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Hecke Operators on the q-Analogue of Group Cohomology

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Abstract. We construct the q-analogue of a certain class of group cohomology and introduce the action of Hecke operators on such cohomology. We also show that such an action determines a representation of a Hecke ring in each of the associated group cohomology spaces.

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1. Introduction

Hecke operators play an important role in number theory, especially in connection with the theory of automorphic forms. More specifically, Hecke operators arise usually as endomorphisms of the space of certain automorphic forms and are used as a tool for the investigation of multiplicative properties of Fourier coefficients of such automorphic forms. On the other hand, it is well-known that various types of automorphic forms are closely connected with the cohomology of the corresponding arithmetic groups. A classic example is the Eichler-Shimura isomorphism (cf. [1], [7]) which provides a correspondence between holomorphic modular forms and elements of the parabolic cohomology space of the associated Fuchsian group. Because of such intimate connections, it would be natural to study the Hecke operators on the cohomology of arithmetic groups associated to automorphic forms as was done in a number of papers (see e.g. [3], [5], [4], [9]). Hecke operators on the cohomology of more general groups were also investigated by Rhie and Whaples in [6].

Various objects that are studied in the theory of quantum groups can be obtained from more traditional mathematical objects using quantizations or deformations. One way of achieving this is by constructing the q-analogue of a known mathematical object, which will reduce to the original object when q assumes a specific value. In [2], Kapranov introduced the q-analogue of homological algebra. Among other things, he introduced the homology of complexes that is associated to an N-th root of unity $q \neq 1$ for some positive integer N and whose boundary map d satisfies $d^N = 0$ instead of the usual $d^2 = 0$. His homology reduces to the usual one when $N = 2$ and $q = -1$.

In this paper we follow the method of Kapranov to construct the q-analogue of the cohomology of groups with coefficients in complex vector spaces and introduce the action of Hecke operators on such cohomology spaces. We then show that such an action determines a representation of a Hecke ring in the associated group cohomology space.

2. Group cohomology

In this section we construct the q-analogue of group cohomology associated to group representations in complex vector spaces following closely the approach of Kapranov [2] used for the construction of his q-analogue of homology.

Let Γ be a group, and let $\mathcal{M}$ be a complex vector space that is a Γ-module. Thus $\mathcal{M}$ is a representation space of Γ, and for each $\gamma \in \Gamma$ the map $v \mapsto \gamma \cdot v$ is a $\mathbb{C}$-linear map. We denote by $C^p(\Gamma, \mathcal{M})$ the vector space over $\mathbb{C}$ consisting of all maps $c : \Gamma^{p+1} \to \mathcal{M}$ that satisfy

$$c(\gamma\gamma_0, \gamma\gamma_1, \dots, \gamma\gamma_p) = \gamma \cdot c(\gamma_0, \gamma_1, \dots, \gamma_p) \tag{2.1}$$

for all $\gamma, \gamma_0, \dots, \gamma_p \in \Gamma$. Let N be a positive integer, and let $q \in \mathbb{C}$ be an N-th root of unity with $q \neq 1$. Given a function $f : \Gamma^{p+1} \to \mathcal{M}$ and an integer k with $0 \leq k \leq p+1$, we denote by $\delta_k f$ the function $\delta_k f : \Gamma^{p+2} \to \mathcal{M}$ given by

$$(\delta_k f)(\gamma_0, \gamma_1, \dots, \gamma_{p+1}) = f(\gamma_0, \dots, \widehat{\gamma}_k, \dots, \gamma_{p+1}), \tag{2.2}$$

where $\widehat{\gamma}_k$ means suppressing the component γ_k. Thus we obtain the maps $\delta_k : C^p(\Gamma, \mathcal{M}) \to C^{p+1}(\Gamma, \mathcal{M})$ satisfying the relation

$$\delta_j \circ \delta_k = \delta_{k-1} \circ \delta_j. \tag{2.3}$$

for $0 \leq j < k \leq p+1$. We now define the map $d_q : C^p(\Gamma, \mathcal{M}) \to C^{p+1}(\Gamma, \mathcal{M})$ by

$$d_q c = \sum_{k=0}^{p+1} q^k (\delta_k c) \tag{2.4}$$

for all $c \in C^p(\Gamma, \mathcal{M})$, so that we have

$$(d_q c)(\gamma_0, \gamma_1, \dots, \gamma_{p+1}) = \sum_{k=0}^{p+1} q^k \cdot c(\gamma_0, \dots, \widehat{\gamma}_k, \dots, \gamma_{p+1}),$$

and the map d_q reduces to the coboundary map for the usual group cohomology if $q = -1$ and $N = 2$ (cf. [5, §1.2]).

