Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 42 (2001), No. 1, 165-178.

A Series of Kählerian Invariants and Their Applications to Kählerian Geometry

Dedicated to Professor David E. Blair on his 60th birthday

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Abstract. We introduce a series of invariants on Kähler manifolds and prove a series of general inequalities involving these invariants for Kähler submanifolds in complex space forms. We also determine Kähler submanifolds in complex space forms which satisfy the equality cases of these inequalities.

MSC 2000: 53C40, 53C42 (primary); 53C55 (secondary)

1. Introduction

Let M^n be a Kähler manifold of complex dimension n. Denote by J the complex structure on Kähler manifolds. For each plane section $\pi \subset T_xM$, $x \in M$, we denote by $K(\pi)$ the sectional curvature of the plane section π. Let $e_1, \dots, e_n,\ e_{1^*} = Je_1, \dots,\ e_{n^*} = Je_n$ be a field of orthonormal frames on M. Then the scalar curvature τ of M is defined by

$$\tau = \sum_{i<j} K(e_i, e_j), \quad i, j = 1, \dots, n, 1^*, \dots, n^*, \tag{1.1}$$

where $K(e_i, e_j)$ is the sectional curvature of the section spanned by e_i and e_j.

A plane section $\pi \subset T_xM$ is called *totally real* if $J\pi$ is perpendicular to π. For each real number k we define an invariant δ_k^r by

$$\delta_k^r(x) = \tau(x) - k \inf K^r(x), \quad x \in M, \tag{1.2}$$

0138-4821/93 \$ 2.50

where $\inf K^r(x) = \inf_{\pi^r}\{K(\pi^r)\}$ and π^r runs over all totally real plane sections in T_xM. (This type of invariants is similar to the invariants introduced in [3,4,5]. For some recent results involving this type of invariants, see for instance [6,8,9,12]).

A Kähler manifold $\tilde{M}^m(4c)$ of constant holomorphic sectional curvature $4c$ is called a *complex space form.* There are three types of complex space forms: *elliptic, hyperbolic,* or *flat* according as the holomorphic sectional curvature is positive, negative, or zero.

Let $CP^m(4c)$ be a complex projective m-space endowed with the Fubini-Study metric of constant holomorphic sectional curvature $4c$. Then $CP^m(4c)$ is a complete and simply-connected elliptic complex space form.

Complex Euclidean space $\mathbf{C}^m$ endowed with the usual Hermitian metric is a complete and simply-connected flat complex space form.

Let D_m be the open unit ball in $\mathbf{C}^m$ endowed with the natural complex structure and the Bergman metric of constant holomorphic sectional curvature $4c$, $c < 0$. Then D_m is a complete and simply-connected hyperbolic complex space form.

By a *Kähler submanifold* of a Kähler manifold we mean a complex submanifold with the induced Kähler structure [7,10]. For a Kähler submanifold M^n of a Kähler manifold $\tilde{M}^{n+p}$ we denote by h and A the second fundamental form and the shape operator of M^n in $\tilde{M}^{n+p}$, respectively. For the Kähler submanifold we consider an orthonormal frame $e_1, \ldots, e_n,\ e_{1^*} = Je_1, \ldots,\ e_{n^*} = Je_n$ of the tangent bundle and an orthonormal frame $\xi_1, \ldots, \xi_p,\ \xi_{1^*} = J\xi_1, \ldots,\ \xi_{p^*} = J\xi_p$ of the normal bundle.

With respect to such an orthonormal frame, the complex structure J on M is given by

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \tag{1.3}$$

where I_n denotes an identity matrix of degree n.

For a Kähler submanifold M^n in $\tilde{M}^{n+p}$ the shape operator of M^n satisfies

$$A_{J\xi_r} = JA_r, \quad JA_r = -A_rJ, \quad \text{for } r = 1, \ldots, n, 1^*, \ldots, p^*, \tag{1.4}$$

where $A_r = A_{\xi_r}$. From (1.3) and (1.4) it follows that the shape operator of M^n takes the form:

$$A_\alpha = \begin{pmatrix} A'_\alpha & A''_\alpha \\ A''_\alpha & -A'_\alpha \end{pmatrix}, \quad A_{\alpha^*} = \begin{pmatrix} -A''_\alpha & A'_\alpha \\ A'_\alpha & A''_\alpha \end{pmatrix}, \quad \alpha = 1, \ldots, p, \tag{1.5}$$

where A'_α and A''_α are $n \times n$ matrices. Condition (1.5) implies that every Kähler submanifold M^n is minimal, *i.e.,* $\operatorname{trace} A_\alpha = \operatorname{trace} A_{\alpha^*} = 0$, $\alpha = 1, \ldots, p$.

Now we introduce the notion of strongly minimal Kähler submanifolds.

Definition 1. *A Kähler submanifold M^n of a Kähler manifold $\tilde{M}^{n+p}$ is called strongly minimal if it satisfies*

$$\operatorname{trace} A'_\alpha = \operatorname{trace} A''_\alpha = 0, \quad \text{for } \alpha = 1, \ldots, p,$$

with respect to some orthonormal frame: $e_1, \ldots, e_n,\ e_{1^*} = Je_1, \ldots,\ e_{n^*} = Je_n,\ \xi_1, \ldots, \xi_p,$ $\xi_{1^*} = J\xi_1, \ldots, \xi_{p^*} = J\xi_p$.

