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Stretched $\mathfrak{m}$-primary Ideals

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Abstract. The multiplicity of an $\mathfrak{m}$-primary ideal I of a Cohen-Macaulay local ring $(A, \mathfrak{m})$ of dimension d can be written as $e(I) = \lambda(I/I^2) - (d-1)\lambda(A/I) + K - 1$ for some integer $K \geq 1$. In the case $K = 1, 2,$ the Hilbert function of I and the depth of the associated graded ring of A with respect to I are very well understood. In this paper we are dealing with the case $K = 3$ and we determine the possible Hilbert functions of stretched ideals whose Cohen-Macaulay type is not too big. Our main result extends to a considerable extent a deep result of J. Sally who proved that the associated graded ring of a Gorenstein local ring with embedding dimension equal to $e(\mathfrak{m}) + d - 3$, is Cohen-Macaulay.

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Introduction

Let $(A, \mathfrak{m})$ be a local Cohen-Macaulay ring of dimension d and multiplicity e. A good measure of the singularity at $(A, \mathfrak{m})$ is the Hilbert function of $(A, \mathfrak{m})$. This is the Hilbert function $H_A(t)$ of the associated graded ring $gr_{\mathfrak{m}}A = \oplus_{i \geq 0}(\mathfrak{m}^i/\mathfrak{m}^{i+1})$ of A, which assigns to each non-negative integer t the dimension over $A/\mathfrak{m}$ of the vector space $\mathfrak{m}^t/\mathfrak{m}^{t+1}$. The main problem is that, in general, the ring $gr_{\mathfrak{m}}A$ has few, if any, good properties so that what is known about Hilbert functions for nice graded algebras is often not applicable. So the task is twofold: try to get information about the Hilbert function in spite of bad properties of $gr_{\mathfrak{m}}A$ and try to find properties of $(A, \mathfrak{m})$ which lead to reasonable associated graded rings $gr_{\mathfrak{m}}A$.

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One way to recognize some Cohen-Macaulay local ring with good associated graded ring is the use of the fact that v, the embedding dimension of A, satisfies $d \le v \le e+d-1$. If we put $h := v-d$, the embedded codimension of A, then $e \ge h+1$ and we can write $e = h+K$ for some integer $K \ge 1$. If $K=1$, then $e=h+1$ and $gr_{\mathfrak{m}}A$ is Cohen-Macaulay with Hilbert series $P_A(z) := \sum_{t\ge 0} H_A(t)z^t = (1+hz)/(1-z)^d$ (see [11]). If $K=2$ then $gr_{\mathfrak{m}}A$ has depth at least $d-1$ and $P_A(z) = (1+hz+z^s)/(1-z)^d$ for some integer s, $2 \le s \le h+1$ (see [8]). Finally if $K=3$ and A is Gorenstein, then $gr_{\mathfrak{m}}A$ is Cohen-Macaulay and $P_A(z) = (1+hz+z^2+z^3)/(1-z)^d$ (see [13]). The proof of this far reaching result needs several pages of very tricky and clever computations leading to the description of a suitable minimal set of generators of the powers of the maximal ideal of A. In the following, we will often refer to this result which will be simply called *Sally's theorem.*

In [10] we proved that if $K=3$ and the Cohen-Macaulay type τ of A verifies $\tau < h$, then $gr_{\mathfrak{m}}A$ has depth at least $d-1$ and $P_A(z) = (1+hz+z^t+z^s)/(1-z)^d$ where $2 \le t \le s \le \tau+2$ and $t < s$ if $\tau = 1$. This strongly extends Sally's theorem but does not complete the list of the possible Hilbert functions in the case $K=3$. In [10] we asked whether $gr_{\mathfrak{m}}(A)$ has depth at least $d-1$ for $\tau \ge h$.

Recently H. Wang gave a negative answer to this question by producing a 2-dimensional Cohen-Macaulay local ring of multiplicity $e = h+3$, $\tau = h$ and $\operatorname{depth}(gr_{\mathfrak{m}}(A)) = 0$. In fact if $A = k[[x,y,z,u,v]]/(z^2, zu, zv, uv, zx-u^3, zy-v^3)$, then $P_A(z) = (1+3x+3x^3-x^4)/(1-x)^2$, hence $e = 6 = h+3$, $d=2$, but $\operatorname{depth}(gr_{\mathfrak{m}}(A)) = 0$.

Since, by geometric reasons, one needs to blow up $\mathfrak{m}$-primary ideals different from $\mathfrak{m}$, we are lead to consider the more general problem of determining the Hilbert function of an $\mathfrak{m}$-primary ideal I of the local Cohen-Macaulay ring $(A, \mathfrak{m})$. This is the Hilbert function of the associated graded ring $G := gr_I(A) = \oplus_{i\ge 0}(I^i/I^{i+1})$ of A with respect to the I-adic filtration, namely the function $H_I(t) = \lambda(I^t/I^{t+1})$ where $\lambda(M)$ stands for the length of the (A/I)-module M. One knows that $e(I) \ge \lambda(I/I^2) - (d-1)\lambda(A/I)$, hence, if we denote by h the embedding codimension of I, which is by definition the integer $h := \lambda(I/I^2) - d\lambda(A/I)$, we have $e(I) \ge h + \lambda(A/I)$ and we can write $e(I) = h + \lambda(A/I) + K - 1$ for some integer $K \ge 1$.

If $K=1$, the associated graded ring G is Cohen-Macaulay and its Hilbert series is $P_I(z) := \sum_{j\ge 0} H_I(j)z^j = (\lambda(A/I)+hz)/(1-z)^d$ (see [16]). If $K=2$, then $\operatorname{depth}(G) \ge d-1$ and $P_I(z) = (\lambda(A/I)+hz+z^s)/(1-z)^d$ for some s, $2 \le s \le h - \lambda(A/I)$ (see [7]). The next step is the case $K=3$, which means $e(I) = h + \lambda(A/I) + 2$, and the task is to extend Sally's theorem to this more general setting.

The first remark is that, in the primary case, the assumption A is Gorenstein does not imply that G is Cohen-Macaulay. For example, if we consider the Gorenstein local ring $A = k[[t^5, t^6, t^9]]$ and its $\mathfrak{m}$-primary ideal $I = (t^5, t^6)$, then it easy to see that $P_I(z) = (2+z+z^2+z^4)/(1-z)$ so that $e(I) = h + \lambda(A/I) + 2$ but the associated graded ring G is not Cohen-Macaulay.

To throw light upon the problem, we must recall that a main ingredient in the proof of Sally's theorem is that, under the given assumptions, for any ideal J generated by a maximal superficial sequence in $\mathfrak{m}$, the ideal $(\mathfrak{m}/J)^2$ is a principal ideal. In the terminology of [12], A is a stretched Gorenstein local ring. This suggests the following definition: given an $\mathfrak{m}$-primary ideal I, we say that I is *stretched* if there exists an ideal J generated by a maximal superficial sequence in I, such that

a) $I^2 \cap J = IJ$ and

b) $H_{I/J}(2) = 1$.

This definition depends upon the choice of J, nevertheless it works nice for our proposal. Condition a), which holds true for every J if $I = \mathfrak{m}$, is what we need in order to preserve the embedding codimension modulo a superficial element.

Stretched $\mathfrak{m}$-primary ideals and their main properties are considered in Section 2 of the paper. We first prove that the associated graded ring G of a stretched ideal I is Cohen-Macaulay if and only if $I^{K+1} = JI^K$ (see Theorem 2.6). Thus it is natural to consider those stretched ideals I such that $I^{K+1} \subseteq JI^{K-1}$, a property which has relevant consequences investigated in Section 3. We know by [10], Theorem 3.2, that the maximal ideal of a Cohen-Macaulay local ring $(A, \mathfrak{m})$ has this property if $e = h + 3$ and $\tau < h$, where τ is the Cohen-Macaulay type of A. Hence, as in [7], for a stretched ideal I we set $\tau(I) := \lambda((J : I) \cap I/J)$ and we say that $\tau(I)$ is the Cohen-Macaulay type of I.

