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Failure of Splitting from Module-Finite Extension Rings*[1]

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1. Introduction

A conjecture raised by J. Koh asks whether a module-finite extension of commutative rings $R \hookrightarrow S$ in which S, viewed as an R-module, has finite projective dimension, splits as a map of R-modules. This question is a natural generalization of the Direct Summand Conjecture (D.S.C.) of Hochster. This conjecture asserts that in the case where R is a regular ring (and consequently S has finite projective dimension over R) this inclusion splits as a map of R-modules. Its validity is known for equicharacteristic rings and for rings of mixed characteristic in dimensions one and two but remains open in dimension three and higher [11]. Koh's question replaces the condition of R being regular for the weaker condition of S having finite projective dimension over R. It has been shown in several cases that by weakening the hypothesis of R being regular, and replacing it by the condition that certain modules have finite projective dimension, the conclusions of many theorems fail to hold. For instance, the rigidity of Tor fails under the weaker hypothesis [9], as well as the positivity of the intersection multiplicity, $\chi_R(M, N)$, for modules M, N of finite projective dimension over R [10].

Throughout this paper all rings will be commutative with identity. In the first part we will construct several new examples in which we will show that Koh's conjecture is false for rings of prime characteristic as well as for rings of mixed characteristic. These examples will be constructed, in the prime characteristic case, as pushouts of quotients, by quadratic polynomials, of Stanley-Reisner rings over the field $\mathbb{Z}_2$, associated to a three-dimensional

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simplicial complex whose underlying space is the union of three tetrahedra touching at three points (Figure 1–2). These counterexamples, as the ones constructed in [6], also lift to mixed characteristic. Unlike the examples previously constructed, we will be able to provide a direct proof, and one which does not use computers, of the fact that the inclusion map does not split. Besides, these examples are conceptually clearer and have smaller Krull dimension.

In the second part of this paper we will prove some cases of the Direct Summand Conjecture in the following set up. As it was shown in [6], the D.S.C. is equivalent to the following statement. Let (R, m) be a local regular ring and S a ring of the form T/J where J is an ideal of T such that $J \cap R = (0)$ (so that $R \hookrightarrow S = T/J$ is injective) where T is an R-algebra of the form

$$T = R[Z_1, \ldots, Z_n]/(f_1(Z_1), \ldots, f_n(Z_n)),$$

where each $f_i(Z_i)$ is a monic polynomial on the variable Z_i alone. In this set up, the splitting of $R \hookrightarrow S$ is equivalent to the fact that $\mathrm{Ann}_T(J) \not\subseteq mT$ [6]. We will show how to reduce the problem to the case where J is a principal ideal of T, and that in this case $\mathrm{Ann}_T(J) \not\subseteq mT$ if the projective dimension of J, as an R-module, is less than two.

2. New counterexamples to Koh's conjecture

In this section we construct two new counterexamples to Koh's conjecture for rings of characteristic equal to two and will show how to lift these examples to mixed characteristic. We will construct a module-finite extension $R \hookrightarrow S$ in which S will be a ring, which viewed as an R-module, will have torsion free rank equal to two and projective dimension equal to one.

2.1. First counterexample

Let us consider the abstract simplicial complex Δ with vertices

$$V = \{x_{12}, x_{22}, x_{32}, t_1, t_2, t_3, y_1, y_2, y_3\},$$

and facets (faces of maximal degree) given by

$$F_1 = \{t_1, t_2, y_3, x_{32}\},\ F_2 = \{y_1, x_{12}, t_2, t_3\},\ F_3 = \{t_1, t_3, x_{22}, y_2\},$$

(see Figure 1), i.e., Δ consists of all subsets (simplexes) of $F_1,\ F_2,\ F_3$.