The main purpose of this paper is to prove the following.

Theorem 1. *For any Kähler submanifold M^n of complex dimension $n \geq 2$ in a complex space form $\tilde{M}^{n+p}(4c)$, we have*

$$\inf K^r \leq c. \tag{1.6}$$

The equality case of (1.6) *holds identically if and only if M^n is a totally geodesic Kähler submanifold.*

Theorem 2. *For any Kähler submanifold M^n of complex dimension $n \geq 2$ in a complex space form $\tilde{M}^{n+p}(4c)$, the following statements hold.*

(1) *For each $k \in (-\infty, 4]$, δ_k^r satisfies*

$$\delta_k^r \leq (2n^2 + 2n - k)c. \tag{1.7}$$

(2) *Inequality* (1.7) *fails for every $k > 4$.*
(3) *$\delta_k^r = (2n^2 + 2n - k)c$ holds identically for some $k \in (-\infty, 4)$ if and only if M^n is a totally geodesic Kähler submanifold of $\tilde{M}^{n+p}(4c)$.*
(4) *The Kähler submanifold M^n satisfies $\delta_4^r = (2n^2 + 2n - 4)c$ at a point $x \in M^n$ if and only if there exists an orthonormal basis*

$$e_1, \ldots, e_n,\ e_{1^*} = Je_1, \ldots, e_{n^*} = Je_n,\ \xi_1, \ldots, \xi_p,\ \xi_{1^*} = J\xi_1, \ldots, \xi_{p^*} = J\xi_p$$

of $T_x\tilde{M}^{n+p}(4c)$ such that, with respect to this basis, the shape operator of M^n takes the following form:

$$A_\alpha = \begin{pmatrix} A'_\alpha & A''_\alpha \\ & \\ A''_\alpha & -A'_\alpha \end{pmatrix}, \quad A_{\alpha^*} = \begin{pmatrix} -A''_\alpha & A'_\alpha \\ & \\ A'_\alpha & A''_\alpha \end{pmatrix}, \tag{1.8}$$

$$A'_\alpha = \begin{pmatrix} \begin{matrix} a_\alpha & b_\alpha \\ b_\alpha & -a_\alpha \end{matrix} & 0 \\ 0 & 0 \end{pmatrix}, \quad A''_\alpha = \begin{pmatrix} \begin{matrix} a^*_\alpha & b^*_\alpha \\ b^*_\alpha & -a^*_\alpha \end{matrix} & 0 \\ 0 & 0 \end{pmatrix} \tag{1.9}$$

for some $n \times n$ matrices A'_α, A''_α, $\alpha = 1, \ldots, p$.

Theorem 3. *A complete Kähler submanifold M^n $(n \geq 2)$ in $CP^{n+p}(4c)$ satisfies*

$$\delta_4^r = 2(n^2 + n - 2)c \tag{1.10}$$

identically if and only if

(1) *M^n is a totally geodesic Kähler submanifold, or*
(2) *$n = 2$ and M^2 is a strongly minimal Kähler surface in $CP^{2+p}(c)$.*

Theorem 4. *A complete Kähler submanifold M^n $(n \geq 2)$ of $\mathbf{C}^{n+p}$ satisfies $\delta_4^r = 0$ identically if and only if*

(1) *M^n is a complex n-plane of $\mathbf{C}^{n+p}$, or*
(2) *M^n is a complex cylinder over a strongly minimal Kähler surface M^2 in $\mathbf{C}^{n+p}$ (i.e., M is the product submanifold of a strongly minimal Kähler surface M^2 in $\mathbf{C}^{p+2}$ and the identity map of the complex Euclidean $(n-2)$-space $\mathbf{C}^{n-2}$).*

In Section 5 we provide some nontrivial examples of strongly minimal Kähler submanifolds in complex space forms. In the last section we show that every strongly minimal Kähler surface in complex space form is framed-Einstein.

2. Proof of Theorem 1

For each nonzero tangent vector X of M^n we denote by $H(X)$ the holomorphic sectional curvature of X, that is, $H(X)$ is the sectional curvature of the plane section spanned by X and JX. From the definitions of sectional and holomorphic sectional curvatures, we have (see [2, p.517])

$$\begin{aligned} K(X,Y) + K(X,JY) = \frac{1}{4}\Big\{&H(X+JY) + H(X-JY) \\ &+ H(X+Y) + H(X-Y) - H(X) - H(Y)\Big\}, \end{aligned} \tag{2.1}$$

for orthonormal vectors X and Y with $g(X,JY)=0$.

Let T^1M^n denote the unit sphere bundle of M consisting of all unit tangent vectors on M. For each $x \in M^n$, we put

$$W_x = \Big\{(X,Y) : X, Y \in T_x^1M^n \text{ such that } g(X,Y) = g(X,JY) = 0\Big\}. \tag{2.2}$$

Then W_x is a closed subset of $T_x^1M^n \times T_x^1M^n$. It is easy to verify that if $\{X,Y\}$ spans a totally real plane section, then both $\{X+JY, X-JY\}$ and $\{X+Y, X-Y\}$ also span totally real plane sections.

