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Singularities and Duality in the Flat Geometry of Submanifolds of Euclidean Spaces

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Abstract. We study some properties of submanifolds in Euclidean space concerning their generic contacts with straight lines and hyperplanes and expose some duality relation associated to these contacts.

Introduction

J. W. Bruce and V. M. Zakalyukin prove in [5] that the hypersurface dual to a caustic whose generating family is affine in its parameters is isomorphic to certain strata of the bifurcation diagram of a family of projections obtained through this generating family. This can be translated, in the case of the family of height functions on a generic surface M in $\mathbb{R}^3$, into the fact, explored in [4], that the subset of normals over the parabolic set of M (i.e., the caustic of the family of height functions) and that of principal asymptotic directions (bifurcation set of the family of orthogonal projections of M onto planes) are dual as subsets of the 2-sphere. Similar results for surfaces in $\mathbb{R}^4$ were obtained by J. W. Bruce and A. C. Nogueira in [3].

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Our goal in the present paper is to generalize this geometrical interpretation of the duality to the submanifolds of any dimension in euclidean space. For this purpose we characterize the singularities of the height functions and orthogonal projections on generic submanifolds from the geometrical viewpoint. We then conclude as a consequence of the duality result obtained by Bruce and Zakalyukin that, as it was to be expected as the natural generalization of the situation for curves and surfaces in 3-space, *the dual of the set of binormals of a submanifold M is the set of its asymptotic directions.* (We actually include a direct geometrical proof of this fact for the smooth part of the bifurcation set of the height functions family).

We also obtain a duality relation, which is not a consequence of the results in [5], concerning the lowest codimensional multilocal phenomena. This also generalizes the results of [4] relative to bitangencies of curves and surfaces in $\mathbb{R}^3$.

The method we follow is based on the consideration of canal hypersurfaces. So, we prove the results for hypersurfaces first and then extend them for higher codimensional submanifolds by comparing their contacts with hyperplanes and lines with those of their canal hypersurfaces.

The concepts of binormal and asymptotic directions, which are central in this setting, were introduced in [10] for submanifolds of codimension 2. We remind here their definitions and basic facts, extending them to the general case of submanifolds of any codimension.

We shall use some basic concepts of singularity theory throughout the paper, for a general background we recommend [1] and [7].

1. Some results for hypersurfaces

The family of height functions of an embedding, $f : M \to \mathbb{R}^{n+1}$, of a hypersurface M in Euclidean space is defined as

$$\begin{array}{rcccl} \lambda(f) & : & M \times S^n & \longrightarrow & \mathbb{R} \\ & & (p, v) & \longmapsto & \langle f(p), v \rangle = f_v(p). \end{array}$$

It is not difficult to see that f_v has a singularity at $p \in M$ if and only if v is the normal vector to $f(M)$ at $f(p)$. Moreover, if $\Gamma : M \to S^n$ denotes the normal Gauss map over this hypersurface, we have that a point $p \in M$ is a degenerate (non-Morse) singularity for some height function f_v if and only if $v = \Gamma(p)$ and p is a singular point of Γ. In fact, the point p is a singularity of type A_k (with the notation of Arnol'd [1]) of $f_{\Gamma(p)}$ if and only if the Boardman symbol of order k of the germ of the map Γ at p is $\sum^{\overbrace{1, \ldots, 1}^{k-1}, 0}$. (See [13]).

So the bifurcation set (or caustic) of the family $\lambda(f)$,

$$\sum(\lambda(f)) = \{v \in S^n : \exists p \in M \text{ such that } \frac{\partial \lambda(f)}{\partial p}(p, v) = \frac{\partial^2 \lambda(f)}{\partial p^2}(p, v) = 0\},$$

is precisely given by the set of unit normal vectors at the parabolic points (i.e., points at which the Gaussian curvature vanishes) of M.

Given a unit vector $\alpha \in S^n$, we can also take the orthogonal projection of M into the hyperplane orthogonal to the direction α in $\mathbb{R}^{n+1}$,

$$\begin{aligned} h_\alpha : M &\longrightarrow \mathbb{R}^n \\ p &\longmapsto p - \langle \alpha, p \rangle \alpha. \end{aligned}$$

It is not difficult to see that the map h_α has a singularity at a point $p \in M$ if and only if α is a tangent direction of M at p. We recall that a tangent direction to M at p is said to be *asymptotic* if the normal curvature of M at p vanishes in this direction. A *principal asymptotic direction* is a principal direction of curvature corresponding to a vanishing principal curvature at a parabolic point. Denote the family of projections of M into hyperplanes of $\mathbb{R}^{n+1}$ associated to the embedding f by,

$$\begin{array}{rccc} P(f) & : & M \times S^n & \longrightarrow & \mathbb{R}^n \\ & & (p, \alpha) & \longmapsto & h_\alpha(p). \end{array}$$

By a *generic* $f : M \to \mathbb{R}^{n+1}$ we shall understand an embedding f such that both families $\lambda(f)$ and $P(f)$ are locally versal at all the points (see [1] for the definition of versality). For $n \leq 6$ this is equivalent to requiring a finite number of transversality conditions on appropriate multijets of f (see [8]) and thus we have that the generic embeddings form an open and dense subset. Then we have that if M is a generically embedded submanifold of dimension $n \leq 6$, $\lambda(f)$ is a family of functions whose singularities may have at most codimension n and hence they can only be simple singularities of types A_k, D_k, E_k (see Arnold's list [1]).