Let us denote by $R_0 = \mathbb{Z}_2[\Delta]$ the Stanley-Reisner ring

$$R_0 = \frac{\mathbb{Z}_2[x_{i2}, y_i, t_i]}{I_\Delta}$$

corresponding to Δ where

$$I_\Delta = (x_{i2}x_{j2},\ x_{i2}t_i,\ x_{i2}y_j,\ y_iy_j,\ y_it_i,\ t_1t_2t_3),\ i = 1, 2, 3,\ j = 1, 2, 3,\ i \neq j,$$

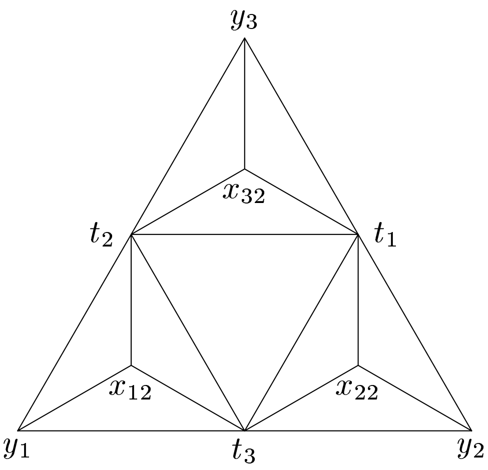

Figure 1

is the ideal associated to the complex Δ (generated by all square-free monomials in the variables of the set of vertices V which are not elements of Δ [7]).

Let

$$R = R_0[x_{11,}x_{21}, x_{31}]/H_0 \tag{2.1}$$

be the ring obtained by first adjoining to R_0 the variables x_{11}, x_{21}, x_{31}, and then killing the ideal H_0 of R_0 generated by the homogeneous elements

$$\begin{aligned} h_1 &= t_1x_{11} + x_{32}y_3 + x_{22}y_2, \\ h_2 &= t_2x_{21} + x_{12}y_1 + x_{32}y_3, \\ h_3 &= t_3x_{31} + x_{12}y_1 + x_{22}y_2. \end{aligned}$$

Remark 2.1. As it is well known, the primary decomposition for the ideal I_Δ in R_0 is given by

$$I_\Delta = \text{ß}_{F_1} \cap \text{ß}_{F_2} \cap \text{ß}_{F_3}$$

where ß_{F_i} are the ideals of R_0 generated by the facets of Δ

$$\text{ß}_{F_1} = (y_1, y_2, x_{12}, x_{22}, t_3),\ \ \text{ß}_{F_2} = (y_2, y_3, x_{22}, x_{32}, t_1),\ \ \text{ß}_{F_3} = (y_1, y_3, x_{12}, x_{32}, t_2).$$

The following lemma shows that this decomposition lifts naturally to a primary decomposition for the ideal $H_0 + I_\Delta$ in $R_2 = \mathbb{Z}_2[x_{ij}, y_i, t_i],\ i = 1, 2, 3,\ j = 1, 2$.

Lemma 2.1. *If* $\text{ß}_{F_1}, \text{ß}_{F_2},\ \text{ß}_{F_3}, I_\Delta, H_0$ *denote the ideals as above, regarded as ideals of* R_2*, then*

(i) *The ideals* $Q_1 = (\text{ß}_{F_1} + H_0),\ Q_2 = (\text{ß}_{F_2} + H_0),\ Q_3 = (\text{ß}_{F_3} + H_0)$ *are prime.*

(ii) *The primary decomposition of* $I_\Delta + H_0$ *is given by*

$$I_\Delta + H_0 = Q_1 \cap Q_2 \cap Q_3.$$

Proof of (i). Let us observe first that

$$R_2/Q_1 \cong \mathbb{Z}_2[x_{11}, x_{21}, x_{31}, x_{32}, y_3, t_1, t_2]/Q_1^{'}$$

where $Q_1' = (x_{11}t_1 + x_{21}t_2,\ x_{32}y_3 + x_{21}t_2)$. In order to show that Q_1 is prime, it suffices to do this after we localize at t_2. But after localization Q_1' becomes a principal ideal generated by the polynomial $x_{11}t_1 + x_{32}y_3$ which is clearly irreducible in $(R_2)_{t_2}$ and consequently Q_1 is a prime ideal in R_2. In a similar way we conclude that $Q_2,\ Q_3$ are also prime ideals. $\square$