We define a function $\hat{H} : W_x \to \mathbf{R}$ by

$$\hat{H}(X,Y) = H(X) + H(Y), \quad (X,Y) \in W_x. \tag{2.3}$$

Suppose that (X_m, Y_m) is a point in W_x such that $\hat{H}$ attains an absolute maximum value, say m_x, at (X_m, Y_m). Then (2.1) implies

$$K(X_m, Y_m) + K(X_m, JY_m) \leq \frac{1}{4}\hat{H}(X_m, Y_m). \tag{2.4}$$

On the other hand, it is known that every holomorphic sectional curvature $H(X)$ of a Kähler submanifold M^n in complex space form $\tilde{M}^{n+p}(4c)$ satisfies $H(X) \leq 4c$ (cf. [10]). Thus, we obtain from (2.4) that

$$K(X_m, Y_m) + K(X_m, JY_m) \leq 2c,$$

which implies inequality (1.6).

Now, suppose that the equality case of (1.6) holds identically on M^n. Then (2.1) gives

$$\begin{aligned} H(X+JY)+&H(X-JY)+H(X+Y)+H(X-Y) \\ &-H(X)-H(Y)\geq 8c, \end{aligned} \tag{2.5}$$

for any orthonormal vectors X, Y with $g(X, JY) = 0$. Put

$$H_1 = H(X+JY)+H(X-JY), \quad H_2 = H(X+Y)+H(X-Y), \\ H_3 = H(X)+H(Y).$$

Case (a). $H_3 \geq H_1, H_2$.
In this case, (2.5) implies

$$H(X+Y)+H(X-Y) \geq 8c. \tag{2.6}$$

Combining this with $H \leq 4c$, we obtain

$$H(X+Y) = H(X-Y) = 4c, \tag{2.7}$$

for any orthonormal vectors X, Y with $g(X, JY) = 0$. Since every tangent vector of a Kähler manifold M^n with $n \geq 2$ must lie in a totally real 2-plane, every nonzero vector in T_xM^n can be expressed as the sum of two orthonormal vectors X, Y with $g(X, JY) = 0$. Therefore, from (2.7) we conclude that M^n has constant holomorphic sectional curvature $4c$. Therefore, M^n is a totally geodesic Kähler submanifold in $\tilde{M}^{n+p}(4c)$.

Case (b). $H_2 \geq H_1, H_3$.
In this case, after replacing X and Y by $(X+Y)/2$ and $(X-Y)/2$ respectively, we obtain from (2.5) that

$$H(X+Y+JX+JY)+H(X+Y-JX-JY)+H_3-H_2 \geq 8c, \tag{2.8}$$

Since $H_2 \geq H_3$ and $H \leq 4c$, we obtain

$$8c \geq H(X+Y+JX+JY)+H(X+Y-JX-JY) \geq 8c. \tag{2.9}$$

Consequently, we have

$$H(X+Y+JX+JY) = H(X+Y-JX-JY) = 4c$$

for any orthonormal vectors X, Y with $g(X, JY) = 0$. Since every nonzero tangent vector can be expressed as $X+Y+J(X+Y)$ for some orthonormal vectors X, Y with $g(X, JY) = 0$, we conclude that M^n has constant holomorphic sectional curvature $4c$. Hence, the immersion of M^n is totally geodesic, too.

Case (c). $H_1 \geq H_2, H_3$.

This case can be proved in the same way as case (b).

Consequently, in all of three cases, the equality of (1.6) implies that M^n is a totally geodesic Kähler submanifold.

Conversely, if M^n is a totally geodesic Kähler submanifold of $\tilde{M}^{n+p}(4c)$, then M^n has constant holomorphic sectional curvature $4c$; and thus it has constant totally real sectional curvature c. In particular, we have $\inf K^r = c$. □

3. Proof of Theorem 2

Let M^n be a Kähler submanifold of complex dimension n in a complex space form $\tilde{M}^{n+p}(4c)$. Let R denote the Riemann curvature tensor of M. Then, by Gauss equation, we have

$$\begin{aligned} \langle R(X,Y)Z,W\rangle &= \langle h(X,W),h(Y,Z)\rangle - \langle h(X,Z),h(Y,W)\rangle \\ &+c\{\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle + \langle JY,Z\rangle\langle JX,W\rangle \\ &- \langle JX,Z\rangle\langle JY,W\rangle + 2\langle X,JY\rangle\langle JZ,W\rangle\}. \end{aligned} \tag{3.1}$$

Since every Kähler submanifold of a Kähler manifold is minimal, Gauss's equation implies that the scalar curvature of M^n satisfies

$$2\tau = 4n(n+1)c - ||h||^2, \tag{3.2}$$

where $||h||^2$ is the squared norm of the second fundamental form. From (3.2) we obtain

$$\tau \leq (2n^2+2n)c, \tag{3.3}$$

with the equality holding if and only if M^n is a totally geodesic Kähler submanifold.

