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# Isoptics of Pairs of Nested Closed Strictly Convex Curves and Crofton-Type Formulas

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**Abstract.** In this paper we present some geometric properties of isoptics of pairs of nested closed strictly convex curves. The theory of isoptics provides a simple geometric method to prove some generalizations of well-known integral formulas of Crofton-type.

## 1. Introduction

In this paper we consider a pair of two nested strictly convex curves $C_1$ and $C_2$ such as in Figure 1.1. Choose a coordinate system with the origin $O$ in the interior of $C_2$. Assume that the curves $C_i$ are given by the equation $z_i(t) = p_i(t)e^{it} + \dot{p}_i(t)ie^{it}$, $i = 1, 2$, where $p_1, p_2$ are the support functions of the curves $C_1$ and $C_2$, respectively. Consider the tangent line $k_1$ to the curve $C_1$ at a point $z_1(t)$ and the tangent line $k_2'$ to $C_2$ parallel to $k_1$ in the manner shown in Figure 1.1. Rotate the tangent line $k_2'$ in a clockwise direction to the position $k_2$ in such a way that the tangent lines $k_1$ and $k_2$ form an angle $\alpha$, $\alpha \in (0, \pi)$. Then $k_2$ is the tangent line to $C_2$ at the point $z_2(t + \alpha)$. Let $z_\alpha(t)$ denote the intersection point of the tangent lines $k_1$ and $k_2$. The curve $C_\alpha : z = z_\alpha(t)$, where $\alpha$ is fixed, is said to be the *$\alpha$-isoptic of the first kind* of the pair $C_1$ and $C_2$. If we rotate the tangent line $k_2'$ in the counterclockwise direction we get a point $z = \tilde{z}_\alpha(t)$. The curve $z = \tilde{z}_\alpha(t)$ is said to be the *$\alpha$-isoptic of the second kind* of the pair $C_1$ and $C_2$. Note that according to the above definitions there are exactly two isoptics of the same kind passing through each point exterior to the curve $C_1$.

Consider the isoptics of the first kind. We fix $\alpha \in (0, \pi)$. Then

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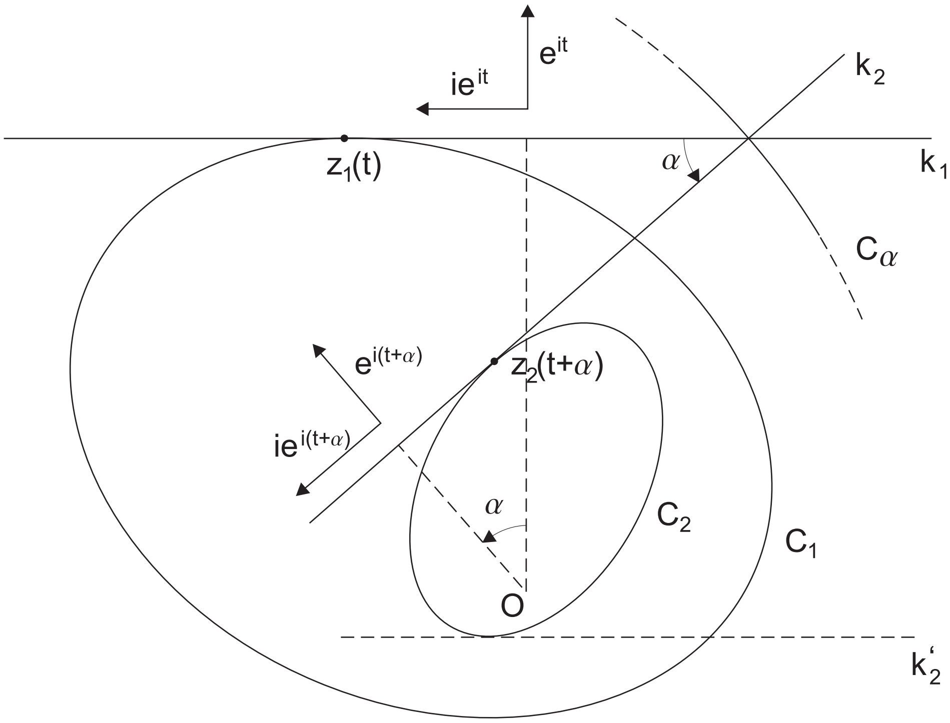