For a generic embedding $f : M \to \mathbb{R}^{n+1}$, the subset $S_1(\Gamma)$ of parabolic points of corank 1 of the Gauss mapping Γ is a smooth $(n-1)$-submanifold of M. Now, given a parabolic point $p \in S_1(\Gamma)$ there is a unique asymptotic direction α of M at p. This direction can also be characterized by the fact, shown in [10], that it generates the kernel of the Hessian quadratic form of the height function $f_{\Gamma(p)}$. So we have a one-to-one correspondence between normals at parabolic points and asymptotic directions over the subset $S_1(\Gamma)$.

Denote

$\mathcal{B}_1(f) = \{v \in S^n : v \text{ is a normal vector at a point } p \text{ of type } \sum^{1,0} \text{ of } \Gamma(f)\}$,

$\mathcal{A}_1(f) = \{\alpha \in S^n : \alpha \text{ is a principal asymptotic direction at a point } p \text{ of type } \sum^{1,0} \text{ of } \Gamma(f)\}$.

For a generic M both subsets define smooth (n-1)-submanifolds of S^n. In fact, we know that the subset $S_{1,0}(\Gamma(f)) = \{p \in S_1(\Gamma) : p \text{ is of type} \sum^{1,0}\}$ is an open subset of $S_1(\Gamma)$ and the restriction of Γ to $S_{1,0}(\Gamma(f))$ is a local diffeomorphism onto $\Gamma(S_{1,0}(\Gamma(f)))$. Thus $\mathcal{B}_1(f)$ is an $(n-1)$ smooth submanifold of S^n. On the other hand, we can define $\Psi : S_{1,0}(\Gamma(f)) \longmapsto S^n$, given by $\Psi(p) = \alpha$, the unique asymptotic direction of p, which is a local diffeomorphism and thus $\mathcal{A}_1(f)$ is also a $(n-1)$-submanifold of S^n.

Suppose that Y is a smooth $(n-1)$-submanifold of S^n. Given any point $y \in Y$, there is a unique maximal $(n-1)$-sphere, $S_y \in S^n$ which is tangent to Y at y. As y varies in Y the poles of S_y describe another submanifold Y^* of S^n that is called *dual* of Y. Clearly, Y^* can be regarded as lying in $\mathbb{R}P^{n-1}$. In the case that Y has singular points then Y^* is defined as the closure of the dual of the smooth part of Y. In the case that Y and Y^* are smooth it can be seen that $(Y^*)^* = Y$.

Theorem 1. *The sets $\mathcal{B}_1(f)$ and $\mathcal{A}_1(f)$ are hypersurfaces and $\mathcal{B}_1(f)^* = \mathcal{A}_1(f)$ and $\mathcal{A}_1(f)^* = \mathcal{B}_1(f)$.*

Proof. Take $v \in \mathcal{B}_1(f)$, so there is $p \in M$ such that p is a fold point of f_v, or in other words p is a point of type $\Sigma^{1,0}$ of Γ. We can take now a neighbourhood V of v and an orthonormal frame $\{e_1, \dots, e_n\}$ in a neighborhood U of p in M such that e_1 is the asymptotic direction at p. Then the fact that p is of type $\sum^{1,0}$ implies that $e_1(p)$ is transversal to $S_{1,0}(\Gamma) = \{x \in M : x \text{ is of type } \Sigma^{1,0} \text{ for } \Gamma\}$. And since $D\Gamma(p)$ leaves its eigenspaces invariant, we have that $T_v\mathcal{B}_1(f)$ must be generated by the vectors $e_2(p), \dots, e_n(p)$. In this way, the pole of the hypersphere obtained by sectioning S^n with the hyperplane $<v> \oplus T_v\mathcal{B}_1(f)$ (tangent to $\mathcal{B}_1(f)$ at v) has the direction of $e_1(p)$ and we get that the dual of $\mathcal{B}_1(f)$ is contained in $\mathcal{A}_1(f)$.