Now, suppose that π is a given totally real plane section $\pi \subset T_xM$. We choose an orthonormal basis $e_1,\dots,e_n,e_{1^*},\dots,e_{n^*},\xi_1,\dots,\xi_p,\xi_{1^*},\dots,\xi_{p^*}$ such that $\pi = \mathrm{Span}\,\{e_1,e_2\}$. With respect to such a basis, we have

$$A_\alpha = \begin{pmatrix} A'_\alpha & A''_\alpha \\ A''_\alpha & -A'_\alpha \end{pmatrix}, \quad A_{\alpha^*} = \begin{pmatrix} -A''_\alpha & A'_\alpha \\ A'_\alpha & A''_\alpha \end{pmatrix}, \quad \alpha = 1,\dots,p, \tag{3.4}$$

where A'_α and A''_α are $n \times n$ matrices. Applying (3.1), (3.2) and (3.4) we have

$$\begin{aligned} 4n(n+1)c - 2\tau &= 4\sum_{\alpha=1}^{p}\{||A'_\alpha||^2 + ||A''_\alpha||^2\} \\ &\geq 4\sum_{\alpha=1}^{p}\Big\{(h_{11}^\alpha)^2 + (h_{22}^\alpha)^2 + 2(h_{12}^\alpha)^2 + (h_{11}^{\alpha^*})^2 + (h_{22}^{\alpha^*})^2 + 2(h_{12}^{\alpha^*})^2\Big\} \\ &\geq -8\sum_{\alpha=1}^{p}\Big\{h_{11}^\alpha h_{22}^\alpha - (h_{12}^\alpha)^2 + h_{11}^{\alpha^*}h_{22}^{\alpha^*} - (h_{12}^{\alpha^*})^2\Big\} \\ &= -8K(\pi) + 8c. \end{aligned} \tag{3.5}$$

From (3.5) we obtain

$$\tau - 4\,K(\pi) \le (2n^2 + 2n - 4)c. \tag{3.6}$$

Since inequality (3.6) holds for any totally real plane sections, we get

$$\tau - 4\,\inf K^r \le (2n^2 + 2n - 4)c. \tag{3.7}$$

From (3.2) and (3.7) we obtain, for any positive number p, that

$$(p+1)\tau - 4\,\inf\,K^r \le \big\{(p+1)(2n^2 + 2n) - 4\big\}\,c. \tag{3.8}$$

Therefore, we obtain

$$\delta_k^r \le (2n^2 + 2n - k)c, \tag{3.9}$$

for any $k \in (0,4)$. Combining (3.3), (3.7) and (3.9) we obtain inequality (1.7) for $k \in [0,4]$. Inequality (1.7) with $k < 0$ follows from (3.3) and Theorem 1.

For statement (2), we consider the complex quadric Q_2 in $CP^3(4c)$ defined by

$$Q_2 = \Big\{(z_0, z_1, z_2, z_3) \in CP^3(4c) : z_0^2 + z_1^2 + z_2^2 + z_3^2 = 0\Big\}, \tag{3.10}$$

where $\{z_0, z_1, z_2\}$ is a homogeneous coordinate system of $CP^3(4c)$. It is well-known that the scalar curvature τ and $\inf\,K^r$ of Q_2 are given by $\tau = 8c, \inf\,K^r = 0$. Thus we have

$$\delta_k^r = 8c \tag{3.11}$$

for any k. Since $(2n^2+2n-k)c = (12-k)c$ for $n = 2$, (3.10) implies $\delta_k^r > (2n^2+2n-k)c$ for $k > 4$. Hence, inequality (1.7) fails for each $k > 4$.

In order to prove statement (3), let us assume M^n is a Kähler submanifold of $\tilde{M}^{n+p}(4c)$ satisfying $\delta_k^r = (2n^2 + 2n - k)c$ identically for some $k \in (-\infty, 4)$. We divide the proof into three cases:

If $\delta_0^r = (2n^2 + 2n)c$, then (3.2) implies that M^n is totally geodesic.

If $\delta_k^r = (2n^2 + 2n - k)c$ for some $k \in (0,4)$, then (1.7) and the definition of δ_k^r yield

$$(2n^2 + 2n - k)c = \Big(1 - \frac{k}{4}\Big)\delta_0^r + \frac{k}{4}\delta_4^r \le (2n^2 + 2n - k)c, \tag{3.12}$$

which implies in particular that $\delta_0^r = (2n^2 + 2n)c$. Therefore, M is a totally geodesic Kähler submanifold.

If $\delta_k^r = (2n^2 + 2n - k)c$ for some $k \in (-\infty, 0)$, then, (3.3), together with the definition of δ_k^r, and Theorem 1 imply

$$(2n^2 + 2n - k)c = \tau - k\,\inf K^r \le (2n^2 + 2n - k)c. \tag{3.13}$$

In particular, this gives $\delta_0^r = (2n^2 + 2n)c$. Therefore, M is totally geodesic.
Conversely, it is easy to verify that every totally geodesic Kähler submanifold of $\tilde{M}^{n+p}(4c)$ satisfies $\delta_k^r = (2n^2 + 2n - k)c$ identically for any k.

For the proof of statement (4), we assume M^n is a Kähler submanifold satisfying $\delta_4^r = (2n^2 + 2n - 4)c$. From the proof of statement (1), we know that the inequalities in (3.5) become equalities. Thus, the second fundamental form of M^n must satisfy

$$h_{11}^r + h_{22}^r = 0, \quad h_{1j}^r = h_{2j}^r = h_{jk}^r = 0,$$
$$r = 1, \dots, p, 1^*, \dots, p^*, \quad j, k = 3, \dots, n.$$

From this we conclude that the shape operator of M^n takes the form (1.8 - 1.9), with respect to some orthonormal basis $e_1, \dots, e_n, Je_1, \dots, Je_n, \xi_1, \dots, \xi_p, J\xi_1, \dots, J\xi_p$.