As in the case of the maximal ideal, we are able to prove that the associated graded ring of a stretched ideal I such that $\tau(I) < h + 1 - \lambda(A/I)$, has a small amount of good behaviour in that $I^{K+1} \subseteq JI^2$ (see Theorem 2.7). Hence, if $K = 3$, then $I^{K+1} \subseteq JI^{K-1}$. In Section 3, by using a deep result of [7] (see also [4], [3] and [2]), we prove that a stretched ideal I such that $I^{K+1} \subseteq JI^{K-1}$ has $\operatorname{depth}(G) \geq d-1$ and $P_I(z) = \left(\lambda(A/I) + hz + z^2 + \cdots + z^{K-1} + z^s\right)/(1-z)^d$ for some $s \geq K$ (see Proposition 3.1). Thus, in order to complete the description of the Hilbert series of a stretched ideal I such that $I^{K+1} \subseteq JI^{K-1}$, we may reduce the problem to the one-dimensional case and need to look for a good upper bound for the degree s of the h-polynomial of I. This is done in Proposition 3.3 for the general case and in Theorem 3.7 in our favorite case, $K = 3$, when we are able to prove that $s \leq \tau + 2$. This gives the main result of the paper which is formulated and proved in Theorem 3.12. It says that if I is an integrally closed stretched ideal such that $K = 3$ and $I^4 \subseteq JI^2$, then $\operatorname{depth}(G) \geq d - 1$ and $P_I(z) = \left(\lambda(A/I) + hz + z^2 + z^s\right)/(1 - z)^d$ for some integer s such that $3 \leq s \leq \tau + 2$. The desired extension of Sally's theorem is now a trivial consequence of this result.

Some of the results of this paper have been conjectured after (and confirmed by) explicit computations performed by using the computer algebra system CoCoA[1].

1. Preliminaries

Let $(A, \mathfrak{m})$ be a local ring of dimension d and residue field k. If I is an $\mathfrak{m}$-primary ideal, the Hilbert Function $H_I(\ )$ of I is defined as

$$H_I(t) = H_G(t) = \lambda_{A/I}(I^t/I^{t+1})$$

where G is the associated graded ring of A with respect to the I-adic filtration, that is $G := gr_I(A) = \oplus_{t\geq 0}(I^t/I^{t+1})$.

The generating function of this numerical function is the power series

$$P_I(z) = \sum_{t\geq 0} H_I(t) z^t.$$

This series is called the Hilbert series of I. It is well known that this series is rational and that, even more, there exists a polynomial $h_I(z)$ with integer coefficients such that $h_I(1) \neq 0$ and

$$P_I(z) = \frac{h_I(z)}{(1-z)^d}.$$

The polynomial $h_I(z) = h_0 + h_1 z + \dots h_s z^s$ is called the *h-polynomial* of I and the vector $(h_0, h_1, \dots, h_s)$ the *h-vector* of I.

It is clear that we have

$$h_0 = H_I(0) = \lambda(A/I), \qquad h_1 = \lambda(I/I^2) - d\,\lambda(A/I).$$

The integer h_1 will be indicated by $h(I)$ or simply by h if the ideal I is clear from the context. For every $i \geq 0$ we let

$$e_i(I) := \frac{h_I^{(i)}(1)}{i!} \qquad \text{and} \qquad \binom{X+i}{i} := \frac{(X+i)\dots(X+1)}{i!}.$$

In particular $e_0(I) = h_I(1)$ and the polynomial

$$p_I(X) := \sum_{i=0}^{d-1} (-1)^i e_i(I) \binom{X+d-i-1}{d-i-1}$$

has rational coefficients and degree $d-1$. Further, for every $n \gg 0$, $p_I(n) = H_I(n)$. The polynomial $p_I(X)$ is called the *Hilbert polynomial* of I and the integer $e_0(I) = h_I(1)$ is called the *multiplicity* of I, often indicated simply by $e(I)$. If A is artinian and I is any non-zero ideal of A, then $e(I) = h_I(1) = \lambda(A)$.

We recall that if A has positive dimension, an element x in I is called superficial for I if there exists an integer $c > 0$ such that

$$(I^n : x) \cap I^c = I^{n-1}$$

for every $n > c$.

It is easy to see that a superficial element x is not in I^2 and that x is superficial for I if and only if $x^* := \overline{x} \in I/I^2$ does not belong to the relevant associated primes of G. Hence, if the residue field is infinite, superficial elements always exist. Further, if A has positive depth, every superficial element for I is also a regular element in A.

A sequence $x_1, \dots, x_r$ in the local ring $(A, \mathfrak{m})$ is called a superficial sequence for I, if x_1 is superficial for I and $\overline{x_i}$ is superficial for $I/(x_1, \dots, x_{i-1})$ for $2 \leq i \leq r$. By passing, if needed, to the local ring $A[X]_{(\mathfrak{m},X)}$, we may assume that the residue field is infinite. Hence if $\operatorname{depth}(A) \geq r$, there exists a superficial sequence $x_1, \dots, x_r$ for I and every superficial sequence is also a regular sequence in A. Such a sequence has the right properties for a good behaviour of the numerical invariants under reduction modulo the ideal it generates.

Let $x_1, \dots, x_r$ be a superficial sequence for I and put $\overline{I} := I/(x_1, \dots, x_r)$. Then, for $i = 0, \dots, d-r$, we have $e_i(I) = e_i(\overline{I})$. Further the following conditions are equivalent:

$$\begin{aligned}\operatorname{depth}(G) \geq r \iff P_I(z) = \tfrac{P_{\overline{I}}(z)}{(1-z)^r} \iff h_I(z) = h_{\overline{I}}(z) \\ \iff I^j \cap (x_1, \dots, x_r) = I^{j-1}(x_1, \dots, x_r) \text{ for every } j \geq 1.\end{aligned}$$

If J is the ideal generated by a maximal superficial sequence in I, we know by [16] that

$$e(I) = \lambda(I/I^2) - (d-1)\lambda(A/I) + \lambda(I^2/JI) = h + \lambda(A/I) + \lambda(I^2/JI)$$

so that $\lambda(I^2/JI)$ does not depend on J and we put $K := \lambda(I^2/JI) + 1$. Then we have

$$e(I) = h + K + \lambda(A/I) - 1.$$

Let us now recall a construction due to Ratliff and Rush (see [6]). For every n we have a chain of ideals

$$I^n \subseteq I^{n+1} : I \subseteq I^{n+2} : I^2 \subseteq \cdots \subseteq I^{n+k} : I^k \subseteq \cdots$$

This chain stabilizes at an ideal which we will denote by

$$\widetilde{I^n} := \bigcup_{k \geq 1} (I^{n+k} : I^k).$$

For every i and j we have $I^i \subseteq \widetilde{I^i}$ and $\widetilde{I^i}\widetilde{I^j} \subseteq \widetilde{I^{i+j}}$. Further, if x is superficial for I and a non zero-divisor, it is an easy consequence of the Artin Rees lemma that for every integer $j \gg 0$ we have $I^j : x = I^{j-1}$. From this we easily get $I^i = \widetilde{I^i}$, for $i \gg 0$.

Finally, for every $n \geq 0$, we have $\widetilde{I^{n+1}} : x = \widetilde{I^n}$ which implies that G has positive depth if and only if the equality $I^i = \widetilde{I^i}$ holds for every $i \geq 0$.