Proof of (ii). Let $ß_{F_1}^e, H_0^e, I_\Delta^e$ denote the expansions of $ß_{F_1}, H_0$, and I_Δ to $(R_2)_{Q_1}$. We observe that the ideal $Q_1(R_2)_{Q_1}$ is equal to the sum of the ideals $ß_{F_1}^e$ and H_0^e. To see this, note that each generator

$$y_1t_1/1,\ y_2y_3/1,\ x_{32}t_3/1,\ x_{12}t_1/1,\ x_{22}x_{32}/1$$

of $ß_{F_1}^e$ is contained in I_Δ^e, and since t_1, y_3, x_{32} are units in $(R_2)_{Q_1}$, then the result follows.

In a similar way we prove that $Q_j(R_2)_{Q_j}$ is equal to $(I_\Delta + H_0)^e$ in $(R_2)_{Q_j}$, for $j = 2, 3$. This implies that the primary components of $I_\Delta + H_0$ are precisely $Q_1,\ Q_2,\ Q_3$ and therefore

$$I_\Delta + H_0 = Q_1 \cap Q_2 \cap Q_3.$$ $\square$

From this lemma we immediately obtain the following corollary.

Corollary 2.2. *The Krull dimension of $R \simeq R_2/(H_0 + I_\Delta)$ is equal to* 5.

Lemma 2.3. *Let C be the cokernel of the matrix*

$$\mathrm{X} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix}.$$

Then the sequence

$$0 \to R^2 \xrightarrow{X} R^3 \to C \to 0 \tag{2.2}$$

is a free resolution of C.

Proof. It suffices to see that X is injective. For this we use the Buchsbaum-Eisenbud criterion for exactness: in this case we just have to check that $\operatorname{depth}_\Delta R \geq 1$, where Δ denotes the ideal generated by the 2×2 minors of the matrix X. Let z be the sum of the three minors of X, i.e.,

$$z = (x_{11}x_{22} - x_{21}x_{12}) + (x_{11}x_{32} - x_{31}x_{12}) + (x_{21}x_{32} - x_{31}x_{22}).$$

In order to show that z is not a zero divisor, it is enough to verify that it is not contained in any of the primes Q_i, $i = 1, 2, 3$. But in R/Q_1 the element $\overline{z} = x_{32}(x_{11} + x_{21})$ is clearly not zero since both x_{32} and $x_{11} + x_{21}$ are not in Q_1. In a similar way we prove that z is not in any of the other two primes. $\square$

Remark 2.2. The fact that X is injective can also be shown by direct computation as follows: let (a, b) be an element of the first syzygy module $\mathrm{Syz}_1(\mathrm{C})$, of C. Thus

$$\begin{aligned} &1. \quad ax_{11} + bx_{12} = 0 \\ &2. \quad ax_{21} + bx_{22} = 0 \\ &3. \quad ax_{31} + bx_{32} = 0, \end{aligned}$$

in $R = R_2/(I_\Delta + H_0)$. Multiplying the first equation by t_1 we obtain $ax_{11}t_1 + bx_{12}t_1 = 0$, and since $t_1x_{12} \in I_\Delta + H_0$, then $bt_1x_{12} = 0$.

Therefore $ax_{11}t_1 = 0$, i.e., $ax_{11}t_1 \in I_\Delta + H_0$. From this it follows that $ax_{11}t_1/1 \in Q_1(R_2)_{Q_1}$. Since $x_{11}/1, t_1/1$ are units in this ring we get that $a \in Q_1$. Similarly, we show that $a \in Q_2$, and $a \in Q_3$. This forces a to be 0 in R. Therefore, bx_{12}, bx_{22}, and bx_{32} are contained in $I_\Delta + H_0$, and in particular bx_{12} is contained in $Q_2(R_2)_{Q_2}$. But $x_{12}/1$ is a unit in $(R_2)_{Q_2}$ and consequently $b \in Q_2$. A similar reasoning shows that b is contained in Q_1 and Q_3 what forces $b = 0$ in R.