If ν is a positive integer, we set

$$[\nu]_q = \frac{1-q^\nu}{1-q} = 1 + q + \cdots + q^{\nu-1},$$

$$[\nu]_q! = [\nu]_q[\nu-1]_q \cdots [2]_q[1]_q.$$

Lemma 2.1. *Given an element $c \in C^p(\Gamma, \mathcal{M})$, we have*

$$(d_q)^m c = [m]_q! \cdot \sum_{i_1 \leq \cdots \leq i_m} q^{i_1+\cdots+i_m} \delta_{i_m} \cdots \delta_{i_1} c, \tag{2.5}$$

for each positive integer m.

Proof. We shall use induction on m. Since (2.5) obviously holds for $m = 1$, we begin the induction process by assuming that it is true for $m > 1$. Then, for each $c \in C^p(\Gamma, \mathcal{M})$, we have

$$\begin{aligned}(d_q)^{m+1} c &= [m]_q! \cdot \sum_{i_1 \leq \cdots \leq i_m} q^{i_1+\cdots+i_m} \cdot \sum_j q^j \delta_j \cdot \delta_{i_m} \cdots \delta_{i_1} c \\ &= [m]_q! \cdot \sum_{i_1 \leq \cdots \leq i_m} q^{i_1+\cdots+i_m} \cdot \Bigg(\sum_{j<i_1} + \sum_{i_1 \leq j < i_2} + \cdots \\ &\qquad \cdots + \sum_{i_{m-1} \leq j < i_m} + \sum_{i_m \leq j} \Bigg) q^j \cdot \delta_j \delta_{i_m} \cdots \delta_{i_1} c.\end{aligned}$$

However, replacing j by i_{m+1}, we have

$$\begin{aligned}&\sum_{i_1 \leq \cdots \leq i_m} q^{i_1+\cdots+i_m} \cdot \sum_{i_m \leq j} q^j \cdot \delta_j \delta_{i_m} \cdots \delta_{i_1} c \\ &\qquad = \sum_{i_1 \leq \cdots \leq i_m \leq i_{m+1}} q^{i_1+\cdots+i_m+i_{m+1}} \cdot \delta_{i_{m+1}} \delta_{i_m} \cdots \delta_{i_1} c.\end{aligned}$$

On the other hand, using (2.3), we obtain

$$\begin{aligned}&\sum_{i_1 \leq \cdots \leq i_m} q^{i_1+\cdots+i_m} \cdot \sum_{i_{m-1} \leq j < i_m} q^j \cdot \delta_j \delta_{i_m} \cdots \delta_{i_1} c \\ &\qquad = \sum_{i_1 \leq \cdots \leq i_m} q^{i_1+\cdots+i_m} \cdot \sum_{i_{m-1} \leq j < i_m} q^j \cdot \delta_{i_m - 1} \delta_j \cdots \delta_{i_1} c.\end{aligned}$$

Replacing i_m by $i_{m+1} + 1$ and j by i_m, we see that the right hand side of the above equation reduces to

$$q \cdot \sum_{i_1 \leq \cdots \leq i_m \leq i_{m+1}} q^{i_1+\cdots+i_m+i_{m+1}} \cdot \delta_{i_{m+1}} \delta_{i_m} \cdots \delta_{i_1} c.$$