In the following we assume that $(A, \mathfrak{m})$ is a Cohen-Macaulay local ring of dimension d and I is an $\mathfrak{m}$-primary ideal in A. We let J be the ideal generated by $x_1, \dots, x_d$, a maximal superficial sequence for I. Then $e(I) = e(I/J) = \lambda(A/J)$ and we define for every $j \geq 0$

$$\rho_j := \lambda(\widetilde{I^{j+1}}/J\widetilde{I^j}), \quad a_j := \lambda(\widetilde{I^j}/I^j), \quad v_j := \lambda(I^{j+1}/JI^j).$$

For example we have $a_0 = 0$, $v_0 = \lambda(I/J) = \lambda(A/J) - \lambda(A/I) = e(I) - \lambda(A/I)$ and

$$v_1 = K - 1 = e(I) - \lambda(A/I) - h. \tag{1}$$

When the ring A has dimension one, we have more relevant properties of the integers already introduced. Hence, from now on, we are assuming $d = 1$ and we let $J = xA$, where x is a superficial element of the $\mathfrak{m}$-primary ideal I. Further we let as above

$$h_I(z) = h_0 + h_1 z + \cdots + h_s z^s$$

where $h_s \neq 0$. Hence, in the following, s is the degree of the h-polynomial of I. We have

$$h_i = H_I(i) - H_I(i-1) \tag{2}$$

for every $i \geq 1$ so that $e(I) = h_I(1) = \sum_{i=0}^{s} h_i = H_I(s)$.

It is well known, see [10], that for every $j \geq 0$ we have

$$e(I) = H_I(j) + v_j \tag{3}$$

and

$$\rho_j + a_j = v_j + a_{j+1}. \tag{4}$$

Since we have seen that $e(I) = H_I(s)$, from (3) we get $v_s = 0$. On the other hand from (2) and (3) we get

$$h_i = v_{i-1} - v_i \tag{5}$$

for every $i \geq 1$, hence $v_{s-1} = h_s > 0$. Since $v_j = 0$ implies $v_t = 0$ for every $t \geq j$, we have

$$\begin{cases} v_j > 0 & \text{if } j \leq s-1 \\ v_j = 0 & \text{if } j \geq s \end{cases} \tag{6}$$

so that s is exactly the reduction number of I. We have $I^{s+1} = xI^s$, from which we get $I^{s+t} = x^t I^s$ for every $t \geq 0$.

Let j be an integer, $j \geq s$, and let t be a positive integer such that $\widetilde{I^j} = I^{j+t} : I^t$; we have

$$\widetilde{I^j} = I^{j+t} : I^t \subseteq I^{j+t} : x^t = x^{j+t-s} I^s : x^t = x^{j-s} I^s \subseteq I^j,$$

so that $a_j = 0$ for every $j \geq s$. Hence $a_0 = a_s = 0$ and by (5) and (4) we get

$$e_1(I) = \sum_{j=0}^{s} j h_j = \sum_{j=0}^{s-1} v_j = \sum_{j=0}^{s-1} \rho_j. \tag{7}$$

Exactly in the same way as we proved Proposition 1.4 in [10], we can prove now the following result which gives a lower bound for the integers a_j in terms of the integers v_i.

Proposition 1.1. *Let $d = 1$ and x be a superficial element of the $\mathfrak{m}$-primary ideal I. Further let $j \geq 1$ and $t \geq 0$ be integers such that $j + t \leq s$. Then*

$$a_j \geq v_{j+t-1} - v_{j-1} + t.$$

By using this result we get a rough but useful upper bound for the reduction number s of I. This result should be compared with Proposition 1.5 in [10].

Proposition 1.2. *Let $d = 1$ and x be a superficial element of the $\mathfrak{m}$-primary ideal I. Then*

$$s \leq 1 + \lambda\left((xA : I) \cap I + \widetilde{I^2}/xA\right) + \lambda(\widetilde{I}/I).$$

Proof. We have

$$(xA : I) \cap I + \widetilde{I^2} \supset \widetilde{I^2} \supset I^2 \supset xI$$

and

$$(xA : I) \cap I + \widetilde{I^2} \supset xA \supset xI.$$

From this we get $\lambda(A/I) + \lambda\left((xA : I) \cap I + \widetilde{I^2}/xA\right) = v_1 + a_2 + \lambda\left((xA : I) \cap I + \widetilde{I^2}/\widetilde{I^2}\right)$.

By using the above proposition with $j = 2$ and $t = s - 2$, we get $v_1 + a_2 \geq v_{s-1} + s - 2$. On the other hand the map

$$A/\widetilde{I} \xrightarrow{x} (xA : I) \cap I + \widetilde{I^2}/\widetilde{I^2}$$

is injective and $v_{s-1} \geq 1$. Hence

$$\lambda(A/I) + \lambda\left((xA : I) \cap I + \widetilde{I^2}/xA\right) \geq v_{s-1} + s - 2 + \lambda(A/\widetilde{I}) \geq s - 1 + \lambda(A/\widetilde{I})$$

and the conclusion follows. □

2. Stretched ideals

Let I be an $\mathfrak{m}$-primary ideal of the d-dimensional Cohen-Macaulay local ring $(A, \mathfrak{m})$. We have seen in the first section that the Hilbert series of I can be written as

$$P_I(z) = \sum_{t \geq 0} H_I(t) z^t = \frac{h_I(z)}{(1-z)^d}$$

where $h_I(z) = h_0 + h_1 z + \ldots h_s z^s$ is the h-polynomial of I. We have

$$h(I) = h_1 = \lambda(I/I^2) - d\lambda(A/I)$$

hence, when $I = \mathfrak{m}$, we get $h_1 = \lambda(\mathfrak{m}/\mathfrak{m}^2) - d$. This is the reason why this integer is often called the *embedded codimension* of A.

One of the main properties of this integer is that it does not change modulo a superficial sequence. In the case of an $\mathfrak{m}$-primary ideal, this property does not hold anymore, but one needs the additional assumption $I^2 \cap J = IJ$, which trivially holds if $I = \mathfrak{m}$.

Lemma 2.1. *Let I be an $\mathfrak{m}$-primary ideal and let J be the ideal generated by a maximal superficial sequence $(x_1, \ldots, x_d)$ for I. Given an integer $r \leq d$, we let $\overline{I} = I/(x_1, \ldots, x_r)$ and $\overline{A} = A/(x_1, \ldots, x_r)$. If $I^2 \cap J = IJ$ then*

$$h(I) = h(\overline{I}).$$

Proof. If $x \in J$ is a superficial element for I and $I^2 \cap J = IJ$, it is clear that

$$(I/xA)^2 \cap (J/xA) = (I/xA)(J/xA),$$

hence we may assume $r = 1$ and then we let $x = x_1$. We have a short exact sequence

$$0 \to (I^2 + xA)/I^2 \to I/I^2 \to \overline{I}/\overline{I}^2 \simeq I/(I^2 + xA) \to 0.$$

Now it is clear that $(I^2 + xA)/I^2 \simeq xA/(I^2 \cap xA) = xA/xI \simeq A/I$, where the equality $I^2 \cap xA = xI$ comes from the fact that $x_1, x_2, \ldots, x_d$ form a regular sequence in A.

Since $\overline{A}/\overline{I} \simeq A/I$, from the above exact sequence we get

$$h(\overline{I}) = \lambda(\overline{I}/\overline{I}^2) - (d-1)\lambda(\overline{A}/\overline{I}) = \lambda(I/I^2) - d\lambda(A/I) = h(I). \qquad \square$$

Under the light of this lemma, we come to the main definition of this note. It was inspired by the work of J. Sally (see [12]) and motivated the title of this paper.

Definition 2.2. *Given an $\mathfrak{m}$-primary ideal I, we say that I is stretched if there exists an ideal J generated by a maximal superficial sequence for I, such that*

a) $I^2 \cap J = IJ$,

b) $H_{I/J}(2) = 1$.

For example the maximal ideal of a Cohen-Macaulay local ring of embedding dimension $e+d-2$ or of a Gorenstein local ring of embedding dimension $e+d-3$ is stretched.

We remark that the condition $H_{I/J}(2) = 1$ depends upon the choice of J. An example already appeared in [12], but the construction there heavily relies on the fact that the residue field was not algebraically closed, beeing the field of real numbers. We give here an other example which does not depend on the residue field k.

