits_{S^{n-1}} |\langle u, v\rangle| g_\epsilon(v)\, d\omega(v), \qquad u \in S^{n-1},$$

where ω denotes the spherical Lebesgue measure on S^{n-1}, has a continuous even solution g_ϵ on S^{n-1}. As shown in [5], $\|g_\epsilon\|_{\max} \le \|h(\mathbf{I}_{B(\epsilon)}, \cdot)\|_r$, where $\|\cdot\|_{\max}$ is the maximum norm on S^{n-1} and $\|\cdot\|_r$ is a certain norm involving derivatives up to order at most $n+3$. From the uniform convergence of the derivatives just mentioned, it follows that $\|g_\epsilon - g_0\|_{\max} \le \|h(\mathbf{I}_{B(\epsilon)}, \cdot) - h(B^n, \cdot)\|_r \to 0$ for $\epsilon \to 0$, where g_0 is a positive constant. Hence, if ϵ is sufficiently small, the function g_ϵ is positive, and hence the isoperimetrix $\mathbf{I}_{B(\epsilon)}$ is a zonoid. The assertion of the second part of Theorem 3 now follows from Lemma 1, if one observes that $B(\epsilon)$ is not an ellipsoid.

4. Proof of Theorem 2

Let $(e_1, \dots, e_n)$ be an orthonormal basis of $\mathbb{R}^n$, with respect to the chosen scalar product. We need the inequality

$$\sum_{\epsilon_j = \pm 1} |\epsilon_1 \xi_1 + \ldots + \epsilon_n \xi_n| \ge \gamma(n) \sum_{j=1}^{n} |\xi_j| \qquad \text{with } \gamma(n) := 2\binom{n-1}{\left[\frac{n-1}{2}\right]}, \tag{11}$$

for $\xi_1, \dots, \xi_n \in \mathbb{R}$, for which we first give a proof. For reasons of homogeneity and symmetry, it suffices to prove (11) for $(\xi_1, \dots, \xi_n)$ taken from the simplex

$$\Delta := \left\{ (\xi_1, \dots, \xi_n) \in \mathbb{R}^n : \xi_j \ge 0, \sum \xi_j = 1 \right\}.$$

Denote the left-hand side of (11) by $F(\xi_1, \dots, \xi_n)$. Since F is a convex function and the restriction $F|\Delta$ is invariant under the affine symmetry group of Δ, the function $F|\Delta$ attains its minimum at the points of a nonempty compact convex set containing the centroid of Δ. It follows that

$$\begin{aligned} F(\xi_1, \dots, \xi_n) &\ge F\left(\frac{1}{n}, \dots, \frac{1}{n}\right) = \frac{1}{n} \sum_{\epsilon_j = \pm 1} |\epsilon_1 + \ldots + \epsilon_n| \\ &= \frac{1}{n} \sum_{j=0}^{n} \binom{n}{j} |n - 2j| = \gamma(n), \end{aligned}$$

where the last equation is proved by induction.

By $Q := \operatorname{conv}\{\pm e_1, \dots, \pm e_n\}$ we denote the crosspolytope.

Lemma 2. *If Z is a zonoid with centre at the origin and $\lambda > 0$ is a real number satisfying*

$$Q \subset Z \subset \lambda Q, \tag{12}$$

then $\lambda \geq \lambda_{\min} := 2^{-n} n\gamma(n)$.

Proof. Let the zonoid Z satisfy (12). Its support function has a representation

$$h(Z,x) = \int_{S^{n-1}} |\langle u, x\rangle| \, d\rho(u), \qquad x \in \mathbb{R}^n,$$

with an even measure ρ on the unit sphere S^{n-1}. Using (11), we get

$$\begin{aligned}
&\sum_{\epsilon_j=\pm 1} h(Z, \epsilon_1 e_1 + \ldots + \epsilon_n e_n) \\
&\quad = \int_{S^{n-1}} \sum_{\epsilon_j=\pm 1} |\epsilon_1\langle u, e_1\rangle + \ldots + \epsilon_n\langle u, e_n\rangle| \, d\rho(u) \\
&\quad \geq \gamma(n) \int_{S^{n-1}} \sum_{j=1}^{n} |\langle u, e_j\rangle| \, d\rho(u) = \gamma(n) \sum_{j=1}^{n} h(Z, e_j) \\
&\quad \geq \gamma(n) \sum_{j=1}^{n} h(Q, e_j) = n\gamma(n).
\end{aligned}$$

The right-hand inclusion of (12) implies

$$\sum_{\epsilon_j=\pm 1} h(Z, \epsilon_1 e_1 + \ldots + \epsilon_n e_n) \leq \lambda \sum_{\epsilon_j=\pm 1} h(Q, \epsilon_1 e_1 + \ldots + \epsilon_n e_n) = 2^n \lambda.$$

Both inequalities together yield the assertion of Lemma 2.