Similarly, we obtain

$$\begin{aligned}\sum_{i_1\le\cdots\le i_m} q^{i_1+\cdots+i_m} \cdot \sum_{i_{m-2}\le j<i_{m-1}} q^j \cdot \delta_j\delta_{i_m}\cdots\delta_{i_1}c \\ = \sum_{i_1\le\cdots\le i_m} q^{i_1+\cdots+i_m} \cdot \sum_{i_{m-2}\le j<i_{m-1}} q^j \cdot \delta_{i_m-1}\delta_{i_{m-1}-1}\delta_j\cdots\delta_{i_1}c \\ = q^2 \cdot \sum_{i_1\le\cdots\le i_m\le i_{m+1}} q^{i_1+\cdots+i_m+i_{m+1}} \cdot \delta_{i_{m+1}}\delta_{i_m}\cdots\delta_{i_1}c\end{aligned}$$

by replacing i_m by $i_{m+1}+1$, i_{m-1} by i_m+1, and j by i_{m-1}. Treating the other summations similarly, we see that

$$(d_q)^{m+1}c = [m]_q! \cdot (1+q+\cdots+q^m) \cdot \sum_{i_1\le\cdots\le i_m\le i_{m+1}} q^{i_1+\cdots+i_m+i_{m+1}} \cdot \delta_{i_{m+1}}\delta_{i_m}\cdots\delta_{i_1}c.$$

Now (2.5) follows from the fact that

$$[m]_q! \cdot (1+q+\cdots+q^m) = [m]_q! \cdot [m+1]_q = [m+1]_q!,$$

and hence the induction is complete. □

Remark 2.2. Lemma 2.1 is essentially the same as Lemma 0.3 in [2] which was given without proof.

Corollary 2.3. *For each integer p the composite*

$$(d_q)^N = d_q \circ \cdots \circ d_q : C^p(\Gamma, \mathcal{M}) \to C^{p+N}(\Gamma, \mathcal{M})$$

of the N maps

$$C^p(\Gamma, \mathcal{M}) \xrightarrow{d_q} C^{p+1}(\Gamma, \mathcal{M}) \xrightarrow{d_q} \cdots \xrightarrow{d_q} C^{p+N}(\Gamma, \mathcal{M})$$

is equal to zero.

Proof. This follows immediately from Lemma 2.1 and the relation $[N]_q! = [N]_q \cdot [N-1]_q!$ with $[N]_q = (1-q^N)/(1-q)$. □

For $1 \le k \le N$ consider the maps

$$(d_q)^k : C^p(\Gamma, \mathcal{M}) \to C^{p+k}(\Gamma, \mathcal{M}), \quad (d_q)^{N-k} : C^{p-N+k}(\Gamma, \mathcal{M}) \to C^p(\Gamma, \mathcal{M}),$$

where we assume that $C^\nu(\Gamma, \mathcal{M}) = \{0\}$ and $d_q = 0$ on $C^\nu(\Gamma, \mathcal{M})$ for $\nu < 0$. Then we have

$$(d_q)^k \circ (d_q)^{N-k} = (d_q)^N = 0,$$

and hence it follows that $\operatorname{Im}(d_q)^{N-k} \subset \operatorname{Ker}(d_q)^k$. This allows us to define the q-analogue of the cohomology of the group Γ with coefficients in $\mathcal{M}$ as follows.

Definition 2.4. *Given an integer p, the p-th q-cohomology of Γ with coefficients in $\mathcal{M}$ consists of the N complex vector spaces*

$$_1H^p(\Gamma, \mathcal{M}), {_2H^p}(\Gamma, \mathcal{M}), \dots, {_NH^p}(\Gamma, \mathcal{M}),$$

where

$$_kH^p(\Gamma, \mathcal{M}) = \frac{Ker[(d_q)^k : C^p(\Gamma, \mathcal{M}) \to C^{p+k}(\Gamma, \mathcal{M})]}{Im[(d_q)^{N-k} : C^{p-N+k}(\Gamma, \mathcal{M}) \to C^p(\Gamma, \mathcal{M})]} \tag{2.6}$$

for $k = 1, \dots, N$.

3. Hecke operators

In this section, we discuss Hecke operators acting on the q-cohomology spaces of a group associated to its representation in a complex vector space. First, we shall review the definition of Hecke rings (see e.g. [3], [8] for details). Let G be a fixed group, and let Γ and Γ' be subgroups of G. Γ and Γ' are said to be commensurable if $\Gamma \cap \Gamma'$ has finite index in both Γ and Γ', in which case we write $\Gamma \sim \Gamma'$. We denote by $\widetilde{\Gamma}$ the commensurability subgroup of Γ, that is,

$$\widetilde{\Gamma} = \{\alpha \in G \mid \alpha\Gamma\alpha^{-1} \sim \Gamma\}.$$