Example 2.3. Let us consider the ring

$$A = k\left[\left[t^6, t^7, t^{11}, t^{15}\right]\right]$$

and let I be the maximal ideal of A. Then both t^6 and $t^6 + t^7$ are superficial elements in I, but $H_{I/t^6A}(2) = 2$ and $H_{I/(t^6+t^7)A}(2) = 1$.

In the following, a stretched $\mathfrak{m}$-primary ideal I will come always equipped with an ideal J generated by a maximal superficial sequence for I such that $I^2 \cap J = IJ$ and $H_{I/J}(2) = 1$.

We collect some easy properties of stretched ideals.

Lemma 2.4. *For a stretched ideal I the following properties hold:*

1) $e(I) \geq \lambda(A/I) + h + 1$.
2) *For every $n \geq 1$ we have $I^{n+1} = JI^n + (a^n b)$, where $a, b \in I$, $a, b \notin J$.*
3) $I = (b) + (J : a) \cap I$.
4) *For every $n \geq 2$ we have $v_n \leq v_{n-1}$.*
5) *For every $n \geq 2$ we have $H_{I/J}(n) \leq 1$.*
6) *For every $n \geq 1$ we have $a^n b\, \mathfrak{m} \subseteq I^{n+2} + JI^n$.*
7) *If $I \cap (J : a) \neq I \cap (J : I)$, then $v_2 < v_1$.*

Proof. 1) We have $H_{I/J}(0) = \lambda(A/I)$, $H_{I/J}(1) = h(I/J) = h(I) = h$ and $H_{I/J}(2) = 1$. This implies

$$e(I) = e(I/J) \geq \lambda(A/I) + h + 1.$$

2) We have $H_{I/J}(2) = 1$ hence

$$1 = \lambda\left((I/J)^2/(I/J)^3\right) = \lambda(I^2/I^3 + J \cap I^2) = \lambda(I^2/I^3 + JI).$$

This implies $I^2 = I^3 + JI + (ab)$ for some $a, b \in I$, $a, b \notin J$. By Nakayama we get $I^2 = JI + (ab)$ and the conclusion follows by easy induction on n.

3) We have $aI \subseteq I^2 = IJ + (ab)$, hence

$$I \subseteq (b) + (J : a) \cap I \subseteq I.$$

4) and 5) By part 2) and for every $n \geq 2$ we have epimorphisms

$$I^n/JI^{n-1} \xrightarrow{a} I^{n+1}/JI^n \to 0$$

and

$$(I/J)^n/(I/J)^{n+1} \xrightarrow{a} (I/J)^{n+1}/(I/J)^{n+2} \to 0.$$

From this we see that $v_n \leq v_{n-1}$ and $H_{I/J}(n+1) \leq H_{I/J}(n)$. This proves 4) and 5).

6) We have seen that

$$1 = \lambda(I^2/I^3 + JI) = \lambda(IJ + (ab)/I^3 + IJ).$$

This implies $ab\mathfrak{m} \subseteq I^3 + IJ$ which proves the assertion for $n = 1$.

If $n \geq 2$, we have by induction

$$a^n b\ \mathfrak{m} \subseteq a(a^{n-1}b\ \mathfrak{m}) \subseteq a(I^{n+1} + JI^{n-1}) \subseteq I^{n+2} + JI^n.$$

7) Let $x \in I \cap (J : a)$, $x \notin I \cap (J : I)$. Then $ax \in J \cap I^2 = JI$ and there exists $y \in I$ such that $xy \notin J$. Since $xy \in I^2$, the element $\overline{xy} \in I^2/JI$ is non-zero and is killed by a because $axy \in JI^2$. This proves that $I^2/JI \xrightarrow{a} I^3/JI^2 \to 0$ is not injective, so that the conclusion follows. □

We remark that by 1) of the above lemma we have $e(I) - h - \lambda(A/I) \geq 1$, so that, by (1), $K \geq 2$. Further, if I is stretched, then by 5) we have

$$P_{I/J}(z) = \lambda(A/I) + hz + z^2 + \cdots + z^{e(I)-h+1-\lambda(A/I)}.$$

Hence K is the least integer such that $I^{K+1} \subseteq J$. Summing up, for a stretched ideal I, the integer $K = e(I) - h + 1 - \lambda(A/I)$ satisfies the conditions

$$K \geq 2, \quad I^K \not\subseteq J, \quad I^{K+1} \subseteq J, \quad v_1 = K - 1.$$

We also remark that since $I^{K+1} \subseteq J$, then $I^{K+1} \subseteq J \cap I^2 = JI$. One cannot improve this result since Sally in [12] gave an example of a stretched ideal I such that $I^{K+1} \not\subseteq JI^2$. We prove in the following lemma that the condition $I^{K+1} \subseteq JI^j$ for some $j \geq 2$ has strong implications on the associated graded ring G.

From now on, we will tacitly add to the equipment of a given stretched ideal I the couple of integers a and b as in 2) of the above lemma.

Lemma 2.5. *Let I be a stretched ideal and let j be an integer, $0 \leq j \leq K$. Then we have:*

i) $I^{j+1} \cap J = JI^j + (a^K b)$.

ii) $I^{K+1} \subseteq JI^j$ *if and only if* $I^{n+1} \cap J = JI^n$ *for every* $n \leq j$.

Proof. i) We have $I^{K+1} \cap J = I^{K+1}$, hence, if $j = K$, the conclusion follows by Lemma 2.4, 2). We use descending induction on j. It is sufficient to prove $I^{j+1} \cap J \subseteq JI^j + (a^K b)$, because the other inclusion follows by assumption.

Let $j < K$; by Lemma 2.4, 2), we have $I^{j+1} = JI^j + (a^j b)$ so that

$$I^{j+1} \cap J = JI^j + (a^j b) \cap J.$$

Since $j + 1 \leq K$, $I^{j+1} \not\subseteq J$, hence $a^j b \notin J$ so that

$$(a^j b) \cap J \subseteq (a^j b\mathfrak{m}) \cap J.$$

By Lemma 2.4, 6), and using the inductive assumption, we finally get

$$I^{j+1} \cap J \subseteq JI^j + (I^{j+2} + JI^j) \cap J = JI^j + (I^{j+2} \cap J) =$$
$$= JI^j + JI^{j+1} + (a^K b) = JI^j + (a^K b).$$

ii) By part i) and for every $n \leq j$, we have $I^{n+1} \cap J = JI^n + (a^K b)$. Now, if $I^{K+1} \subseteq JI^j$, we get $I^{n+1} \cap J \subseteq JI^n + JI^j = JI^n$ for every $n \leq j$. Conversely, if $I^{j+1} \cap J = JI^j$, we get

$$I^{K+1} \subseteq J \cap I^{j+1} \subseteq JI^j. \qquad \square$$

The next result determines which stretched ideals have Cohen-Macaulay associated graded rings. It extends one of the main results in [12].

Theorem 2.6. *Let I be a stretched ideal. Then G is Cohen-Macaulay if and only if*

$$I^{K+1} = JI^K.$$

Proof. By [15], G is Cohen-Macaulay if and only if $I^{n+1} \cap J = JI^n$ for every $n \geq 0$, hence if G is Cohen-Macaulay then $I^{K+1} \cap J = JI^K$ and the conclusion follows since $I^{K+1} \subseteq J$.

Conversely, by ii) of the above lemma, we certainly have $I^{n+1} \cap J = JI^n$ for every $n \leq K$. The same conclusion holds also if $n \geq K$ because, in that case, $I^{n+1} = JI^n$. $\square$

Let now add one more numerical invariant to a stretched ideal I, namely its Cohen-Macaulay type.

Let I be a stretched ideal; we set

$$\tau(I) := \lambda\left((J : I) \cap I/J\right)$$

and we say that $\tau(I)$ is the *Cohen-Macaulay type* of I. If the ideal I is clear from the context, we will simply write τ instead of $\tau(I)$.