We now see that $\mathcal{A}_1(f)^* = \mathcal{B}_1(f)$. Consider $\alpha \in \mathcal{A}_1(f)$, so there is some p in M such that $\Gamma(f)$ has a singularity of type $\Sigma^{1,0}$ at p, or in other words the height function in the direction $\Gamma(f)(p)$ has a non-Morse singularity at p. We can also take a neighbourhood W of α in $\mathcal{A}_1(f)$ and an orthogonal coordinate neighbourhood $(U, \{e_1(p), \dots, e_n(p)\})$ at p in such way that $e_1(p) = \alpha$. Moreover let $\Psi(U \cap S_{1,0}(\Gamma(f))) \subset W$, where Ψ is the diffeomorphism that assigns to each point q in $S_{1,0}(\Gamma(f))$ the unique asymptotic direction of M at q.

Now, given $u \in T_\alpha\mathcal{A}_1(f)$, we have curves $\beta : (-\varepsilon, \varepsilon) \longrightarrow W$ such that $\beta(0) = \alpha$, $\beta'(0) = u$, and $\delta : (-\varepsilon, \varepsilon) \longrightarrow U \cap S_{1,0}(\Gamma(f))$ is a curve such that $\delta(0) = p$ and $\delta'(0) = w$, where w is the unique vector such that $D\Psi(p)(w) = u$. Since $\Psi(\delta(t)) \in \mathcal{A}_1(f)\forall t$, this implies that $v(t) = \Gamma(\delta(t))$ provides a height function with a non-Morse singularity at $\delta(t)$, $\forall t \in (-\varepsilon, \varepsilon)$. We then have that

(i) $\langle \Psi(\delta(t)), v(t) \rangle = 0$ and

(ii) $\langle \Psi(\delta(t)), v'(t) \rangle = 0$,

for $v'(t) = D\Gamma(\delta(t)) \cdot (\delta'(t))$ and $\delta(t) \in S_{1,0}(\Gamma)$.

From (i) we get $<D\Psi(\delta(t))(\delta'(t)), v(t)> = 0$, which in particular for $t = 0$ tells us that $<D\Psi(p)(\delta'(0)), v(0)> = <u, v> = 0$. And hence $v \perp T_\alpha\mathcal{A}_1(f)$, so $\alpha^* = v$. □

Now, a consequence of this theorem and the Theorem 3 in [5] is that $\alpha \in \mathcal{A}_1(f)$ if and only if the projection h_α of M into the hyperplane orthogonal to the principal asymptotic direction α has at $p \in M$ a Σ^1 non-transverse singularity.

A normal form for h_α in a neighbourhood of p was given in Proposition 5 in [5]:

$$(x_1, \dots, x_n) \longmapsto (x_1^3 + \sum_{i=2}^{n} \pm x_1 x_i^2,\ x_2, \dots, x_n)$$

Moreover, Theorem 1 above can be seen as a special case of the following one, which is a consequence of the more sophisticated methods exposed in [5].

Theorem 2. ([5], Theorem 3) *Let $\mathcal{B}(f)$ and $\mathcal{A}(f)$ denote respectively the closures of $\mathcal{B}_1(f)$ and $\mathcal{A}_1(f)$. These subsets are dual in S^n.*

2. Canal hypersurfaces, binormals and asymptotic directions

Suppose now that $f : M \to \mathbb{R}^{n+1}$ is a codimension 2 embedding of a submanifold M. We can also define in this case the families $\lambda(f)$ and $P(f)$ as above, that is, $\lambda(f)(p,v) = f_v(p)$ and $P(f)(p,\alpha) = h_\alpha(p)$.

We observe that the singular set of the family $\lambda(f)$

$$\sum(\lambda(f)) = \{(p,v) \in M \times S^n : \frac{\partial\lambda(f)}{\partial p}(p,v) = 0\}$$

can be identified with the circle bundle over M whose fibre at a point p is the unit circle in the normal plane of $f(M)$ at $f(p)$. This is also known as the *canal hypersurface* of M, which can actually be considered as a hypersurface in $\mathbb{R}^{n+1}$ by means of an embedding

$$\begin{array}{rrcl} \tilde{f} : & CM & \longrightarrow & \mathbb{R}^{n+1} \\ & (p,v) & \longmapsto & f(m) + \epsilon v. \end{array}$$

for ϵ small enough.