Figure 1.1

$$z_\alpha(t) = z_1(t) + \lambda(t) i e^{it} = z_2(t+\alpha) + \mu(t) i e^{i(t+\alpha)}. \tag{1.1}$$

In this case $\mu < 0$, however $\lambda$ is arbitrary. It is easy to check that

$$\lambda(t) = -\dot{p}_1(t) - \cot\alpha\, p_1(t) + p_2(t+\alpha)\frac{1}{\sin\alpha} \tag{1.2}$$

$$\mu(t) = -p_1(t)\frac{1}{\sin\alpha} - \dot{p}_2(t+\alpha) + \cot\alpha\, p_2(t+\alpha). \tag{1.3}$$

Hence we get an equation of an $\alpha$-isoptic of the first kind

$$z_\alpha(t) = p_1(t)e^{it} + \left(p_2(t+\alpha)\frac{1}{\sin\alpha} - p_1(t)\cot\alpha\right) i e^{it}. \tag{1.4}$$

Similarly,

$$\tilde{z}_\alpha(t) = z_1(t) + \tilde{\lambda}(t) i e^{it} = z_2(t+\alpha-\pi) + \tilde{\mu}(t) i e^{i(t+\alpha-\pi)}. \tag{1.5}$$

Then

$$\tilde{\lambda}(t) = -\dot{p}_1(t) - p_2(t+\alpha-\pi)\frac{1}{\sin\alpha} - p_1(t)\cot\alpha, \tag{1.6}$$

$$\tilde{\mu}(t) = p_1(t)\frac{1}{\sin\alpha} + p_2(t+\alpha-\pi)\cot\alpha - \dot{p}_2(t+\alpha-\pi), \tag{1.7}$$

and

$$\tilde{z}_\alpha(t) = p_1(t)e^{it} + \left(-p_2(t+\alpha-\pi)\frac{1}{\sin\alpha} - p_1(t)\cot\alpha\right) i e^{it}. \tag{1.8}$$

Note that in both cases the isoptic of the pair of nested strictly convex curves is at least of the class $C^1$.

From now on, we consider only the isoptics of the first kind, unless otherwise stated. We have

$$\dot{z}_\alpha(t) = -\lambda(t)e^{it} + \varrho(t)ie^{it},$$

where

$$\varrho(t) = p_1(t) + \dot{p}_2(t+\alpha)\frac{1}{\sin\alpha} - \dot{p}_1(t)\cot\alpha. \tag{1.9}$$

Let $q(t) = z_1(t) - z_2(t+\alpha)$. Then

$$q(t) = \sin^2\alpha(\varrho(t) - \lambda(t)\cot\alpha)e^{it} - \sin^2\alpha(\lambda(t) + \varrho(t)\cos\alpha)ie^{it}. \tag{1.10}$$

It is easy to check that

$$|\dot{z}_\alpha(t)|^2 = \frac{1}{\sin^2\alpha}|q(t)|^2. \tag{1.11}$$

Since the considered curves are nested and $\alpha \in (0,\pi)$ then from formula (1.11) it follows that the isoptic $C_\alpha$ is always regular, i.e. $|\dot{z}_\alpha(t)| \neq 0$.

**Corollary 1.1.** *The length $|q(t)|$ is constant if and only if $t = as + s_0$, where $s$ is the natural parameter on the isoptic.*

## 2. Sine theorem for a pair of curves

Let $C_1$ and $C_2$ be a pair of two nested strictly convex curves such as in Figure 1.1 and $C_\alpha$ its $\alpha$-isoptic of the first kind. Define the angles $\varphi$ and $\psi$ formed by the tangent lines to $C_1$ and $C_2$ at $z_1(t)$ and $z_2(t+\alpha)$ with the tangent line to the isoptic $C_\alpha$ at the point $z_\alpha(t)$, respectively.