If $I = \mathfrak{m}$, then it is well known that the integer $\lambda\left((J : \mathfrak{m}) \cap m/J\right) = \lambda\left((J : \mathfrak{m})/J\right)$ does not depend on the ideal J generated by a maximal superficial sequence for $\mathfrak{m}$ and is called the Cohen-Macaulay type of A. This explains the above terminology.

As in the case of the maximal ideal (see [10]), the associated graded ring of a stretched ideal I, whose Cohen-Macaulay type is not too big, has a small amount of good behaviour in that $I^{K+1} \subseteq JI^2$.

Theorem 2.7. *Let I be a stretched ideal. If $\tau < h + 1 - \lambda(A/I)$, then*

$$v_2 = K - 2 \quad \text{and} \quad I^3 \cap J = JI^2.$$

In particular $I^{K+1} \subseteq JI^2$.

Proof. We have $v_2 = \lambda(I^3/JI^2) = \lambda(I^3/J\cap I^3)+\lambda(J\cap I^3/JI^2) = \lambda(I^3+J/J)+\lambda(J\cap I^3/JI^2)$. Let us consider the chain

$$I^3 + J = J + (a^2b) \supset J + (a^3b) \supset \cdots \supset J + (a^{K-1}b) \supset J + (a^Kb) = J.$$

This chain has $K-2$ steps and all the inclusions are strict because if $a^ib \in J + (a^{i+1}b)$ for some $2 \le i \le K-1$, then $a^ib \in J$ so that $I^{i+1} = JI^i + (a^ib) \subseteq J$ with $i+1 \le K$, a contradiction.

Hence we have $\lambda(I^3 + J/J) \ge K-2$ so that

$$v_2 \ge K - 2 + \lambda(J \cap I^3/JI^2)$$

and the two assertions of the theorem will follow if we prove that $v_2 \le K-2$, or equivalently $v_2 < v_1$.

By Lemma 2.4, 3), we have $I = (b) + (J : a) \cap I$ and by Lemma 2.4, 7), we need to prove

$$I \cap (J : a) \ne I \cap (J : I).$$

Let us assume by contradiction that $I \cap (J : a) = I \cap (J : I)$. We have a chain

$$J \subseteq (J : I) \cap I \subseteq (J : I) \cap I + (b) = I.$$

Hence, since

$$\lambda(I/J) = e - \lambda(A/I) = h + K - 1,$$

if we prove

$$\lambda\left((J : I) \cap I + (b)/(J : I) \cap I\right) \le K - 2 + \lambda(A/I), \tag{8}$$

we get $h+K-1 \le \tau+K-2+\lambda(A/I)$, a contradiction to the assumption $\tau < h+1-\lambda(A/I)$.

In order to prove (8), we consider the following chain of length $K-1$ connecting $(J : I)\cap I$ to $(J : I) \cap I + (b)$.

$$\begin{aligned}(J : I) \cap I = (J : I) \cap I + (a^{K-1}b) \subseteq (J : I) \cap I + (a^{K-2}b) \subseteq \dots\\ \dots \subseteq (J : I) \cap I + (ab) \subseteq (J : I) \cap I + (b).\end{aligned}$$

By Lemma 2.5, 6), and for every $n \ge 1$, we have

$$\mathfrak{m}a^nb \subseteq I^{n+2} + J = (a^{n+1}b) + J$$

so that

$$\lambda\left((J : I) \cap I + (ab)/(J : I) \cap I) + (a^{K-1}b)\right) \le K - 2.$$

On the other hand

$$(J : I) \cap I + (ab) = (J : I) \cap I + I^2 = (J : I) \cap I + I((J : I) \cap I + (b)) = (J : I) \cap I + bI,$$

so that

$$((b) + (J : I) \cap I) \,/\, (bI + (J : I) \cap I) \simeq (b)/\,(bI + (J : I) \cap (b))\,.$$

This implies

$$\lambda\left((b)+(J:I)\cap I\right)/\left(bI+(J:I)\cap I\right)\leq \lambda((b)/bI)\leq \lambda(A/I).$$

This proves (8) and the theorem. $\square$

The above result extends Theorem 2.5. in [12] where the conclusion follows in the case $I=\mathfrak{m}$ and $\tau=1$.

The following example shows that, in the above theorem, the condition $\tau<h+1-\lambda(A/I)$ cannot be relaxed.

Example 2.8. Let $A=k[[t^7,t^8,t^{13},t^{19},t^{25}]]$ and let $I=\mathfrak{m}$. Then $d=1$, $h=4$, $K=3$, $\tau=4$ and $x:=t^7$ is a superficial element of $\mathfrak{m}$. We have $P_A(z)=(1+4z+z^3+z^5)/(1-z)$ and $P_{A/xA}(z)=(1+4z+z^2+z^3)/(1-z)$ so that $\mathfrak{m}$ is stretched and $v_2=2=K-1$.

We remark that if $I=\mathfrak{m}$ then $h\geq\lambda(A/I)=1$, so that $h-\lambda(A/I)+1$ is always a positive integer. But for an $\mathfrak{m}$-primary ideal I this integer can be negative. In this case one could expect that v_2 is as small as possible, namely $v_2=K-2$. The following examples show that we can have both $v_2>K-2$ and $v_2=K-2$.

Example 2.9. Let

$$A=k\left[\left[t^7,t^8,t^{12},t^{13},t^{18}\right]\right]$$

and let $I=(t^7,t^8,t^{13})$. It is easy to check that I is stretched with $J=(t^7)$, $K=3$, $h_I(z)=3+2z+z^3+z^5$. Hence $h=2<\lambda(A/I)=3$, $v_2=2=K-1$.

On the other hand, if we consider in A the $\mathfrak{m}$-primary ideal $I=(t^7,t^8,t^{18})$, then I is stretched with $J=(t^7)$, $K=3$, $h_I(z)=3+2z+z^2+z^6$, $h=2<\lambda(A/I)=3$ and $v_2=1=K-2$.

We end this section by proving that stretched ideals I such that $I^{K+1}\subseteq JI^2$ have $H_I(2)$ completely determined by h and $\lambda(A/I)$. We need the following lemma.

Lemma 2.10. *Let I be an $\mathfrak{m}$-primary ideal and J be an ideal generated by a maximal superficial sequence $(x_1,\dots,x_d)$ for I. Given a positive integer l, let us assume that $I^{n+1}\cap J=JI^n$ for every $n\leq l$. Then*

$$h_I(z)\equiv h_{I/J}(z)\mod(z^{l+1}).$$

Proof. We prove the assertion by induction on $d=\dim A$, the case $d=0$ being trivial. Let $d\geq 1$ and put $x:=x_1$, $\overline{I}=I/xA$ and $\overline{J}=J/xA$. We have $\dim A/xA=d-1$ and $\overline{I}^{n+1}\cap\overline{J}=\overline{J}\overline{I}^n$ for every $n\leq l$. By the inductive assumption the h-polynomials of $\overline{I}$ and $\overline{I}/\overline{J}\simeq I/J$ coincide in degree $n\leq l$. Hence it is enough to prove that the h-polynomials of I and $\overline{I}$ coincide modulo (z^{l+1}). Since $I^{n+1}\cap J=JI^n$ for every $n\leq l$, it follows by [15] that

$$I^{n+1}\cap xA=xI^n\quad\text{for every}\quad n\leq l.$$

By an extension of a well known result of Singh, for every n we have

$$H^1_{\overline{I}}(n)=\sum_{j=0}^{n}H_{\overline{I}}(j)=H_I(n)+\lambda(I^{n+1}:x/I^n).$$

Therefore for every $n \le l$, we get $H^1_{\overline{I}}(n) = H_I(n)$ which implies

$$\frac{h_{\overline{I}}(z)}{(1-z)^d} \equiv \frac{h_I(z)}{(1-z)^d} \mod(z^{l+1}).$$

Thus $h_I(z) \equiv h_{\overline{I}}(z) \mod(z^{l+1})$, as desired. □

Corollary 2.11. *Let I be a stretched ideal.*

a) *If $I^{K+1} \subseteq JI^l$ for some $l \le K$, then $h_I(z) \equiv \lambda(A/I) + hz + z^2 + \cdots + z^l \bmod (z^{l+1})$.*

b) *If $I^{K+1} \subseteq JI^2$, then $H_I(2) = \binom{d+1}{2}\lambda(A/I) + hd + 1$.*

Proof. For the first assertion, we remark that, by Lemma 2.5, we have $I^{n+1} \cap J = JI^n$ for every $n \le l$. Therefore, by Lemma 2.10, we get $h_I(z) \equiv h_{I/J}(z)$ modulo z^{l+1}. Since by Lemma 2.4, 5), we have $h_{I/J}(z) = \lambda(A/I) + hz + z^2 + \cdots + z^K$, the conclusion follows.