Given $\alpha \in \widetilde{\Gamma}$, the double coset $\Gamma\alpha\Gamma$ has a decomposition of the form

$$\Gamma\alpha\Gamma = \coprod_{1 \leq i \leq d} \Gamma\alpha_i \tag{3.1}$$

for some elements $\alpha_1, \dots, \alpha_d \in \widetilde{\Gamma}$, where $\coprod$ denotes the disjoint union. Given a semigroup Δ with $\Gamma \subset \Delta \subset \widetilde{\Gamma}$, we denote by $R(\Gamma, \Delta)$ the free $\mathbb{Z}$-module generated by the double cosets $\Gamma\alpha\Gamma$ with $\alpha \in \Delta$. Then $R(\Gamma, \Delta)$ has a ring structure with multiplication defined as follows. Let $\alpha, \beta \in \Delta \subset \widetilde{\Gamma}$, and suppose we have decompositions of the form

$$\Gamma\alpha\Gamma = \coprod_{1 \leq i \leq d} \Gamma\alpha_i, \quad \Gamma\beta\Gamma = \coprod_{1 \leq i \leq e} \Gamma\beta_i$$

with $\alpha_i, \beta_j \in \widetilde{\Gamma}$. If $\{\Gamma\epsilon_1\Gamma, \dots, \Gamma\epsilon_r\Gamma\}$ is the set of all distinct double cosets contained in $\Gamma\alpha\Gamma\beta\Gamma$, then we define the product $(\Gamma\alpha\Gamma) \cdot (\Gamma\beta\Gamma)$ by

$$(\Gamma\alpha\Gamma) \cdot (\Gamma\beta\Gamma) = \sum_{k=1}^{r} m(\Gamma\alpha\Gamma, \Gamma\beta\Gamma; \Gamma\epsilon_k\Gamma) \cdot (\Gamma\epsilon_k\Gamma),$$

where $m(\Gamma\alpha\Gamma, \Gamma\beta\Gamma; \Gamma\epsilon_k\Gamma)$ denotes the number of elements in the set

$$\{(i, j) \mid \Gamma\alpha_i\beta_j = \Gamma\epsilon_k\}$$

for $1 \leq k \leq r$.

Now for each $\gamma \in \Gamma$ we have $\Gamma\alpha\Gamma\gamma = \Gamma\alpha\Gamma$, and therefore it follows that

$$\Gamma\alpha\Gamma = \coprod_{1\le i\le d} \Gamma\alpha_i = \coprod_{1\le i\le d} \Gamma\alpha_i\gamma.$$

Thus for $1 \le i \le d$, we see that

$$\alpha_i\gamma = \xi_i(\gamma)\cdot\alpha_{i(\gamma)} \tag{3.2}$$

for some element $\xi_i(\gamma) \in \Gamma$, where $(\alpha_{1(\gamma)}, \dots, \alpha_{d(\gamma)})$ is a permutation of $(\alpha_1, \dots, \alpha_d)$.

Lemma 3.1. *For $1 \le i \le d$, we have*

$$i(\gamma\gamma') = i(\gamma)(\gamma'), \quad \xi_i(\gamma\gamma') = \xi_i(\gamma)\cdot\xi_{i(\gamma)}(\gamma') \tag{3.3}$$

for all $\gamma, \gamma' \in \Gamma$.

Proof. For each i and $\gamma, \gamma' \in \Gamma$ we see that

$$\begin{aligned}(\alpha_i\gamma)\gamma' &= \xi_i(\gamma)\cdot\alpha_{i(\gamma)}\gamma'\\ &= \xi_i(\gamma)\cdot\xi_{i(\gamma)}(\gamma')\cdot\alpha_{i(\gamma)(\gamma')}.\end{aligned}$$

Therefore the lemma follows by comparing this with $\alpha_i(\gamma\gamma') = \xi_i(\gamma\gamma')\alpha_{i(\gamma\gamma')}$. $\square$

Given a nonnegative integer p and a Γ-module $\mathcal{M}$, let $C^p(\Gamma, \mathcal{M})$ be the complex vector space described in Section 2. For an element $c \in C^p(\Gamma, \mathcal{M})$ and a double coset $\Gamma\alpha\Gamma$ with $\alpha \in \widetilde{\Gamma}$ that has a decomposition as in (3.1), we consider the map $c' : \Gamma^{p+1} \to \mathcal{M}$ given by

$$c'(\gamma_0, \dots, \gamma_p) = \sum_{i=1}^{d} \alpha_i^{-1}\cdot c(\xi_i(\gamma_0), \dots, \xi_i(\gamma_p)),$$

where the maps $\xi_i : \Gamma \to \Gamma$ are determined by (3.2).