Define $[v,w] = ad - bc$, when $v = a + bi$ and $w = c + di$. Following these notations we get

$$\sin\varphi = \frac{[\dot{z}_\alpha(t), ie^{it}]}{|\dot{z}_\alpha(t)|} = \frac{-\lambda(t)}{|\dot{z}_\alpha(t)|} = \frac{|z_1(t) - z_\alpha(t)|}{|\dot{z}_\alpha(t)|}. \tag{2.1}$$

Note that here we have $\lambda < 0$. Similarly, we get

$$\sin\psi = \frac{|z_2(t+\alpha) - z_\alpha(t)|}{|\dot{z}_\alpha(t)|}. \tag{2.2}$$

Hence we obtain the so-called sine theorem

**Theorem 2.1.**

$$\frac{|q|}{\sin\alpha} = \frac{|z_1(t) - z_\alpha(t)|}{\sin\varphi} = \frac{|z_2(t+\alpha) - z_\alpha(t)|}{\sin\psi} = |\dot{z}_\alpha(t)|. \tag{2.3}$$

A theorem analogous to the one above holds for isoptics of the second kind.

## 3. Convexity of isoptics

From now on, in considerations involving the curvature, we always assume that the curves $C_1$ and $C_2$ are of class $C^2$ and of positive curvature. It is easy to establish the following useful formulas:

$$\dot{\lambda}(t) = -\frac{1}{k_1(t)} + \varrho(t), \tag{3.1}$$

$$\dot{\varrho}(t) = -\lambda(t) - \frac{1}{k_1(t)}\cot\alpha + \frac{1}{\sin\alpha}\cdot\frac{1}{k_2(t+\alpha)}, \tag{3.2}$$

where $k_1(t)$ and $k_2(t)$ are the curvature functions of curves $C_1$ and $C_2$, respectively. Then

$$\begin{aligned}&[\dot{z}_\alpha(t), \ddot{z}_\alpha(t)] = \\ &= 2\lambda^2(t) + 2\varrho^2(t) + \frac{\lambda(t)}{k_1(t)}\cot\alpha - \frac{\lambda(t)}{k_2(t+\alpha)}\cdot\frac{1}{\sin\alpha} - \frac{\varrho(t)}{k_1(t)}.\end{aligned} \tag{3.3}$$

On the other hand,

$$[q(t), \dot{q}(t)] = -\frac{-\lambda(t)}{k_1(t)}\cot\alpha + \frac{\lambda(t)}{k_2(t+\alpha)}\cdot\frac{1}{\sin\alpha} + \frac{\varrho(t)}{k_1(t)}. \tag{3.4}$$

Hence

$$k_\alpha(t) = \frac{[\dot{z}_\alpha(t), \ddot{z}_\alpha(t)]}{|\dot{z}_\alpha(t)|^3} = \frac{\sin\alpha}{|q(t)|^3}\left(2|q(t)|^2 - [q(t), \dot{q}(t)]\right). \tag{3.5}$$

Finally, we get

**Theorem 3.1.** *An isoptic $C_\alpha$ is convex if and only if*

$$\left|\frac{d}{dt}\left(\frac{q(t)}{|q(t)|}\right)\right| < 2. \tag{3.6}$$

An analogous theorem is valid for the isoptics of second kind.

Reconsider formula (3.3). Since $\lambda\cot\alpha - \varrho = \frac{\mu}{\sin\alpha}$, we have then

$$[\dot{z}_\alpha(t), \ddot{z}_\alpha(t)] = 2\lambda^2(t) + 2\varrho^2(t) - \frac{1}{\sin\alpha}\left(\frac{-\mu(t)}{k_1(t)} + \frac{\lambda(t)}{k_2(t+\alpha)}\right). \tag{3.7}$$

**Corollary 3.1.** *An isoptic $C_\alpha$ is convex if and only if*

$$2|q(t)|^2 > \sin\alpha\left(\frac{\lambda(t)}{k_2(t)} - \frac{\mu(t)}{k_1(t)}\right) \tag{3.8}$$