We prove b). By a) we have $h_I(z) \equiv \lambda(A/I) + hz + z^2 \mod(z^3)$. The assertion follows by an easy computation of the coefficient of z^2 in the power series $P_I(z) = \frac{h_I(z)}{(1-z)^d} = (\lambda(A/I) + hz + z^2 + \cdots) \left(\sum_{j\ge 0} \binom{d+j-1}{j} z^j\right)$. □

This corollary extends Corollary 2.6 in [12].

3. Stretched ideals I with $I^{K+1} \subseteq JI^{K-1}$

In this section we are dealing with stretched ideals I such that $I^{K+1} \subseteq JI^{K-1}$. We first prove that the depth of the associated graded ring of such ideals is at least $d-1$. This is a consequence of a deep result essentially contained in [9] (see Proposition 3.5 and Proposition 3.6) but concretely formulated in a series of subsequent papers (see [2], [3], [4], [7]).

Proposition 3.1. *Let I be a stretched ideal. We have*

a) $I^{K+1} \subseteq JI^{K-1} \iff \lambda(I^K/JI^{K-1}) = 1$.

b) *If* $I^{K+1} \subseteq JI^{K-1}$, *then* depth $(G) \ge d-1$ *and* $h_I(z) = \lambda(A/I)+hz+z^2+\cdots+z^{K-1}+z^s$ *for some $s \ge K$.*

Proof. For the first assertion, if $\lambda(I^K/JI^{K-1}) = 1$, then $\mathfrak{m}I^K \subseteq JI^{K-1}$ and in particular $I^{K+1} \subseteq JI^{K-1}$. Conversely, if $I^{K+1} \subseteq JI^{K-1}$, then by Lemma 2.4, 6), we get $a^{K-1}b\ \mathfrak{m} \subseteq I^{K+1} + JI^{K-1} = JI^{K-1}$. Since $I^K = JI^{K-1} + (a^{K-1}b)$ and $I^K \not\subseteq J$, the assertion follows.

We prove now b). Let $I^{K+1} \subseteq JI^{K-1}$; by Corollary 2.11 we have $h_I(z) = \lambda(A/I) + hz + z^2 + \cdots + z^{K-1} \mod(z^K)$ and by Lemma 2.5 we have $I^{n+1} \cap J = JI^n$ for every $n \le K-1$. Further by a) we have $\lambda(I^K/JI^{K-1}) = 1$. Then, by [7] Theorem 3.2, we get depth$(G) \ge d-1$ and $h_I(z) = \lambda(A/I) + hz + z^2 + \cdots + z^{K-1} + z^s$ for some integer $s \ge K$. □

In view of the above result and in order to complete the description of the Hilbert series of a stretched ideal I such that $I^{K+1} \subseteq JI^{K-1}$, we may reduce the problem to the one-dimensional case and need to look for a good upper bound of the reduction number s of the given stretched ideal I. It will be crucial to compute the numerical invariants related to the Ratliff-Rush filtration of I. We recall that we have put

$$a_j := \lambda(\widetilde{I^j}/I^j), \quad \rho_j := \lambda(\widetilde{I^{j+1}}/x\widetilde{I^j}), \quad v_j := \lambda(I^{j+1}/xI^j)$$

where x is a superficial element of I such that $I^2 \cap xA = xI$ and $H_{I/xA}(2) = 1$. We recall also that we have put $K := e(I) - h + 1 - \lambda(A/I)$.

Proposition 3.2. *Let $d = 1$ and let I be a stretched ideal such that $I^{K+1} \subseteq xI^{K-1}$. Then*

a) $a_j = s - K$ *for* $j = 2, \dots, K$.

b)
$$\rho_j = \begin{cases} s - 1 - a_1 & \text{if } j = 1 \\ K - j & \text{if } 2 \le j \le K - 1 \\ 0 & \text{if } j \ge K. \end{cases}$$

c) $\widetilde{I^{j+1}} = x\widetilde{I^j} + I^{j+1}$ *for* $j \ge 2$.

Proof. By (3) and Proposition 3.1, b), we have

$$v_n = e(I) - H_I(n) = e(I) - (\lambda(A/I) + h + n - 1) = K - n$$

if $n \le K - 1$ and

$$v_n = e(I) - (\lambda(A/I) + h + K - 2) = 1$$

if $K - 1 \le n \le s - 1$.

Hence, if we put $j = 2$ and $t = s - 2$ in Proposition 1.1, we get $a_2 \ge v_{s-1} - v_1 + s - 2 = 1 - (K - 1) + s - 2 = s - K$.

On the other hand, by Lemma 2.5, one has $I^{n+1} \cap xA = xI^n$ for $n \le K - 1$, so that the following sequence is exact for every $n \le K - 1$:

$$0 \to \widetilde{I^n}/I^n \xrightarrow{x} \widetilde{I^{n+1}}/I^{n+1} \to \widetilde{I^{n+1}}/x\widetilde{I^n} + I^{n+1} \to 0.$$

This implies $s - K \le a_2 \le \cdots \le a_K$. Hence, by using (4) and (7), we get

$$\sum_{j=0}^{s-1} v_j = \sum_{j=0}^{K-1} v_j + s - K \le \sum_{j=0}^{K-1} v_j + a_K = \sum_{j=0}^{K-1} \rho_j \le \sum_{j\ge 0}^{s-1} \rho_j = \sum_{j\ge 0}^{s-1} v_j.$$

It follows that $\rho_K = \cdots = \rho_{s-1} = 0$ and $a_K = s - K$, which gives $a_2 = \cdots = a_K = s - K$. By using the above exact sequences, this implies $\widetilde{I^{n+1}} = x\widetilde{I^n} + I^{n+1}$ for every $n \ge 1$. Hence a) and c) follow. Finally b) easily follows by (4) with a direct computation. □

Using the above result and with the assumption that $I = \widetilde{I}$, we can get an upper bound of the reduction number s of I which does not depend anymore on $\widetilde{I^2}$.

Proposition 3.3. *Let $d = 1$ and let I be a stretched ideal such that $I^{K+1} \subseteq xI^{K-1}$ and $I = \widetilde{I}$. Then*

$$s \le K - 2 + \lambda\left((xA : I^2) \cap I/xA\right).$$

Proof. By Lemma 2.4, for every $n \geq 1$ we have $I^{n+1} = xI^n + (a^n b)$ with $a, b \in I$, $a, b \notin xA$. By using the above proposition we get $\widetilde{I^3} = x\widetilde{I^2} + I^3$. This implies $a\widetilde{I^2} \subseteq \widetilde{I^3} \subseteq x\widetilde{I^2} + (a^2 b)$, hence, since $\widetilde{I^2} \subseteq \widetilde{I} = I$, we have

$$\widetilde{I^2} \subseteq (I^2 + (xA : a)) \cap I = I^2 + ((xA : a) \cap I) \subseteq I^2 + ((xA : I^2) \cap I).$$

This and Proposition 1.2 imply

$$\begin{aligned} s &\leq 1 + \lambda\left((xA : I) \cap I + \widetilde{I^2}/xA\right) \leq 1 + \lambda\left((xA : I^2) \cap I + \widetilde{I^2}/xA\right) \leq \\ &\leq 1 + \lambda\left((xA : I^2) \cap I + I^2/xA\right) = \\ &= 1 + \lambda\left((xA : I^2) \cap I + I^2/(xA : I^2) \cap I\right) + \lambda\left((xA : I^2) \cap I/xA\right). \end{aligned}$$

We will prove that $\lambda\left((xA : I^2) \cap I + I^2/(xA : I^2) \cap I\right) = K - 3$, which gives the conclusion.