Conversely, suppose the shape operator at a point $x \in M^n$ takes the form (1.8-1.9) with respect to some orthonormal basis $e_1, \dots, e_m, Je_1, \dots, Je_n, \xi_1, \dots, \xi_p, J\xi_1, \dots, J\xi_p$. Then the equation of Gauss implies $\inf K^r = K(e_1, e_2)$. Moreover, from (1.8-1.9) and (3.2), we also have

$$4n(n+1)c - 2\tau = 8 \sum_{\alpha=1}^{p} \left\{ a_\alpha^2 + b_\alpha^2 + a_\alpha^{*\,2} + b_\alpha^{*\,2} \right\} = -8K(e_1, e_2) + 8c.$$

Therefore, we obtain $\delta_4^r = (2n^2 + 2n - 4)c$. □

4. Proofs of Theorems 3 and 4

Assume $M^n\,(n \geq 2)$ is a complete Kähler submanifold of $CP^{n+p}(4c)$ which satisfies $\delta_4^r = 2(n^2 + n - 2)c$ identically. Then from Theorem 2 we know that the shape operator of M^n in $CP^{n+p}(4c)$ takes the form (1.8-1.9) with respect to some orthonormal frame $e_1, \dots, e_n, Je_1, \dots, Je_n, \xi_1, \dots, \xi_p, J\xi_1, \dots, J\xi_p$.

Recall that the relative nullity space RN_x of M^n is defined by

$$RN_x = \{X \in T_xM^n : h(X, Y) = 0, \text{ for all } Y \in T_xM^n\}. \tag{4.1}$$

The dimension $\mu(x)$ of RN_x is called the nullity at x. The subset G of M^n where $\mu(x)$ assumes the minimum, say μ, is open in M^n. The scalar μ is called the index of relative nullity of M^n. From (1.8-1.9) we obtain $\mu \geq 2n - 4$. Hence, Corollary 5 of [1, p.436] implies that M^n is a totally geodesic Kähler submanifold unless $n = 2$.

When $n = 2$, the shape operator of M^2 in $CP^{2+p}(4c)$ takes the form:

$$A_\alpha = \begin{pmatrix} A'_\alpha & A''_\alpha \\ & \\ A''_\alpha & -A'_\alpha \end{pmatrix}, \quad A_{\alpha^*} = \begin{pmatrix} -A''_\alpha & A'_\alpha \\ & \\ A'_\alpha & A''_\alpha \end{pmatrix}, \tag{4.2}$$

where

$$A'_\alpha = \begin{pmatrix} a_\alpha & b_\alpha \\ b_\alpha & -a_\alpha \end{pmatrix}, \quad A''_\alpha = \begin{pmatrix} a_\alpha^* & b_\alpha^* \\ b_\alpha^* & -a_\alpha^* \end{pmatrix}. \tag{4.3}$$

for some functions $a_\alpha, b_\alpha, a^*_\alpha, b^*_\alpha,\ \alpha = 1, \ldots, p$. This implies that M^2 is a strongly minimal Kähler surface.

Conversely, let us assume that M^n is either totally geodesic in $CP^{n+p}(4c)$ or a strongly minimal Kähler surface in $CP^{2+p}(4c)$.

If M^n is totally geodesic, then M^n is a $CP^n(4c)$. In this case, we have $\tau = (2n^2+2n)c$ and $\inf K^r = c$. Thus, $\delta^r_4 = 2(n^2+n-2)c$.

If M^n is a strongly minimal Kähler surface with $n = 2$, then statement (4) of Theorem 2 implies that M^2 satisfies (1.10) identically.

For the proof of Theorem 4, we assume M^n is a complete Kähler submanifold of $\mathbf{C}^{n+p}$ satisfying $\delta^r_4 = 0$ identically. Then Theorem 2 implies that the shape operator of M^n in $CP^{n+p}(4c)$ takes the form (1.7-1.8) with respect to some orthonormal frame $e_1, \ldots, e_m, Je_1, \ldots, Je_n, \xi_1, \ldots, \xi_p, J\xi_1, \ldots, J\xi_p$. Using (1.7-1.8) we obtain $\mu \geq 2n-4$, $(n \geq 2)$. Hence, by Theorem 7 of [1, p.439], M^n is a complex cylinder over a strongly minimal Kähler surface, unless M^n is totally geodesic.

The converse is easy to verify. □

5. Examples of strongly minimal Kähler submanifolds

Every totally geodesic Kähler submanifold of a complex space form is trivially strongly minimal. In this section we provide some nontrivial examples of strongly minimal Kähler submanifolds.

Consider the complex quadric Q_2 in $CP^3(4c)$ defined by (3.9). It is known that the scalar curvature τ of Q_2 equals to $8c$ and $\inf K^r = 0$. Thus, we obtain $\delta^r_4 = 8c$. Thus, Q_2 is a non-totally geodesic Kähler submanifold which satisfies (1.10) with $n = 2$. Therefore, according to Theorem 2, Q_2 is a strongly minimal Kähler surface in $CP^3(4c)$.