*for every $t$.*

Since $\mu(t) < 0$ for each $t$ for any isoptic of first kind, we have

$$-\mu(t) = |z_\alpha(t) - z_2(t)| \tag{3.9}$$

and

$$|\lambda(t)| = |z_\alpha(t) - z_1(t)|. \tag{3.10}$$

Assume that in a neighborhood of the point $t$ we have $\lambda(t) > 0$. Then condition (3.8) can be written in the form

$$2|q(t)|^2 > \sin\alpha \left( \frac{|z_\alpha(t) - z_1(t)|}{k_2(t)} + \frac{|z_\alpha(t) - z_2(t)|}{k_1(t)} \right). \tag{3.11}$$

Then, by the virtue of sine theorem,

$$2|q(t)| > \left( \frac{\sin\varphi}{k_2(t)} + \frac{\sin\psi}{k_1(t)} \right). \tag{3.12}$$

Note that the right hand side is equal to the sum of lengths of projections in the direction determined by the vector $q$ of curvature vectors of curves $C_1$ and $C_2$ at points $t$ and $t+\alpha$, respectively. If $\lambda < 0$ then the first member of the right hand side in (3.8) is taken with the minus sign. Consequently, we get

**Theorem 3.2.** *An isoptic $C_\alpha$ is a convex curve if and only if for each $t$ double the length of the vector $q(t)$ is greater then the sum of the length of the projection on the direction of the vector $q(t)$ of the curvature vector of the curve $C_1$ at the point $t$ and the algebraic measure of the projection of the curvature vector of the curve $C_2$ at the point $t+\alpha$ on the direction of the vector $q(t)$.*

Note that this theorem allows us to check the local convexity of the isoptic knowing only the point at which we examine the isoptic. We need not know even the equation of the isoptic.

Similar considerations can be carried out for the isoptics of the second kind.

## 4. Crofton-type formulas

Let $\omega(t)$ be an angle formed by the tangent line to $C_1$ at the point $z_1(t)$ and the segment $z_1(t)z_2(t+\alpha)$. Consider a mapping

$$F(\alpha, t) = z_\alpha(t). \tag{4.1}$$

Then $\frac{\partial F}{\partial \alpha} = -\frac{\mu}{\sin\alpha} i e^{it}$ and $\frac{\partial F}{\partial t} = \left((p+\ddot{p}) + \lambda_{|t}\right) - \lambda e^{it}$. The Jacobian $J(F)$ of the mapping $F$ is equal to

$$J(F) = -\frac{\mu\lambda}{\sin\alpha}. \tag{4.2}$$

If $A = \{(\alpha, t) \in (0,\pi) \times (0, 2\pi) : \omega(t) < \alpha < \pi\}$ then $F$ is a diffeomorphism of the domain $A$ onto the exterior of the curve $C_1$ less some half-line. Moreover, it is easy to see that for a point $F(\alpha, t)$ we have $\lambda > 0$, $\mu < 0$ so $J(F) > 0$. On the other hand, this mapping restricted to a set $B = \{(\alpha, t) : 0 < \alpha < \omega(t)\}$ is a diffeomorphism as well; however in this case $\lambda < 0$ and $\mu < 0$ and so $|J(F)| = \frac{\lambda\mu}{\sin\alpha}$.

For each point $(x, y) \in \Omega$, where $\Omega$ is the exterior of the curve $C_1$, we consider four segments from the point $(x, y)$ tangent to the curves $C_1$ and $C_2$. These segments we denote respectively by $l_1, m_1, m_2, l_2$, where $l_1, l_2$ are tangent to $C_1$ and $m_1, m_2$ are tangent to $C_2$. Let us observe that we have the same formula (1.2) for $l_1$ and $l_2$ and, similarly, formula (1.3) for $m_1$ and $m_2$. This fact will be used in our calculations of integrals. With the above notations we obtain