Lemma 3.2. *The map c' is an element of $C^p(\Gamma, \mathcal{M})$.*

Proof. It suffices to show that c' satisfies the condition (2.1). Using (3.3) and the fact that c satisfies (2.1), we obtain

$$\begin{aligned}c'(\gamma\gamma_0, \dots, \gamma\gamma_p) &= \sum_{i=1}^{d} \alpha_i^{-1}\cdot c(\xi_i(\gamma\gamma_0), \dots, \xi_i(\gamma\gamma_p))\\ &= \sum_{i=1}^{d} \alpha_i^{-1}\cdot c(\xi_i(\gamma)\xi_{i(\gamma)}(\gamma_0), \dots, \xi_i(\gamma)\xi_{i(\gamma)}(\gamma_p))\\ &= \sum_{i=1}^{d} \alpha_i^{-1}\cdot\xi_i(\gamma)\cdot c(\xi_{i(\gamma)}(\gamma_0), \dots, \xi_{i(\gamma)}(\gamma_p))\end{aligned}$$

for $\gamma, \gamma_0, \dots, \gamma_p \in \Gamma$. Since we have $\alpha_i^{-1} \cdot \xi_i(\gamma) = \gamma \cdot \alpha_{i(\gamma)}^{-1}$ by (3.2), we see that

$$\begin{aligned} c'(\gamma\gamma_0, \dots, \gamma\gamma_p) &= \gamma \cdot \sum_{i=1}^{d} \alpha_{i(\gamma)}^{-1} \cdot c(\xi_{i(\gamma)}(\gamma_0), \dots, \xi_{i(\gamma)}(\gamma_p)) \\ &= \gamma \cdot c'(\gamma_0, \dots, \gamma_p); \end{aligned}$$

hence the lemma follows. □

By Lemma 3.2 each double coset $\Gamma\alpha\Gamma$ with $\alpha \in \widetilde{\Gamma}$ determines the $\mathbb{C}$-linear map

$$T_{\Gamma\alpha\Gamma} : C^p(\Gamma, \mathcal{M}) \to C^p(\Gamma, \mathcal{M})$$

defined by

$$T_{\Gamma\alpha\Gamma}(c)(\gamma_0, \dots, \gamma_p) = \sum_{i=1}^{d} \alpha_i^{-1} \cdot c(\xi_i(\gamma_0), \dots, \xi_i(\gamma_p))$$

for $c \in C^p(\Gamma, \mathcal{M})$, where $\Gamma\alpha\Gamma = \coprod_{1 \le i \le d} \Gamma\alpha_i$ and each ξ_i is as in (3.2).

Lemma 3.3. *The map $T_{\Gamma\alpha\Gamma}$ is independent of the choice of representatives of the coset decomposition of $\Gamma\alpha\Gamma$ modulo Γ.*

Proof. For $\alpha \in \widetilde{\Gamma}$ suppose that

$$\Gamma\alpha\Gamma = \coprod_{1 \le i \le d} \Gamma\alpha_i = \coprod_{1 \le i \le d} \Gamma\beta_i.$$

Then we may assume that $\beta_i = \gamma(\alpha_i)\alpha_i$ for each i with $\gamma(\alpha_i) \in \Gamma$. If ξ_i is as in (3.2), we have

$$\beta_i\gamma = \gamma(\alpha_i)\alpha_i\gamma = \gamma(\alpha_i)\xi_i(\gamma) \cdot \alpha_i(\gamma)$$