On the other hand, it is also well-known that Q_2 is an Einstein-Kähler surface with Ricci tensor $S = 4cg$, where g is the metric tensor of Q_2. Thus, the equation of Gauss and (1.5) yield

$$g(A_1^2X, Y) = cg(X, Y), \quad X, Y \in TQ_2. \tag{5.1}$$

Hence, with respect to a suitable choice of $e_1, e_2,\ Je_1, Je_2, \xi_1, J\xi_1$, we have

$$A_1 = \begin{pmatrix} A_1' & A_1'' \\ \\ A_1'' & -A_1' \end{pmatrix}, \quad A_{1^*} = \begin{pmatrix} -A_1'' & A_1' \\ \\ A_1' & A_1'' \end{pmatrix}, \tag{5.2}$$

where

$$A_1' = \begin{pmatrix} \sqrt{c} & 0 \\ 0 & -\sqrt{c} \end{pmatrix}, \quad A_1'' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.3}$$

This also shows that Q_2 is strongly minimal in $CP^3(4c)$.

The following proposition provides a nontrivial example of strongly minimal Kähler surface in $\mathbf{C}^3$.

Proposition 5. *Let N^2 be the complex surface in $\mathbf{C}^3$ defined by*

$$N^2 = \{z \in \mathbf{C}^3 : z_1^2 + z_2^2 + z_3^2 = 1\}. \tag{5.4}$$

Then M is a strongly minimal Kähler surface in $\mathbf{C}^3$.

Proof. Put $f(z) = z_1^2 + z_2^2 + z_3^2 - 1$. Then $\frac{\partial f}{\partial z} = \left(\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \frac{\partial f}{\partial z_3}\right)$ never vanishes on N^2. By differentiating $f(z) = 0$, we get $\frac{\partial f}{\partial z}(x) \cdot Z = 0$ for $Z \in T_x N^2$, $x \in N^2$. Thus, $\xi = (1/||\frac{\partial f}{\partial z}||)\overline{\frac{\partial f}{\partial z}}$ is a unit normal vector field on N^2. Hence, we get

$$g\left(Z, \overline{\frac{\partial f}{\partial z}}\right) = g\left(Z, i\overline{\frac{\partial f}{\partial z}}\right) = 0, \tag{5.5}$$

for $Z \in T_x N^2$, $x \in N^2$. At $x = (a_1 + ib_1, a_2 + ib_2, a_3 + ib_3) \in N^2$, we have

$$\overline{\frac{\partial f}{\partial z}}(x) = 2(a_1 - ib_1, a_2 - ib_2, a_3 - ib_3). \tag{5.6}$$

From these we see that the tangent space $T_x N^2$ is given by

$$\begin{aligned} T_x(N^2) = \Big\{ Z = (u_1 + iv_1, u_2 + iv_2, u_3 + iv_3) : &\sum_{j=1}^{3} (a_j u_j - b_j v_j) = 0 \\ \text{and } \sum_{j=1}^{3} (b_j u_j + a_j v_j) = 0 \Big\}&. \end{aligned} \tag{5.7}$$

If we put $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$, $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$, then (5.7) is equivalent to

$$T_x N^2 = \{Z = u + iv \in \mathbf{R}^3 \oplus i\mathbf{R}^3 : \langle a, u\rangle = \langle b, v\rangle, \langle b, u\rangle + \langle a, v\rangle = 0\}, \tag{5.8}$$

where $\langle\ ,\ \rangle$ denotes the Euclidean inner product on $\mathbf{R}^3$. Clearly, the condition $x \in N^2$ is equivalent to

$$|a|^2 - |b|^2 = 1, \quad \langle a, b\rangle = 0. \tag{5.9}$$

The covariant derivative of the unit normal vector $\xi = (1/||\frac{\partial f}{\partial z}||)\overline{\frac{\partial f}{\partial z}}$ with respect to a tangent vector W is given by

$$\tilde\nabla_W \xi = ||\tfrac{\partial f}{\partial z}||^{-1} \tilde\nabla_W \overline{\tfrac{\partial f}{\partial z}} + \left(D_W ||\tfrac{\partial f}{\partial z}||^{-1}\right) \overline{\tfrac{\partial f}{\partial z}}, \tag{5.10}$$

where $D_W ||\frac{\partial f}{\partial z}||^{-1}$ is the directional derivative of $||\frac{\partial f}{\partial z}||^{-1}$ with respect to vector W. Because

$$\tilde\nabla_W \overline{\tfrac{\partial f}{\partial z}} = \bar W \left(\overline{\tfrac{\partial^2 f}{\partial z_j \partial z_k}}\right), \tag{5.11}$$

and $\overline{\frac{\partial f}{\partial z}}$ is a normal vector field on N^2, (5.10) implies that the shape operator A_ξ of N^2 in $\mathbf{C}^3$ satisfies (cf. for instance [11])

$$A_\xi(W) = -||\tfrac{\partial f}{\partial z}||^{-1}\Big\{\bar{W}\Big(\overline{\tfrac{\partial^2 f}{\partial z_j \partial z_k}}\Big)\Big\}^{tan}, \tag{5.12}$$

where $\{*\}^{tan}$ is the tangential component of $\{*\}$. Hence

$$A_\xi(W) = -2\,||\tfrac{\partial f}{\partial z}||^{-1}\bar{W}^{tan}, \quad W \in T_xN^2. \tag{5.13}$$

Let $X = \alpha + i\beta$ and $Y = \gamma + i\delta$ be vectors in T_xN^2. Then $g(X,Y) = g(X,iY) = 0$ holds if and only if

$$\langle \alpha, \gamma\rangle + \langle \beta, \delta\rangle = 0, \quad \langle \alpha, \delta\rangle = \langle \beta, \gamma\rangle\,. \tag{5.14}$$