Now we prove Theorem 2. The space ℓ_∞^n can be considered as $(\mathbb{R}^n, \|\cdot\|_C)$, where C is the cube with vertices $\pm e_1 \pm \ldots \pm e_n$. The support function of the isoperimetrix $\mathbf{I}_C$ of the Busemann area for this space is given by

$$h(\mathbf{I}_C, u) = |u| \frac{\kappa_{n-1}}{\lambda_{n-1}(C \cap u^\perp)} \qquad \text{for } u \in \mathbb{R}^n \setminus \{0\}.$$

We normalize the isoperimetrix by defining

$$\mathbf{I} := \frac{2^{n-1}}{\kappa_{n-1}} \mathbf{I}_C;$$

then $h(\mathbf{I}, e_i) = 1$ for $i = 1, \ldots, n$. Since $\mathbf{I}$ has the same Euclidean symmetries as C, it follows that $e_i \in \mathbf{I}$ for $i = 1, \ldots, n$ and hence that

$$Q \subset \mathbf{I}. \tag{13}$$

Let $z := e_1 + \ldots + e_n$. We want to show that

$$h(\mathbf{I}, z) < \lambda_{\min} h(Q, z). \tag{14}$$

Here, $h(Q, z) = 1$. Now

$$h(\mathbf{I}, z) = \sqrt{n}\frac{2^{n-1}}{\lambda_{n-1}(C \cap z^{\perp})} = \frac{\sqrt{n}}{S(n)},$$

where $S(n)$ denotes the $(n-1)$-volume of the intersection of the unit cube $\frac{1}{2}C$ with a hyperplane through its centre and orthogonal to a main diagonal. It is given by

$$S(n) = \frac{\sqrt{n}}{2^{n-1}(n-1)!}\sum_{j=0}^{\left[\frac{n}{2}\right]}(-1)^j \binom{n}{j} (n-2j)^{n-1} = \frac{2}{\pi}\sqrt{n}\int_0^{\infty} \left(\frac{\sin x}{x}\right)^n dx$$

(see Chakerian & Logothetti [3], also for references). Using

$$\lim_{n\to\infty} S(n) = \sqrt{\frac{6}{\pi}}$$

([3], p. 238) and Stirling's formula, one shows that (14) is true for all sufficiently large dimensions. By direct computation, (14) is proved for $n = 3, 5, \dots, 9$. For $n = 4$, (14) is true with equality instead of inequality. Probably (14) holds for all $n \neq 4$.

Now let n be a dimension for which (14) is true. By symmetry, (14) holds also if z is replaced by $\pm e_1 \pm \dots \pm e_n$. It follows that the normalized isoperimetrix $\mathbf{I}$ is contained in the interior of the crosspolytope $\lambda_{\min}Q$. By (13), $\mathbf{I}$ contains the crosspolytope Q. Hence, there exist a factor $a > 1$ and a number $\lambda < \lambda_{\min}$ so that

$$Q \subset \operatorname{int} a\mathbf{I} \subset \operatorname{int} \lambda Q.$$

Forming the isoperimetrix is a continuous operation. Hence, in $\mathcal{K}^n$ (the space of convex bodies in $\mathbb{R}^n$, equipped with the Hausdorff metric) there is a neighbourhood U of the cube C so that, for all centred convex bodies $B \in U$, the isoperimetrix $\mathbf{I}_B$ of the Busemann area for $(\mathbb{R}^n, \|\cdot\|_B)$ still satisfies

$$Q \subset \operatorname{int} \left(a\frac{2^{n-1}}{\kappa_{n-1}}\mathbf{I}_B\right) \subset \operatorname{int} \lambda Q.$$

Since $\lambda < \lambda_{\min}$, it follows from Lemma 2 that $\mathbf{I}_B$ cannot be a zonoid. This implies that the Busemann area for $(\mathbb{R}^n, \|\cdot\|_B)$ is not integral-geometric.

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