We have the following maps on CM

a) the bundle map

$$\begin{array}{rrcl} \xi(f) : & CM & \longrightarrow & M \\ & (p,v) & \longmapsto & p \end{array}$$

b) the Gauss map

$$\begin{array}{rrcl} \Gamma(f) : & CM & \longrightarrow & S^n \\ & (p,v) & \longmapsto & v \end{array}$$

c) the Gaussian curvature function

$$\begin{array}{rrcl} \mathcal{K}(f) : & CM & \longrightarrow & \mathbb{R} \\ & (p,v) & \longmapsto & detD\Gamma(f)(p,v). \end{array}$$

It is not difficult to check that p is a degenerate singularity of f_v if and only if (p,v) is a singular point of $\Gamma(f)$ if and only if $\mathcal{K}(f)(p,v) = 0$. We choose now a coordinate system $U \times W$ for CM at (p,v) in such a way that

a) U is an orthogonal coordinate system for M at p (through which we can identify p with the origin of $\mathbb{R}^{n-1}$) such that

i) $f(x) = (x, f_1(x), f_2(x))$

ii) $f_v(x) = f_2(x)$, $\forall x \in U$

iii) $\frac{\partial f_i}{\partial x_j}(0) = 0$ for $i = 1,2$ and $j = 1,\dots,n$, and

b) W is a coordinate system for S^1 at $v = (0,1)$ obtained by identifying some open neighbourhood of v in S^1 with its image through stereographic projection $s : S^1 - (0,1) \longrightarrow \mathbb{R}$.

In these coordinates we have that the Hessian matrices of f_v and $\tilde{f}_v$ at p and (p,v) respectively, satisfy

$$\mathcal{H}(f_v)(p) \oplus I_1 = \mathcal{H}(\tilde{f}_v)(p,v) = D\Gamma(f)(p,v)$$

where I_1 denotes the identity over $\mathbb{R}$ and $D\Gamma(f)(p,v)$ represents the Jacobian matrix of the Gauss map $\Gamma(f)$ at the point (p,v).

According to the work of Montaldi [12], the contact between two submanifolds M_1 and M_2 that are tangent at a given point p can be determined through the $\mathcal{K}$-singularity class of a contact map, defined as the composition of an embedding whose image is M_1 and a submersion that cuts out M_2. In the particular case of a submanifold M and a tangent hyperplane H, this contact map is precisely the height function defined by the orthogonal direction to H on M. For a generic M, most height functions present Morse singularities and the subset of unit vectors leading to height functions with degenerate singularities over M has codimension 1 in the unit hypersphere of the ambient space. It was shown in [10] that on a generic submanifold M embedded with codimension 2 in the Euclidean space it is possible to find at most two normal vectors at each point, for which the corresponding height functions have a degenerate singularity at the given point p. These were called *binormals* and the tangent hyperplane of M at p orthogonal to them, *osculating hyperplane*. The higher contact of these osculating hyperplanes with M must take place along the lines generated by the kernel directions of the Hessian quadratic form of the height function in the corresponding binormal direction (see [10]). These directions were actually characterized in [10] as the *asymptotic* directions of M at p. They satisfy the following,

$$\begin{aligned}\theta \in Ker\, Hess(f_v)(p) &\leftrightarrow \theta \in Ker\, D\Gamma(f)(p,v)\\ &\leftrightarrow \theta \text{ is a principal asymptotic direction for } CM \text{ at } (p,v).\end{aligned}$$

It is not difficult to check now that an asymptotic direction for M at p is the image through $D\xi(f)(p)$ of a principal asymptotic direction for CM at (p,v).

In the case of a higher codimensional submanifold M, the subset of normal directions leading to height functions with degenerate singularities will not be finite in general at a given point of M. In fact, if M is a generic k-codimensional submanifold of $\mathbb{R}^{n+1}$, the unit normal bundle $\Sigma_p M$ of M at p is a $(k-1)$-sphere. Then the unit vectors inducing height functions with degenerate singularities form a singular variety of codimension 1 in $\Sigma_m M$. We call these *degenerate* directions on M at p and reserve the name *binormal* for those $v \in \Sigma_p M$ for which the germ of f_v at p has $\mathcal{A}$-codimension greater than or equal than $k-1$. For this implies that we will just have a finite number of them at a generic point of M. The corresponding *osculating hyperplane* will be the one having the highest generically possible order of contact with M at the considered point. A detailed study of their existence and distribution over a generically embedded surface in 5-space is made in [11]. Clearly, the concepts of degenerate and binormal directions coincide over the submanifolds of codimension 2. In the present paper we shall be concerned with all the degenerate directions at each point of the submanifold.

By exactly the same arguments as in the case of codimension 2, we can say that any degenerate direction v on M has at least one associated *asymptotic* direction $\theta \in Ker(Hess(f_v)(p))$. We notice that each degenerate direction v over a point p of $S_1(\Gamma(f))$ defines a unique asymptotic direction in T_pM. But a given tangent direction may be the generator of the kernel of more than one degenerate normal direction. In particular, if $dim(M) < codim(M) - 1$ the dimension of the set of degenerate directions at each point is higher than that of T_pM and we shall have a whole cone of degenerate directions in N_pM corresponding to the same asymptotic direction in T_pM.