$$\begin{aligned}(4.3)\qquad &\iint_\Omega \left(\frac{\sin(l_2, m_2)}{l_2 m_2} + \frac{\sin(l_2, m_1)}{l_1 m_2}\right) dxdy = \\ &= \int_0^{2\pi}\int_0^{\omega(t)} \frac{\sin\alpha}{(-\lambda)(-\mu)} \cdot \frac{\lambda\mu}{\sin\alpha} d\alpha dt + \int_0^{2\pi}\int_{\omega(t)}^{\pi} \frac{\sin\alpha}{\lambda(-\mu)} \cdot \frac{-\lambda\mu}{\sin\alpha} d\alpha dt = \\ &= \int_0^{2\pi}\int_0^{\pi} d\alpha dt = 2\pi^2.\end{aligned}$$

Similarly, we can prove that

$$(4.4)\qquad \iint_\Omega \left(\frac{\sin(l_1, m_1)}{l_1 m_1} + \frac{\sin(l_2, m_1)}{l_2 m_1}\right) dxdy = 2\pi^2.$$

Let us observe that these formulas are more general then the one in Santalo [5]. By adding the corresponding sides of the above formulas we get the well-known formula

$$\iint_\Omega \left(\frac{\sin(l_2, m_2)}{l_2 m_2} + \frac{\sin(l_2, m_1)}{l_1 m_2} + \frac{\sin(l_1, m_1)}{l_1 m_1} + \frac{\sin(l_2, m_1)}{l_2 m_1}\right) dxdy = 4\pi^2.$$

This demonstrates that the isoptics provide a nice and direct geometric method to prove some integral formulas. In certain cases our method gives a simple way leading to stronger results.

Let $k_1, \hat{k}_1, k_2$ be the curvatures of the curves $C_1$ and $C_2$ at the points $z_1, \hat{z}_1, z_2$ and $\alpha, \beta, \gamma$ be the angles as in Figure 4.1.

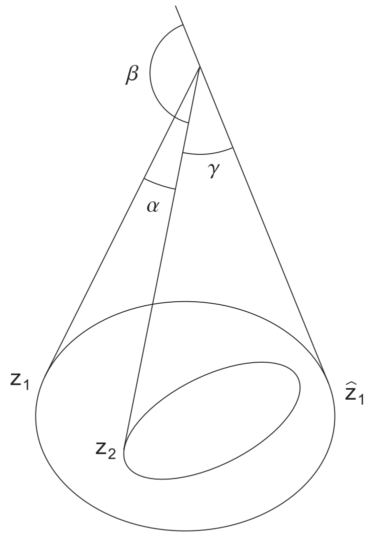


Figure 4.1

Then we obtain

$$\begin{aligned}(4.5)\qquad &\iint_\Omega \left(\frac{\sin\alpha}{l_2 m_2}\left(\frac{1}{k_1}+\frac{1}{k_2}\right)+\frac{\sin\alpha}{l_1 m_2}\left(\frac{1}{\hat{k}_1}+\frac{1}{k_2}\right)\right)dxdy=\\
&=\int_0^{2\pi}\int_0^{\omega(t)}\frac{\sin\alpha}{(-\lambda)(-\mu)}\left(\frac{1}{k_1(t)}+\frac{1}{k_2(t)}\right)\frac{\lambda\mu}{\sin\alpha}d\alpha dt+\\
&+\int_0^{2\pi}\int_{\omega(t)}^{\pi}\frac{\sin\gamma}{\lambda(-\mu)}\left(\frac{1}{k_1(t)}+\frac{1}{k_2(t+\beta)}\right)\frac{-\lambda\mu}{\sin\gamma}d\beta dt=\\
&=\int_0^{2\pi}\int_0^{\omega(t)}\left(\frac{1}{k_1(t)}+\frac{1}{k_2(t+\alpha)}\right)d\alpha dt+\\
&+\int_0^{2\pi}\int_{\omega(t)}\left(\frac{1}{k_1(t)}+\frac{1}{k_2(t+\beta)}\right)d\beta dt=\\
&=\int_0^{2\pi}\int_0^{\pi}\left(\frac{1}{k_1(t)}+\frac{1}{k_2(t+\alpha)}\right)d\alpha dt=\\
&=\int_0^{2\pi}\int_0^{\pi}\left(p(t)+\ddot{p}_1(t)+p_2(t+\alpha)+\ddot{p}_2(t+\alpha)\right)d\alpha dt=\pi(L_1+L_2).\end{aligned}$$