for all $\gamma \in \Gamma$. Hence, using (2.1), we see that

$$\begin{aligned} T_{\Gamma\beta\Gamma}(c)(\gamma_0, \dots, \gamma_p) &= \sum_{i=1}^{d} \beta_i^{-1} \cdot c(\gamma(\alpha_i)\xi_i(\gamma_0), \dots, \gamma(\alpha_i)\xi_i(\gamma_p)) \\ &= \sum_{i=1}^{d} \alpha_i^{-1} \cdot \gamma(\alpha_i)^{-1} \cdot c(\gamma(\alpha_i)\xi_i(\gamma_0), \dots, \gamma(\alpha_i)\xi_i(\gamma_p)) \\ &= \sum_{i=1}^{d} \alpha_i^{-1} \cdot c(\xi_i(\gamma_0), \dots, \xi_i(\gamma_p)) \\ &= T_{\Gamma\alpha\Gamma}(c)(\gamma\gamma_1, \dots, \gamma\gamma_p) \end{aligned}$$

for $c \in C^p(\Gamma, \mathcal{M})$ and $(\gamma_0, \dots, \gamma_p) \in \Gamma^{p+1}$. Therefore it follows that $T_{\Gamma\alpha\Gamma} = T_{\Gamma\beta\Gamma}$, and the proof of the lemma is complete. □

Let $q \in \mathbb{C}$ be an N-th root of unity with $q \neq 1$ for some positive integer N, and let $d_q : C^p(\Gamma, \mathcal{M}) \to C^{p+1}(\Gamma, \mathcal{M})$ be the map defined by (2.4).

Proposition 3.4. *Given a double coset $\Gamma\alpha\Gamma$ with $\alpha \in \widetilde{\Gamma}$, we have*

$$T_{\Gamma\alpha\Gamma} \circ (d_q)^k = (d_q)^k \circ T_{\Gamma\alpha\Gamma}$$

for $1 \leq k \leq N$.

Proof. It suffices to show that $T_{\Gamma\alpha\Gamma}$ commutes with d_q for each double coset $\Gamma\alpha\Gamma$. For $c \in C^p(\Gamma, \mathcal{M})$ and $(\gamma_0, \dots, \gamma_p) \in \Gamma^{p+1}$, assuming that $\Gamma\alpha\Gamma$ has a decomposition as in (3.1) we have

$$\begin{aligned}(d_q \circ T_{\Gamma\alpha\Gamma})(c)(\gamma_0, \dots, \gamma_p) &= \sum_{k=0}^{p+1} q^k \cdot \delta_k(T_{\Gamma\alpha\Gamma}(c))(\gamma_0, \dots, \gamma_p) \\ &= \sum_{k=0}^{p+1} q^k \cdot T_{\Gamma\alpha\Gamma}(\delta_k c)(\gamma_0, \dots, \gamma_p) \\ &= \sum_{k=0}^{p+1} \sum_{i=1}^{d} q^k \cdot \alpha_i^{-1} \cdot (\delta_k c)(\xi_i(\gamma_0), \dots, \xi_i(\gamma_p)),\end{aligned}$$

where δ_k is as in (2.2). On the other hand, we have

$$\begin{aligned}(T_{\Gamma\alpha\Gamma} \circ d_q)(c)(\gamma_0, \dots, \gamma_p) &= \sum_{i=1}^{d} \alpha_i^{-1} \cdot d_q(c)(\xi_i(\gamma_0), \dots, \xi_i(\gamma_p)) \\ &= \sum_{i=1}^{d} \sum_{k=0}^{p+1} \alpha_i^{-1} \cdot q^k \cdot (\delta_k c)(\xi_i(\gamma_0), \dots, \xi_i(\gamma_p)).\end{aligned}$$

Since α_i^{-1} commutes with q^k, we have $T_{\Gamma\alpha\Gamma} \circ d_q = d_q \circ T_{\Gamma\alpha\Gamma}$, and hence the lemma follows. □

It follows from Proposition 3.4 that

$$T_{\Gamma\alpha\Gamma}(\mathrm{Im}(d_q)^{N-k}) \subset \mathrm{Im}(d_q)^{N-k}$$

for $1 \leq k \leq N$. Thus by (2.6) the map $T_{\Gamma\alpha\Gamma} : C^p(\Gamma, \mathcal{M}) \to C^p(\Gamma, \mathcal{M})$ induces the $\mathbb{C}$-linear map

$${}_kT_{\Gamma\alpha\Gamma} : {}_kH^p(\Gamma, \mathcal{M}) \to {}_kH^p(\Gamma, \mathcal{M})$$