3. Singularities of height functions and orthogonal projections in higher codimensions

Given a generic codimension k embedding $f : M \to \mathbb{R}^{n+1}$, we consider the following subsets of the canal hypersurface CM,

$$S_{i_1,\dots,i_k}(\Gamma(f)) = \{(p,v) \in CM : \text{ the germ of } \Gamma(f) \text{ at } (p,v) \text{ is of type } \Sigma^{i_1,\dots,i_k}\}$$

We observe that $(p,v) \in S_{i_1,\dots,i_k}(\Gamma(f))$ provided the height function f_v has a singularity of corank i_1 at the point p. Under appropriate transversality conditions on the k-jets of f we get that $\Gamma(f)$ is a k-generic map and thus the $S_{i_1,\dots,i_k}(\Gamma(f))$ are submanifolds of CM. In particular, for a generic embedding f, the subsets $S_{1,..(k)..,1,0}(\Gamma(f))$ are submanifolds of codimension k in CM. A point $(p,v) \in S_{1,..(k)..,1,0}(\Gamma(f))$ is a singularity of type A_{k+1} of f_v. Moreover, the points of $S_{1,..(k+1)..,1}(\Gamma(f))$ are characterized by the fact that the kernel of $Hess(f_v)(p)$ is tangent to the $(n-k)$-submanifold $S_{1,..(k)..,1,0}(\Gamma(f))$. We want to give next a geometric characterization of these singularities in terms of the normal sections of the submanifold M. Suppose that M has codimension k in $\mathbb{R}^{n+1}$. Given any point $p \in M$ and any unit vector ω tangent to M at p, we denote by γ_ω the curve obtained by intersecting M with the $(k+1)$-Euclidean space spanned by the normal k-plane of M at p and the tangent direction ω. The curve γ_ω is called the *normal section* of M in the direction ω at p.

If we take two linearly independent tangent vectors ω_1 and ω_2 at p, then the $(k+2)$-Euclidean space spanned by N_pM together with ω_1 and ω_2 intersects M in a surface which shall be called the *normal section* of M *along the tangent plane* $\langle\omega_1,\omega_2\rangle$.

Given a curve $\gamma : \mathbb{R} \to \mathbb{R}^{k+1}$ denote its Frenet frame by $\{t, n_1, \dots, n_k\}$ and its Euclidean curvatures by $\{\kappa_i\}_{i=1}^k$. We have the following characterization for the singularities of the height functions f_v^γ on γ.

Lemma 1. *The height function f_v^γ in the direction v has a singularity of type A_j at s_0 if and only if $v \in \langle n_j, \dots, n_k\rangle(s_0)$, $1 \le j \le k$. And f_v^γ has a singularity of type A_{k+1} at s_0 if and only if $v = n_k(s_0)$ and $\kappa_k(s_0) = 0$ (i.e., s_0 is a flattening of γ).*

Proof. It follows from considering the successive derivatives of the function $f_v^\gamma(s)$ and applying the Frenet formulae in exactly the same way that is done in [2, p. 37] for curves in 3-space.□

Let θ be the asymptotic direction for M at a point p and γ_θ the corresponding normal section of M. Then θ spans the kernel of the quadratic form $Hess(f_v)$ at p and we have that $f_v^{\gamma_\theta} = f_v|_{\gamma_\theta}$, so $Ker(Hess(f_v^{\gamma_\theta}) = Ker(Hess(f_v))$. In this case we have that p is an A_j singularity of f_v if and only if p is an A_j singularity of $f_v^{\gamma_\theta}$. Hence we have the following geometric characterization for the singularities of corank 1 of f_v.

Theorem 3. *Given an asymptotic direction θ at a point p of a k-codimensional submanifold M, suppose that v is a binormal direction associated to θ and that p is a singularity of corank 1 of f_v, then we have*

i) *p is an A_j singularity of f_v if and only if $v \in \langle n_j, \dots, n_k\rangle$, where $\{n_i\}_{i=1}^k$ represent the normal vectors in the Frenet frame of the normal section γ_θ of M and $v \cdot n_j \neq 0$.*

ii) *p is an A_k singularity of f_v if and only if $v = n_k$ and p is not a flattening of γ_θ.*

iii) *p is an A_j singularity of f_v, for some $j > k$ if and only if $v = n_k$ and p is a flattening of γ_θ. Moreover, p is of type A_j, $(j > k)$ if and only if θ is tangent to $S_{1,..(j)..,1}(\Gamma(f))$ but transversal to $S_{1,..(j+1)..,1}(\Gamma(f))$.*

Suppose now that M is a submanifold of codimension k in a Euclidean (n+1)-space as above and let p be a singularity of corank 2 of some height function f_v on M. For instance:

a) When $n+1 = 4$ and $k = 2$ this is geometrically characterized by the fact that the curvature ellipse of M at p is a radial segment (see [9]).

b) When $n+1 = 5$ and $k = 3$ it has been proven in [11] that this is equivalent to saying that the point p is 2-singular in the sense of Feldman [6], i.e., the vectors $\{\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2}\}$ are linearly dependent in $\mathbb{R}^5$.