On the other hand, (5.13) implies that $g(A_\xi X, X) + g(A_\xi Y, Y) = 0$ if and only if $g(\bar{X}, X) + g(\bar{Y}, Y) = 0$. From these it follows that X and Y satisfy the conditions $g(A_\xi X, X) + g(A_\xi Y, Y) = 0$ and $g(X,X) = g(Y,Y)$ if and only if $\alpha, \beta, \gamma, \delta$ satisfy

$$||\alpha|| = ||\delta||, \quad ||\beta|| = ||\gamma||. \tag{5.15}$$

Moreover, it follows from (1.4) and (5.13) that the X and Y also satisfy $g(A_{J\xi}X, X) + g(A_{J\xi}Y, Y) = 0$ if and only if $\alpha, \beta, \gamma, \delta$ satisfy $\langle \alpha, \beta\rangle + \langle \gamma, \delta\rangle = 0$.

Consequently, in order to show that there exist two vectors $X, Y \in T_xN^2$ which satisfy $g(X,Y) = g(X,iY) = 0$, $g(X,X) = g(Y,Y)$, $g(A_\xi X, X) + g(A_\xi Y, Y) = 0$ and $g(A_{J\xi}X, X) + g(A_{J\xi}Y, Y) = 0$, it is sufficient to show that there exist four vectors $\alpha, \beta, \gamma, \delta \in \mathbf{R}^3$ satisfying the system:

$$\langle a, \alpha\rangle = \langle b, \beta\rangle\,,\ \langle a, \gamma\rangle = \langle b, \delta\rangle\,, \tag{5.16}$$

$$\langle b, \alpha\rangle + \langle a, \beta\rangle = \langle b, \gamma\rangle + \langle a, \delta\rangle = 0, \tag{5.17}$$

$$\langle \alpha, \gamma\rangle + \langle \beta, \delta\rangle = 0,\ \langle \alpha, \delta\rangle = \langle \beta, \gamma\rangle\,,\ \langle \alpha, \beta\rangle + \langle \gamma, \delta\rangle = 0, \tag{5.18}$$

$$||\alpha|| = ||\delta||, \quad ||\beta|| = ||\gamma||, \tag{5.19}$$

where a, b are vectors in $\mathbf{R}^3$ satisfying $|a|^2 - |b|^2 = 1$ and $\langle a, b\rangle = 0$.

Given two vectors $a, b \in \mathbf{R}^3$ with $|a|^2 - |b|^2 = 1$ and $\langle a, b\rangle = 0$, (5.16)–(5.19) is an underdetermined system which admits some nontrivial solutions $\alpha, \beta, \gamma, \delta \in \mathbf{R}^3$. If we choose a Euclidean coordinate system on $\mathbf{R}^3$ such that $a = (a_1, 0, 0)$ and $b = (0, b_2, 0)$ with $a_1^2 = b_2^2 + 1$ and put

$$\alpha = (\alpha_1, \alpha_2, \alpha_3),\ \beta = (\beta_1, \beta_2, \beta_3),\ \gamma = (\gamma_1, \gamma_2, \gamma_3),\ \delta = (\delta_1, \delta_2, \delta_3),$$

then conditions (5.16) and (5.17) are equivalent to

$$\alpha_1 = \frac{b_2\beta_2}{a_1}, \quad \beta_1 = -\frac{b_2\alpha_2}{a_1}, \quad \gamma_1 = \frac{b_2\delta_2}{a_1}, \quad \delta_1 = -\frac{b_2\gamma_2}{a_1}. \tag{5.20}$$

Thus we get

$$\alpha=\Big(\frac{b_2\beta_2}{a_1},\alpha_2,\alpha_3\Big),\quad \beta=\Big(-\frac{b_2\alpha_2}{a_1},\beta_2,\beta_3\Big),$$
$$\gamma=\Big(\frac{b_2\delta_2}{a_1},\gamma_2,\gamma_3\Big),\quad \delta=\Big(-\frac{b_2\gamma_2}{a_1},\delta_2,\delta_3\Big). \tag{5.21}$$

Substituting (5.21) into (5.18) and (5.19) we obtain

$$\Big(1+\frac{b_2^2}{a_1^2}\Big)(\beta_2\delta_2+\alpha_2\gamma_2)+\alpha_3\gamma_3+\beta_3\delta_3=0, \tag{5.22}$$

$$\Big(1+\frac{b_2^2}{a_1^2}\Big)(\alpha_2\delta_2-\beta_2\gamma_2)+\alpha_3\delta_3+\beta_3\gamma_3=0, \tag{5.23}$$

$$\Big(1-\frac{b_2^2}{a_1^2}\Big)(\alpha_2\beta_2+\gamma_2\delta_2)+\alpha_3\beta_3+\gamma_3\delta_3=0 \tag{5.24}$$

$$\Big(\frac{b_2^2}{a_1^2}\Big)\beta_2^2+\alpha_2^2+\alpha_3^2=\Big(\frac{b_2^2}{a_1^2}\Big)\gamma_2^2+\delta_2^2+\delta_3^2, \tag{5.25}$$

$$\Big(\frac{b_2^2}{a_1^2}\Big)\alpha_2^2+\beta_2^2+\beta_3^2=\Big(\frac{b_2^2}{a_1^2}\Big)\delta_2^2+\gamma_2^2+\gamma_3^2. \tag{5.26}$$