on the p-th q-cohomology space ${}_kH^p(\Gamma, \mathcal{M})$ for each nonnegative integer p and $k = 1, \dots, N$. Let Δ be a semigroup with $\Gamma \subset \Delta \subset \widetilde{\Gamma}$, and consider the associated Hecke ring $R(\Gamma, \Delta)$ described above. Since $R(\Gamma, \Delta)$ is a free $\mathbb{Z}$-module generated by $\{\Gamma\alpha\Gamma \mid \alpha \in \Delta\}$, we can extend the maps ${}_kT_{\Gamma\alpha\Gamma}$ linearly to the entire $R(\Gamma, \Delta)$. Thus an element $X \in R(\Gamma, \Delta)$ of the form $X = \sum m_\alpha \cdot \Gamma\alpha\Gamma$ with $m_\alpha \in \mathbb{Z}$ determines the endomorphism

$$T_X = \sum m_\alpha \cdot {}_kT_{\Gamma\alpha\Gamma}$$

of ${}_kH^p(\Gamma, \mathcal{M})$.

Theorem 3.5. *For $1 \leq k \leq N$ and a nonnegative integer p, the operation of $R(\Gamma, \Delta)$ on ${}_kH^p(\Gamma, \mathcal{M})$ obtained by extending the maps*

$$\Gamma\alpha\Gamma \mapsto {}_kT_{\Gamma\alpha\Gamma}$$

linearly to the entire $R(\Gamma, \Delta)$ is a representation of the Hecke ring $R(\Gamma, \Delta)$ on the right in the p-th q-cohomology space ${}_kH^p(\Gamma, \mathcal{M})$ of Γ with coefficients in $\mathcal{M}$.

Proof. Let $\Gamma\alpha\Gamma$ and $\Gamma\beta\Gamma$ be elements of $R(\Gamma, \Delta)$ with decompositions of the form

$$\Gamma\alpha\Gamma = \coprod_{1 \leq i \leq d} \Gamma\alpha_i, \quad \Gamma\beta\Gamma = \coprod_{1 \leq j \leq e} \Gamma\beta_j,$$

and for each $\gamma \in \Gamma$ denote by $\xi_i(\gamma)$ and $\xi'_j(\gamma)$ the elements of Γ given by

$$\alpha_i\gamma = \xi_i(\gamma)\alpha_{i(\gamma)}, \quad \beta_j\gamma = \xi'_j(\gamma)\beta_{j(\gamma)}$$

for $1 \leq i \leq d$ and $1 \leq j \leq e$ such that $(\alpha_{1(\gamma)}, \dots, \alpha_{d(\gamma)})$ and $(\beta_{1(\gamma)}, \dots, \beta_{e(\gamma)})$ are permutations of $(\alpha_1, \dots, \alpha_d)$ and $(\beta_1, \dots, \beta_e)$, respectively. In order to prove the theorem it suffices to show that

$$T_{(\Gamma\alpha\Gamma)\cdot(\Gamma\beta\Gamma)}c = (T_{\Gamma\beta\Gamma} \circ T_{\Gamma\alpha\Gamma})c$$

for all $c \in C^p(\Gamma, \mathcal{M})$. Let Ω denote the set of elements of Δ such that $\{\Gamma\omega\Gamma \mid \omega \in \Omega\}$ is a complete set of distinct double cosets contained in $\Gamma\alpha\Gamma\beta\Gamma$. Given $\omega \in \Omega$, we assume that the corresponding double coset has a decomposition of the form

$$\Gamma\omega\Gamma = \coprod_{1 \leq k \leq s} \Gamma\omega_k,$$

and denote by $\xi''_k(\gamma) \in \Gamma$ the element given by $\omega_k\gamma = \xi''_k(\gamma)\omega_{k(\gamma)}$ for $\gamma \in \Gamma$ and $1 \leq k \leq s$ with $(\omega_{1(\gamma)}, \dots, \omega_{s(\gamma)})$ a permutation of $(\omega_1, \dots, \omega_s)$. Then we have

$$(\Gamma\alpha\Gamma) \cdot (\Gamma\beta\Gamma) = \sum_{\omega \in \Omega} m(\Gamma\alpha\Gamma, \Gamma\beta\Gamma; \Gamma\omega\Gamma) \cdot \Gamma\omega\Gamma,$$