In general, if $corank(f_v)(p) = 2$ the kernel of $Hess(f_v)(p)$ is a plane. Take two linearly independent generators θ_1 and θ_2 and let S_{θ_1,θ_2} be the normal section along this plane. If we denote $f_v^S = f_v|_{S_{\theta_1,\theta_2}}$ the height function in the direction v restricted to this surface, it follows that $Ker(Hess(f_v))(p) = Ker(Hess(f_v^S))(p)$ and thus we can state the following

Theorem 4. a) *If M is a submanifold of codimension 2 in $\mathbb{R}^{n+1}$ then p is a corank 2 singularity for some height function if and only if p is an inflection point of some normal section of M along a tangent plane of M.*

b) *If M is a submanifold of codimension 3 in $\mathbb{R}^{n+1}$ then p is a corank 2 singularity for some height function if and only if p is a 2-singular point of some normal section of M along a tangent plane of M.*

Remark. For a generic embedding $f : M \to \mathbb{R}^{n+1}$ if $n+1 \leq 6$, the only possible singularities of corank 2 of the height functions on M are those in the series $\{D_j\}_{j=3}^n$. Then it is not difficult to check (through a straightforward calculation in local coordinates) that p is a D_j singularity of f_v if and only if it is a D_j singularity of f_v^S.

We shall consider now the orthogonal projections of higher codimensional submanifolds of $\mathbb{R}^{n+1}$ into hyperplanes. Let $f : M \to \mathbb{R}^{n+1}$ be an embedding of such a submanifold and suppose that θ is an asymptotic direction of M at a point p. As in codimension 2 case, we can choose coordinate systems for f in a neighbourhood U of $p = (0, \dots, 0)$ such that

1) $f(x) = (x_1, \dots, x_r, f_1(x), \dots, f_k(x))$
2) $\frac{\partial f_i}{\partial x_j}(0) = 0, \;\; i = 1, \dots, k, \;\; j = 1, \dots, r$
3) $\theta = e_1 = (1, 0, \dots, 0)$
4) $f_v(x) = f_k(x), \;\; \forall x \in U$, where v is the degenerate direction associated to θ.
5) $f_j(x)$ is a nondegenerate Morse function, $\forall j = 1, \dots, k-1$.

We then have the following

Lemma 2. *Given an asymptotic direction θ of an r-submanifold M embedded with codimension k in $\mathbb{R}^{n+1}$ at a point p, let v be the associated binormal direction. If $(p, v) \in S_{1,0}(\Gamma(f))$*

then the projection h_θ has a non-stable singularity of $\mathcal{A}_e$-codimension 1 at p and it is $\mathcal{A}$-equivalent to one of the normal forms:

$$h_\theta(x_1, \bar{x}) \sim (x_2, \ldots, x_r, x_1x_2, \ldots, x_1x_r, x_1^3 + (\sum_{i=2}^{r} \pm x_i{}^2)x_1)$$

Proof. The proof follows easily from the chosen coordinate system. Also, it is in fact a particular case of Proposition 5 in [5]. □

Now, for each asymptotic direction θ of M, there is a principal asymptotic direction $\tilde{\theta}$ of CM at (p, v), for some normal vector v such that $D\xi(f)(\tilde{\theta}) = \theta$. As before, let us denote by

$\tilde{f}$: the embedding of the canal hypersurface in $\mathbb{R}^{n+1}$,

$\mathcal{B}_1(\tilde{f}) = \{v \in S^n : v$ is a normal vector at a point p of type $\sum^{1,0}$ of $\Gamma(f)\}$,

$\mathcal{A}_1(\tilde{f}) = \{\alpha \in S^n : \alpha$ is a principal asymptotic direction at a point p of type $\sum^{1,0}$ of $\Gamma(f)\}$.

We can see that $\mathcal{B}_1(\tilde{f}) = \mathcal{B}_1(f)$ and that $\mathcal{A}_1(\tilde{f}) = \mathcal{A}_1(f)$, where

$\mathcal{B}_1(f) = \{v \in S^n : f_v$ has a unique A_2 singularity at some point p of $M\}$,

$\mathcal{A}_1(f) = \{\alpha \in S^n : \alpha$ is an asymptotic direction corresponding to a degenerate direction v at a point p such that $(p, v) \in S_{1,0}(\Gamma(f))\}$.