It is easy to verify that

$$\alpha=\Big(\frac{b_2}{\sqrt{a_1^2+b_2^2}},0,1\Big),\quad \beta=\Big(0,\frac{a_1}{\sqrt{a_1^2+b_2^2}},0\Big),$$
$$\gamma=\Big(0,\frac{a_1}{\sqrt{a_1^2+b_2^2}},0\Big),\quad \delta=\Big(-\frac{b_2}{\sqrt{a_1^2+b_2^2}},0,1\Big) \tag{5.27}$$

satisfy system (5.22)–(5.26) (or equivalently (5.16)–(5.19)). Therefore, if we choose $e_1=(\alpha+i\beta)/\sqrt{2}$ and $e_2=(\gamma+i\delta)/\sqrt{2}$, then e_1,e_2 form an orthonormal basis of a totally real plane section such that the shape operator A_ξ with respect to e_1,e_2,Je_1,Je_2 takes the form:

$$A_\xi=\begin{pmatrix} A'_\xi & A''_\xi \\ A''_\xi & -A'_\xi \end{pmatrix},\quad A_{J\xi}=\begin{pmatrix} -A''_\xi & A'_\xi \\ A'_\xi & A''_\xi \end{pmatrix}, \tag{5.28}$$

with trace $A'_\xi=$ trace $A''_\xi=0$. $\square$

6. An additional result

We now introduce the notion of framed-Einstein manifolds as follows:

Definition 2. *A Riemannian n-manifold M is called framed-Einstein if there exist a function γ and an orthonormal frame $\{e_1,\dots,e_n\}$ on M such that the Ricci tensor S of M satisfies $S(e_i,e_i)=\gamma g(e_i,e_i)$ for $i=1,\dots,n$.*

Recall that a Kähler surface M^2 of a Kähler manifold $\tilde{M}^{2+p}$ is called strongly minimal if the shape operator of M^2 takes the form:

$$A_\alpha = \begin{pmatrix} A'_\alpha & A''_\alpha \\ A''_\alpha & -A'_\alpha \end{pmatrix}, \quad A_{\alpha^*} = \begin{pmatrix} -A''_\alpha & A'_\alpha \\ A'_\alpha & A''_\alpha \end{pmatrix}, \quad \alpha = 1, \dots, p, \tag{6.1}$$

$$A'_\alpha = \begin{pmatrix} a_\alpha & b_\alpha \\ b_\alpha & -a_\alpha \end{pmatrix}, \quad A''_\alpha = \begin{pmatrix} a^*_\alpha & b^*_\alpha \\ b^*_\alpha & -a^*_\alpha \end{pmatrix}, \tag{6.2}$$

with respect to some orthonormal frame $e_1, e_2, Je_1, Je_2, \xi_1, \dots, \xi_p, J\xi_1, \dots, J\xi_p$.

For strongly minimal surfaces in complex space forms, we have the following.

Proposition 6. *Let M^2 be a strongly minimal Kähler surface in a complex space form $\tilde{M}^{2+p}(4c)$. Then*

(1) M^2 is a framed-Einstein Kähler surface.

(2) M^2 is an Einstein-Kähler surface if and only if $\sum_{\alpha=1}^p [A'_\alpha, A''_\alpha] = 0$.

Proof. If M^2 is a strongly minimal Kähler surface in a complex space form $\tilde{M}^{2+p}(c)$ whose shape operator satisfies (6.1-6.2) with respect to some orthonormal frame $e_1, e_2, Je_1, Je_2, \xi_1, \dots, \xi_p, J\xi_1, \dots, J\xi_p$, then

$$A_\alpha^2 = \begin{pmatrix} \lambda_\alpha & 0 & 0 & \mu_\alpha \\ 0 & \lambda_\alpha & -\mu_\alpha & 0 \\ 0 & -\mu_\alpha & \lambda_\alpha & 0 \\ \mu_\alpha & 0 & 0 & \lambda_\alpha \end{pmatrix}, \tag{6.3}$$

where $\lambda_a = a_\alpha^2 + b_\alpha^2 + a_\alpha^{*\,2} + b_\alpha^{*\,2}$ and $\mu_\alpha = 2a_\alpha^2 b_\alpha^{*\,2} - 2b_\alpha^2 a_\alpha^{*\,2}$.

On the other hand, from the equation of Gauss we have

$$S(X,Y) = 6\,c\,g(X,Y) - 2\sum_{\alpha=1}^p g(A_\alpha^2 X, Y). \tag{6.4}$$

From (6.3) and (6.4) we obtain

$$S(e_1,e_1) = S(e_2,e_2) = S(Je_1,Je_1) = S(Je_2,Je_2) = 6\,c - 2\sum_{\alpha=1}^p \lambda_\alpha. \tag{6.5}$$

Thus, M^2 is framed-Einstein. From (6.3) and (6.4) we also know that M^2 is Einstein-Kähler if and only if $\sum_{\alpha=1}^p \mu_\alpha = 0$ holds. It is easy to verify that the later condition is equivalent to the condition: $\sum_{\alpha=1}^p [A'_\alpha, A''_\alpha] = 0$. □

Acknowledgement. The author is very grateful to the referee for several valuable suggestions.

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Received April 28, 2000