Since $I^{K+1} \subseteq xA$, we have $a^{K-2}b \in (xA : I^2) \cap I$. Let us consider the chain

$$\begin{aligned} (xA : I^2) \cap I + I^2 = (xA : I^2) \cap I + (ab) \supset (xA : I^2) \cap I + (a^2 b) \supset \cdots \supset \\ \supset (xA : I^2) \cap I + (a^{K-2}b) = (xA : I^2) \cap I. \end{aligned}$$

This has $K - 3$ steps and all the inclusions are strict since $I^K \not\subset xA$. Moreover, by Lemma 2.4, 6), we have

$$\mathfrak{m}a^n b \subseteq I^{n+2} + xI^n \subseteq xA + (a^{n+1}b) \subseteq (xA : I^2) \cap I + (a^{n+1}b).$$

Hence $\lambda\left((xA : I^2) \cap I + I^2/(xA : I^2) \cap I\right) = K - 3$ and the conclusion follows. □

The following example shows that the bound given in Proposition 3.3 is sharp.

Example 3.4. Let

$$A = k\left[\left[t^7, t^8, t^{20}\right]\right] \simeq k\left[[X, Y, Z]\right]/(X^4 - YZ, Y^5 - Z^2)$$

and let $I = \mathfrak{m} = (x, y, z)$ be the maximal ideal of A. Then $(A, \mathfrak{m})$ is Gorenstein of dimension one, $h = 2$, $e(\mathfrak{m}) = 7 = h + 5$ and $\mathfrak{m} = \widetilde{\mathfrak{m}}$. It is clear that x is a superficial element and $\mathfrak{m}$ is stretched with $K = 5$. Moreover $\mathfrak{m}^2 = x\mathfrak{m} + (y^2)$ and $\mathfrak{m}^6 \subseteq x\mathfrak{m}^4$ since $y^6 = x^4 z \in x\mathfrak{m}^4$.

It is easy to check that $\lambda((xA : \mathfrak{m}^2)/xA) = 3$, $h_I(z) = 1 + 2z + z^2 + z^3 + z^4 + z^6$ and hence $s = 6 = K - 2 + \lambda((xA : \mathfrak{m}^2)/xA)$.

Notice that this is an example of a Gorenstein ring with a stretched ideal $\mathfrak{m}$ such that $\mathfrak{m}^{K+1} \subseteq J\mathfrak{m}^{K-1}$, but its associated graded ring G is not Cohen-Macaulay.

We remark that, using the Cohen-Macaulay type of the stretched ideal I, the bound of Proposition 3.3 can be rewritten as

$$s \leq K - 2 + \tau + \lambda((xA : I^2) \cap I/(xA : I)).$$

In the case $K = 3$ and by completely different methods, we can get free of the nasty term $\lambda((xA : I^2) \cap I/(xA : I))$ in the above inequality. We will show that $s \leq \tau + K - 1 = \tau + 2$. We need first a result which is a consequence of the following easy lemma proved in [10], Lemma 1.4.

Lemma 3.5. *Let $L = (x_1, \dots, x_v)$, J and I be ideals of the local ring $(A, \mathfrak{m})$. Let us assume that there exist elements $a, x \in I$ such that*

$$aL \subseteq xL + J.$$

Then there exists $\sigma \in xI^{v-1}$ such that

$$a^v - \sigma \in JI^{v-1} : L.$$

Corollary 3.6. *Let $L = (x_1, \dots, x_v)$ and I be ideals of the local ring $(A, \mathfrak{m})$. If a, x are elements in I such that*

$$aL \subseteq xL + xI^2,$$

then

$$I^2 \cap \left[(xI^2 + IL) : a\right] \subseteq xI^{v+2} : a^{v+1}.$$

Proof. By the above lemma we can find $\sigma \in xI^{v-1}$ such that $a^v - \sigma \in xI^{v+1} : L$. Let $t \in I^2 \cap [(xI^2 + IL) : a]$; then we may write $at = xr + s$ where $r \in I^2$ and $s \in IL \cap I^3$. Since $s \in IL$, we have $(a^v - \sigma)s \in xI^{v+2}$, hence $a^v s \in xI^{v+2}$ because $\sigma s \in (xI^{v-1})I^3 = xI^{v+2}$. Hence $a^{v+1}t = a^v(at) = a^v xr + a^v s \in xI^{v+2}$, as wanted. □

Theorem 3.7. *Let $d = 1$ and let I be a stretched ideal such that $K = 3$, $\widetilde{I} = I$ and $I^4 \subseteq xI^2$. Then $s \leq \tau + 2$.*

Proof. By Lemma 2.4, 2), there exist elements $a, b \in I$, $a, b \notin xA$, such that, for every $n \geq 1$, $I^{n+1} = xI^n + (a^n b)$. By Proposition 3.2, a), we have $a_2 = \lambda(\widetilde{I^2}/I^2) = s - 3$, so that we can write $\widetilde{I^2} = I^2 + (y_1, \dots, y_{s-3})$ where $y_i \notin I^2$ for every i. By Proposition 3.2, c), we have $\widetilde{I^3} = x\widetilde{I^2} + I^3 = x\widetilde{I^2} + (a^2 b)$, hence for every $i = 1, \dots, s-3$, we have $ay_i \in \widetilde{I^3} = x\widetilde{I^2} + (a^2 b)$. This implies that there exist elements $z_1, \dots, z_{s-3} \in I^2$ such that $a(y_i - z_i) \in x\widetilde{I^2}$. If we let $L := (y_1 - z_1, \dots, y_{s-3} - z_{s-3})$, we get $\widetilde{I^2} = I^2 + L$ with $aL \subseteq x\widetilde{I^2} = x(L + I^2)$ and we may apply Corollary 3.6 to get

$$I^2 \cap \left[(xI^2 + IL) : a\right] \subseteq xI^{s-1} : a^{s-2}.$$

We claim that $L \subseteq (xA : I) \cap I$.

Clearly $L \subseteq \widetilde{I^2} \subseteq \widetilde{I} = I$. We prove that $L \subseteq (xA : I)$. By Proposition 3.2, b), we have $\rho_2 = \lambda(\widetilde{I^3}/x\widetilde{I^2}) = 1$ hence, if $LI \not\subseteq xA$, then

$$\widetilde{I^3} = x\widetilde{I^2} + LI = xI^2 + LI.$$

This implies $a^2 b \in I^3 \subseteq \widetilde{I^3} = xI^2 + LI$ so that $ab \in I^2 \cap [(xI^2 + IL) : a] \subseteq xI^{s-1} : a^{s-2}$. The claim follows since this implies $I^s = xI^{s-1}$, a contradiction to the minimality of s.

The claim implies that $(xA : I) \cap I + \widetilde{I^2} = (xA : I) \cap I + I^2$, hence, by Proposition 1.2, we get

$$s \leq 1 + \lambda\left((xA : I) \cap I + \widetilde{I^2}/xA\right) = 1 + \tau + \lambda((xA : I) \cap I + I^2/(xA : I) \cap I).$$

But we have $I^2 = xI + (ab)$ and by Lemma 2.4, 6), $\mathfrak{m}ab \subseteq I^3 + xI \subseteq (xA : I)$. Hence $\lambda((xA : I) \cap I + I^2/(xA : I) \cap I) \leq 1$, as required. $\square$

Example 3.8. Let us consider the local ring $A = k\,[[t^6, t^7, t^{16}, t^{17}]]$ and its maximal ideal $\mathfrak{m}$. This is a one-dimensional Cohen-Macaulay local ring such that $h = 3$ and $x = t^6$ is a superficial element for $\mathfrak{m}$. It is easy to see that $\mathfrak{m}$ is a stretched ideal of multiplicity $h + 3$, that is $K = 3$. Further $\mathfrak{m}^2 = x\mathfrak{m} + (y^2)$ and $y^4 \in x\mathfrak{m}^2$ so that $\mathfrak{m}^4 \subseteq x\mathfrak{m}^2$. We can easily check that $s = \tau + 2$ thus proving that the bound given above is sharp.