Then we have the following result on duality:

Theorem 5. *Given a generic embedding f of a submanifold M in $\mathbb{R}^{n+1}$, such that dim$(M) \geq$ codim$(M) - 1$, the subsets $\mathcal{B}_1(f)$ and $\mathcal{A}_1(f)$ are dual one of each other in S^n. Moreover, if dim$(M) <$ codim$(M) - 1$, then $\mathcal{A}_1(f)$ is the dual of $\mathcal{B}_1(f)$.*

Proof. The result follows from the above observation that $\mathcal{B}_1(f) = \mathcal{B}_1(\tilde{f})$ and that $\mathcal{A}_1(f) = \mathcal{A}_1(\tilde{f})$ and the duality result, obtained in Section 1 for hypersurfaces, applied to the canal hypersurface CM of M.

The restriction on the dimensions is due to the fact that for a generic f, $\mathcal{B}_1(f)$ is a hypersurface in S^n, but the dimension of the stratum $\mathcal{A}_1(f)$ (which is necessarily less than $2dim(M)$) must be less than $n-1$ in case that $dim(M) < codim(M) - 1$, and thus it cannot be a hypersurface in S^n. So, in the terms that we have previously defined the concept of duality, it only makes sense to speak of the dual of $\mathcal{B}_1(f)$ in S^n. □

4. Duality for the multilocal strata

Let M be a hypersurface embedded in $\mathbb{R}^{n+1}$. Denote by $\mathcal{B}_2(f)$ the set of all the normal directions to bitangent hyperplanes of the submanifold M. These normal directions define height functions on M having a double critical value corresponding to two critical points of Morse type. On the other hand denote by $\mathcal{A}_2(f)$ the set of bitangent directions of M. That is, $v \in \mathcal{A}_2(f) \subset S^n$ if and only if the mapping $h_v(p) = p - \langle v, p \rangle v$ has some tangent double point, in other words, there exist nondegenerate critical points p_1 and p_2 of h_v such that $h_v(p_1) = h_v(p_2)$ and $Dh_v(T_{p_1}M) = Dh_v(T_{p_2}M)$. We observe that $\mathcal{B}_2(f)$ and $\mathcal{A}_2(f)$ are, for a generic M, smooth $(n-1)$-dimensional submanifolds of S^n.

Theorem 6. (Duality for hypersurfaces) *Let f be a generic embedding of a hypersurface M in $\mathbb{R}^{n+1}$, then $\mathcal{B}_2(f)^* = \mathcal{A}_2(f)$ and $\mathcal{A}_2(f)^* = \mathcal{B}_2(f)$.*

Proof. Consider the tangent double point stratum $\mathcal{A}_2(f)$. This is a hypersurface $X \subset S^n$ that we can describe as follows: given a bitangent pair $(p,q) \subset M \times M$, i.e., two points of M such that the vectors $\{f(p) - f(q), \frac{\partial f}{\partial x_i}(p), \frac{\partial f}{\partial x_i}(q)\}_{i=1}^n$ form a linear system of rank n in $\mathbb{R}^{n+1}$, and we have that the unit vector $v = \frac{f(p)-f(q)}{\|f(p)-f(q)\|} \in X$.

We can actually find some small enough open subset neighbourhood U of the origin in $\mathbb{R}^n$ and local embeddings,

$$\begin{aligned} f_1 : (U,0) &\longrightarrow (M,p) \\ f_2 : (U,0) &\longrightarrow (M,q) \end{aligned}$$

in such a way that X is locally parametrized at v by

$$\begin{aligned} g : (U,0) &\longrightarrow (X,v) \\ x &\longmapsto \tfrac{f_1(x)-f_2(x)}{\|f_1(x)-f_2(x)\|} \end{aligned}$$

Then the tangent hyperplane to X at v is given by the intersection of S^n with the hyperplane spanned by the vectors $\{ f_1(0) - f_2(0), \frac{\partial f_1}{\partial x_1}(0) - \frac{\partial f_2}{\partial x_1}(0), \ldots, \frac{\partial f_1}{\partial x_n}(0) - \frac{\partial f_2}{\partial x_n}(0)\}$. This hyperplane coincides with the hyperplane spanned by $\{f(p) - f(q), \frac{\partial f}{\partial x_1}(p), \ldots, \frac{\partial f}{\partial x_n}(p)\}$, the tangent hyperplane to M at p which coincides in turn with the tangent hyperplane to M at q. Consequently the poles of this hypersphere are $\pm NM(f(p)) = \pm NM(f(q))$, which is an element of $\mathcal{B}_2(f)$. It is not difficult to see that any element of $\mathcal{B}_2(f)$ is dual to some element of $\mathcal{A}_2(f)^*$. So $\mathcal{A}_2(f)^* = \mathcal{B}_2(f)$.