Taking $m_1$ instead of $m_2$ we obtain an analogous formula. This formula again shows the usefulness of our method to provide a generalization of another well-known formula. Let us calculate the following integrals

$$\begin{aligned}(4.6)\qquad &\iint_\Omega\left(\frac{\sin\alpha}{l_2m_2}\cdot\frac{1}{k_1}\cdot\frac{1}{k_2}+\frac{\sin\gamma}{l_1m_2\cdot}\frac{1}{k_1}\cdot\frac{1}{\hat{k}_2}\right)dxdy=\\
&=\int_0^{2\pi}\int_0^{\pi}\frac{1}{k_1(t)}\cdot\frac{1}{k_2(t+\alpha)}d\alpha dt=\frac{1}{2}\int_0^{2\pi}\int_0^{2\pi}\frac{1}{k_1(t)}\cdot\frac{1}{k_2(t+\alpha)}d\alpha dt=\\
&=\frac{1}{2}L_1L_2.\end{aligned}$$

The formulas (4.5) and (4.6) reduce to well-known formulas (cf. [5]) when $C_1=C_2$. Analogously, we obtain the following integral formulas

$$(4.7)\qquad \iint_\Omega\frac{\sin^2\gamma-\sin^2\alpha}{m_2}=\pi L_1,$$

$$(4.8)\qquad \iint_\Omega\frac{\sin^2\gamma-\sin^2\alpha}{l_2}=\pi L_2.$$

The above formulas generalize our integral formulas (3.5) and (3.6) from [2].

Finally, we prove an integral formula for an annulus. Since the isoptics investigated in this paper can intersect one another, we have to restrict our considerations to certain angles. Let $\omega_M=\max_{t\in<0,2\pi>}\omega(t)$ and $\omega_m=\min_{t\in<0,2\pi>}\omega(t)$. Then for $\beta_2>\beta_1>\omega_M$ (or for $\omega_m>\beta_2>\beta_1$) the isoptics $C_{\beta_2}$ and $C_{\beta_1}$ do not intersect. Fix $\beta_1$ and $\beta_2$ such that $\beta_2>\beta_1>\omega_M$ and consider an annulus $C_{\beta_1}C_{\beta_2}$. Then we have

$$(4.9)\qquad \iint_{C_{\beta_1}C_{\beta_2}}\frac{1}{l_1}dxdy=\int_0^{2\pi}\int_{\beta_1}^{\beta_2}d\alpha dt=$$

$$= \int_0^{2\pi} \int_{\beta_1}^{\beta_2} \left( \frac{1}{\sin^2 \alpha} p_1(t) + \frac{1}{\sin \alpha} \dot{p}_2(t+\alpha) - \frac{\cos \alpha}{\sin^2 \alpha} p_2(t+\alpha) \right) d\alpha dt =$$
$$= L_1(\cot \beta_1 - \cot \beta_2) + L_2 \left( \frac{1}{\sin \beta_2} - \frac{1}{\sin \beta_1} \right).$$

Similarly, we get

$$\iint_{C_{\beta_1} C_{\beta_2}} \frac{1}{m_2} dxdy = L_1 \left( \frac{1}{\sin \beta_2} - \frac{1}{\sin \beta_1} \right) - L_2(\cot \beta_2 - \cot \beta_1). \tag{4.10}$$

Adding the above formulas we get

$$\iint_{C_{\beta_1} C_{\beta_2}} \left( \frac{1}{l_1} + \frac{1}{m_2} \right) dxdy = (L_1 + L_2)(\tan \frac{\beta_2}{2} - \tan \frac{\beta_1}{2}). \tag{4.11}$$

This formula is then a generalization of our integral formula (2.1) given in [3].

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