The following example shows that in the above theorem we cannot delete the assumption $I = \widetilde{I}$.

Example 3.9. We consider

$$A = k\left[\left[t^5, t^6, t^9\right]\right] \quad \text{and} \quad I = (t^5, t^6).$$

Then I is a stretched $\mathfrak{m}$-primary ideal of the one-dimensional ring A. In fact t^5 is a superficial element with $I^2 \cap (t^5) = t^5 I$ and $H_{I/(t^5)}(2) = 1$. Moreover $e(I) = h + \lambda(A/I) + 2$ so that $K = 3$, but $I^4 : (t^5)^3 \neq I$ so that $\widetilde{I} \neq I$.

In this case $I^4 \subseteq t^5 I^2, \tau = 1$, but $h_I(z) = 2 + z + z^2 + z^4$. This example shows also that Sally's theorem does not extend to $\mathfrak{m}$-primary ideals.

The above theorem gives non-trivial examples of $\mathfrak{m}$-primary ideals such that G is Cohen-Macaulay.

Example 3.10. Let $A = k\,[[X, Y, Z]]\,/(XZ - YZ, XZ + Y^3 - Z^2) = k\,[[x, y, z]]$ and let $I = (x, y)$. Then I is stretched with $K = 3$, $J = xA$ and $\tau = 1$. We have $I^4 \subseteq xI^2$, hence $h_I(z) = 2 + z + z^2 + z^3$ so that G is Cohen-Macaulay.

We move now to the higher-dimensional case. Since we need to use the above results where the assumption $I = \widetilde{I}$ is needed and we do not know whether this condition is preserved modulo a superficial element, we are going to use the stronger condition that I is integrally closed.

Theorem 3.11. *Let I be an integrally closed stretched ideal such that $I^{K+1} \subseteq JI^{K-1}$. Then* $\operatorname{depth}(G) \geq d - 1$ *and $h_I(z) = \lambda(A/I) + hz + z^2 + \cdots + z^{K-1} + z^s$ for some integer s with $K \leq s \leq K - 2 + \lambda\left((J : I^2) \cap I/J\right)$.*

Proof. By Proposition 3.1 we have $\operatorname{depth}(G) \geq d - 1$ and $h_I(z) = \lambda(A/I) + hz + z^2 + \cdots + z^{K-1} + z^s$ for some $s \geq K$. We prove by induction on d that $s \leq K - 2 + \lambda\left((J : I^2) \cap I/J\right)$. If $d = 1$ we can apply Proposition 3.3 to get the conclusion after remarking that, by [5], integrally closed implies $I = \widetilde{I}$. Let $d \geq 2$; again by [5] we may assume that there exists $x \in J$ such that x is superficial for I and $\overline{I} := I/xA$ is integrally closed in the Cohen-Macaulay local ring $\overline{A} := A/xA$ which has dimension $d - 1$. Further it is clear that $\overline{I}$ is stretched with respect to $\overline{J} := J/xA$, $h(I) = h(\overline{I})$ and $e(I) = e(\overline{I})$ so that $e(\overline{I}) - h(\overline{I}) - \lambda(\overline{A}/\overline{I}) + 1 =$

$e(I)-h(I)-\lambda(A/I)+1=K$. We also have $(\overline{I})^{K+1}\subseteq \overline{J}(\overline{I})^{K-1}$ so that, by induction, the degree of the h-polynomial $h_{\overline{I}}(z)$ of $\overline{I}$ is bounded above by $K-2+\lambda\left(\left(\overline{J}:\overline{I}^2\right)\cap \overline{I}/\overline{J}\right)$. We have depth$(G)\geq d-1>0$ so that $h_{\overline{I}}(z)=h_I(z)$. The conclusion follows since

$$\left(\overline{J}:\overline{I}^2\right)\cap \overline{I}/\overline{J}=Hom\left(\overline{A}/\overline{I}^2,\overline{I}/\overline{J}\right)=Hom(A/I^2+xA,I/J)=$$

$$=0:_{I/J}(I^2+xA)=\left(J:(I^2+xA)\right)\cap I/J=(J:I^2)\cap I/J. \qquad \square$$

By using exactly the same argument as above we can prove the following result in the special case $K=3$.

Theorem 3.12. *Let I be an integrally closed stretched ideal such that $K=3$ and $I^4\subseteq JI^2$. Then* depth$(G)\geq d-1$ *and $h_I(z)=\lambda(A/I)+hz+z^2+z^s$ for some integer s such that $3\leq s\leq \tau+2$.*

Proof. The proof is exactly the same as in the preceding theorem. One needs only to use Theorem 3.7 instead of Proposition 3.3 for the initial case $d=1$, and the following equality

$$\tau(I)=\lambda\left(Hom(A/J,I/J)\right)=\lambda\left(Hom(\overline{A}/\overline{J},\overline{I}/\overline{J})\right)=\tau(\overline{I})$$

instead of the corresponding one used for the conclusion. $\square$

We remark that the above result does not extend to the case $K\geq 4$. Namely we have seen in Example 3.4 that one can have $s=\tau+K$.

However we ask the following question.

Problem 3.13. *Let I be an integrally closed stretched ideal such that $I^{K+1}\subseteq xI^{K-1}$. Is it true that*

$$s\leq \tau+K\,?$$

We end the paper with the following result which can be considered as the true extension to the $\mathfrak{m}$-primary case of Sally's theorem. It shows that, under suitable assumptions, an integrally closed $\mathfrak{m}$-primary ideal I with $K=3$ is necessarily stretched and verifies $I^4\subseteq JI^2$, so that, by using the above theorem, we can prove that G is Cohen-Macaulay.

Corollary 3.14. *Let I be an integrally closed $\mathfrak{m}$-primary ideal such that $K=3$ and $h>\lambda(A/I)$. If for some ideal J generated by a maximal superficial sequence in I we have $\lambda\left((J:I)\cap I/J\right)=1$, then G is Cohen-Macaulay and*

$$P_I(z)=\frac{\lambda(A/I)+hz+z^2+z^3}{(1-z)^d}.$$

Proof. We know by [5] that for every ideal J generated by a maximal superficial sequence in I we have $I^2\cap J=IJ$. Hence, if we let $\overline{I}:=I/J$, by Lemma 2.1 we have $H_{\overline{I}}(1)=h$. We first prove that I is stretched. It is clear that the h-polynomial of $\overline{I}$ is either $\lambda(A/I)+hz+z^2+z^3$ or $\lambda(A/I)+hz+2z^2$. In the second case, we have $\lambda(\overline{I}^2/\overline{I}^3)=2$ and $I^3\subseteq J$ so that

$$\overline{I}^2/\overline{I}^3 = I^2 + J/I^3 + J = I^2 + J/J \subseteq (J : I) \cap I/J,$$

a contradiction to the assumption $\lambda\left((J : I) \cap I/J\right) = 1$. Hence $H_{\overline{I}}(2) = 1$ and I is stretched with $\tau = 1$.

Since $h > \lambda(A/I)$ we have $\tau = 1 < h + 1 - \lambda(A/I)$ so that by Theorem 2.7 we get $I^4 \subseteq JI^2$. We can apply the above theorem to get $h_I(z) = \lambda(A/I) + hz + z^2 + z^3$; since the h-polynomial of I coincides with that of $\overline{I}$, the associated graded ring G is Cohen-Macaulay, as desired. □

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