On the other hand, given $v \in \mathcal{B}_2(f)$, we can find $(p,q) \in M \times M$ such that $\Gamma(f)(p) = \Gamma(f)(q) = v$ and $f(p) - f(q) \in T_{f(p)}M = T_{f(q)}M$.

By working locally we can parametrize M in small neighbourhoods U and V of p and q respectively

$$\begin{aligned} f_1 : (\mathbb{R}^n,0) &\longrightarrow (U,p) \\ f_2 : (\mathbb{R}^n,0) &\longrightarrow (V,q) \end{aligned}$$

such that $\Gamma(f_1 |_{(R^n \times \{0\})}) = \Gamma(f_2 |_{(\mathbb{R}^n \times \{0\})})$.

So $\mathcal{B}_2(f)$ is parametrized either by $\Gamma(f_1 |_{(\mathbb{R}^n \times \{0\})}) = \gamma_1$ or $\Gamma(f_2 |_{(R^n \times \{0\})}) = \gamma_2$ in a neighbourhood of v. And hence the tangent hypersphere to $\mathcal{B}_2(f)$ at v is given by the intersection of S^n with the hyperplane of $\mathbb{R}^{n+1}$ spanned by $\{\gamma_1(0), \frac{\partial \gamma_1}{\partial x_1}(0), \ldots, \frac{\partial \gamma_1}{\partial x_n}(0)\}$. Now observe that $<(f_1(x) - f_2(x)), \gamma_1(x)> = 0$, $\forall x$. So $\frac{\partial}{\partial x_i}(<f_1 - f_2, \gamma_1>) = <\frac{\partial f_1}{\partial x_i} - \frac{\partial f_2}{\partial x_i}), \gamma_1> + <f_1 - f_2, \frac{\partial \gamma_1}{\partial x_i}>$, $i = 1, \ldots, n$. And since $<\frac{\partial f_1}{\partial x_i}(0) - \frac{\partial f_2}{\partial x_i}(0), \gamma_1(0)> = 0$, $i = 1, \ldots, n$, we get that $<f_1(0) - f_2(0), \frac{\partial \gamma_1}{\partial x_i}(0)> = 0$, $i = 1, \ldots, n$, i.e., $<f(p) - f(q), \frac{\partial \gamma_1}{\partial x_i}(0)> = 0$, $i = 1, \ldots, n$. Consequently $\pm(f(p) - f(q))$ are the poles of this hypersphere. The reciprocal inclusion follows analogously. And we conclude that $\mathcal{B}_2(f)^* = \mathcal{A}_2(f)$. □

Theorem 7. (Duality for submanifolds of codimension higher than 1) *Given any generic embedding f, of an r-manifold M in $\mathbb{R}^{n+1}$ with $2r \geq n-1$, we have $\mathcal{A}_2(f)^* = \mathcal{B}_2(f)$ and $\mathcal{B}_2(f)^* = \mathcal{A}_2(f)$.*

Proof. This result follows from observing that any bitangent affine hyperplane to CM is obtained by translation of some bitangent affine hyperplane of M in the direction of the normal to this hyperplane. And hence they define the same point in S^n. Therefore $\mathcal{B}_2(M) = \mathcal{B}_2(CM)$. On the other hand, if (p, q) is a bitangent pair in M, the corresponding points in CM are $(p+v, q+v)$, where v is the normal vector to the bitangent plane defined by p and q. Then the unit vector in the direction of the segment $\bar{pq}$ gives in both cases the corresponding direction of $\mathcal{A}_2$. Consequently $\mathcal{A}_2(M) = \mathcal{A}_2(CM)$. □

Remark. Given a pair of points $p, q \in M$, a hyperplane H is bitangent to M at p and q if and only if it contains the r-spaces T_pM and T_qM and the straight line defined by the points p and q. Then, in the case that $2dim(M) < n-1$, each couple (p, q) defines a whole k-parameter family ($k = n - (2dim(M)+1)$) and we thus have a whole k-sphere in $\mathcal{B}_2(f)$ corresponding by duality to a single point (direction of $\bar{pq}$) in $\mathcal{A}_2(f)$. In fact, in this case $dim\mathcal{A}_2(f) < n-1$. So $\mathcal{A}_2(f)$ is not a hypersurface in S^n (similarly to what happens with $\mathcal{A}_1(f)$, as we have seen previously), therefore it has no meaning to speak of its dual in the way we have defined it in this paper. In any case we still have:

The dual of $\mathcal{B}_2(f)$ is $\mathcal{A}_2(f)$.

We must point out that this is also the case for the canal hypersurface of such an M, but these hypersurfaces are non-generic (as hypersurfaces!) in the sense that a small perturbation transforms them into generic "non-canal" hypersurfaces for which both the subsets $\mathcal{A}_1(f)$ and $\mathcal{A}_2(f)$ have codimension 1 in S^n.

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