with

$$m(\Gamma\alpha\Gamma, \Gamma\beta\Gamma; \Gamma\omega\Gamma) = \#\{(i,j) \mid \Gamma\alpha_i\beta_j = \Gamma\omega_k\},$$

where $\#$ denotes the number of elements. Thus, for $c \in C^p(\Gamma, \mathcal{M})$ and $(\gamma_0, \dots, \gamma_p) \in \Gamma^{p+1}$, we have

$$\begin{aligned} T_{(\Gamma\alpha\Gamma)\cdot(\Gamma\beta\Gamma)}(c)(\gamma_0, \dots, \gamma_p) &= \sum_{\omega \in \Omega} m(\Gamma\alpha\Gamma, \Gamma\beta\Gamma; \Gamma\omega\Gamma) \cdot T_{\Gamma\omega\Gamma}(c)(\gamma_0, \dots, \gamma_p) \\ &= \sum_{\omega \in \Omega} m(\Gamma\alpha\Gamma, \Gamma\beta\Gamma; \Gamma\omega\Gamma) \cdot \sum_{k=1}^{s} \omega_k^{-1} \cdot c(\xi''_k(\gamma_0), \dots, \xi''_k(\gamma_p)). \end{aligned}$$

Note, however, that there is an element $\mu_k \in \Gamma$ such that $\Gamma\omega_k = \Gamma\omega\mu_k$ for each $k \in \{1, \dots, s\}$. Thus, using the fact that

$$\alpha_i \beta_j \mu_k = \alpha_i \cdot \xi'_j(\mu_k) \cdot \beta_{j(\mu_k)} = \xi_i(\xi'_j(\mu_k)) \cdot \alpha_{i(\xi'_j(\mu_k))} \cdot \beta_{j(\mu_k)},$$

we see that $\Gamma\alpha_i\beta_j = \Gamma\omega$ if and only if $\Gamma\alpha_{i'}\beta_{j'} = \Gamma\omega_k$, where $i' = i(\xi'_j(\mu_k))$ and $j' = j(\mu_k)$. Hence it follows that

$$m(\Gamma\alpha\Gamma, \Gamma\beta\Gamma; \Gamma\omega\Gamma) = \#\{(i,j) \mid \Gamma\alpha_i\beta_j = \Gamma\omega_k\}$$

for each k. Using this and the fact that

$$\alpha_i\beta_j\gamma = \xi_i(\xi'_j(\gamma)) \cdot \alpha_{i(\xi'_j(\gamma))} \cdot \beta_{j(\gamma)}$$

for $\gamma \in \Gamma$, we obtain

$$\begin{aligned} T_{(\Gamma\alpha\Gamma)\cdot(\Gamma\beta\Gamma)}(c)(\gamma_0, \dots, \gamma_p) &= \sum_{\omega \in \Omega} \sum_{k=1}^{s} m(\Gamma\alpha\Gamma, \Gamma\beta\Gamma; \Gamma\omega_k\Gamma) \cdot \omega_k^{-1} \cdot c(\xi''_k(\gamma_0), \dots, \xi''_k(\gamma_p)) \\ &= \sum_{i=1}^{d} \sum_{j=1}^{e} (\alpha_i\beta_j)^{-1} c(\xi_i(\xi'_j(\gamma_0)), \dots, \xi_i(\xi'_j(\gamma_p))) \\ &= \sum_{i=1}^{d} \sum_{j=1}^{e} \beta_j^{-1} \cdot \alpha_i^{-1} \cdot c(\xi_i(\xi'_j(\gamma_0)), \dots, \xi_i(\xi'_j(\gamma_p))) \\ &= \sum_{j=1}^{e} \beta_j^{-1} \cdot T_{\Gamma\alpha\Gamma}(c)(\xi'_j(\gamma_0), \dots, \xi'_j(\gamma_p)) \\ &= (T_{\Gamma\beta\Gamma} \circ T_{\Gamma\alpha\Gamma})(c)(\gamma_0, \dots, \gamma_p). \end{aligned}$$

Therefore, it follows that

$$T_{(\Gamma\alpha\Gamma)\cdot(\Gamma\beta\Gamma)} = T_{\Gamma\beta\Gamma} \circ T_{\Gamma\alpha\Gamma},$$

and the proof of the theorem is complete. □

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