With notation as above, we have the following:

Theorem 2.4. *Let $f \in \mathrm{Hom}_R(R^2,R)$ be the homomorphism that sends e_1, e_2, the standard basis of R^2, to the elements $x_{11}t_1 + x_{12}y_1$ and 0, respectively. Then the R-module*

$$S = (R \oplus R^3)/(f(e_i), -X(e_i)), \quad i = 1, 2,$$

can be endowed with an R-algebra structure satisfying the following properties:

(i) *The structural homomorphism $R \to R \cdot \overline{1} \subset S$ is injective and makes S into a module finite extension algebra of R of projective dimension equal to one, $\mathrm{pd}_R(S) = 1$.*

(ii) *The inclusion $R \hookrightarrow S$ does not split as a map of R-modules.*

Proof of (i). Let T be the quotient of the polynomial ring $R[X_1, X_2, X_3]$ given by

$$T = R[X_1, X_2, X_3]/(X_1X_2, X_1X_3, X_2X_3, X_1^2 - t_1X_1, X_2^2 - t_2X_2, X_3^2 - t_3X_3).$$

Notice first that T is a free R-module generated with basis given by $\{1, u_1, u_2, u_3\}$, where u_i denotes the class of X_i, $i = 1, 2, 3$. (For any relation of linear dependence $a_0 + a_1u_1 + a_2u_2 + a_3u_3 = 0$, after we kill X_2, X_3 in T we get $a_0 + a_1u_1 = 0$ in $R[X_1]/(X_1^2 - t_1X_1)$, which implies that $a_0 = a_1 = 0$. In a similar manner one shows that $a_2 = 0$, and $a_3 = 0$.) Thus, $T \simeq R \oplus R^3$. Let N denote the R-submodule of T generated by the images in T of

$$f(e_1) - X(e_1) \quad \text{and} \quad f(e_2) - X(e_2), \quad \text{i.e., } N = Rv_1 + Rv_2,$$

where

$$v_1 = (x_{12}y_1 + x_{11}t_1) - x_{11}u_1 - x_{21}u_2 - x_{31}u_3,$$

$$v_2 = -x_{12}u_1 - x_{22}u_2 - x_{32}u_3.$$

Note that N is an ideal of T. For this, it is enough to check that $u_iN \subset N$. But for each $i = 1, 2, 3$ one can readily check that

$$\begin{aligned} u_iv_1 &= [(x_{12}y_1 + x_{11}t_1) - t_ix_{i1}]u_i = y_iv_2, \\ \text{and} \quad u_iv_2 &= 0. \end{aligned}$$

Since S and T/N are isomorphic as R-modules, S acquires, via this isomorphism, the structure of an R-algebra. □

Proof of (ii). S is constructed in the same way as in [6], i.e., as the pushout that makes the following diagram commutative

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \xrightarrow{i} & S & \longrightarrow & C & \longrightarrow & 0 \\ & & {\scriptstyle f}\uparrow & & {\scriptstyle g}\uparrow & & {\scriptstyle id}\uparrow & & \\ 0 & \longrightarrow & R^2 & \xrightarrow{\mathrm{X}} & R^3 & \longrightarrow & C & \longrightarrow & 0 \end{array}.$$

Here C denotes the cokernel of the matrix X, as in the previous lemma, g the canonical map sending e_i into u_i, $i = 1, 2, 3$, and id the identity map on C. The exactness of the second row is precisely the content of the previous lemma. From this, it readily follows that i is injective and that $\mathrm{pd}_R(S) = \mathrm{pd}_R(C) = 1$.

Finally, in order to prove that the inclusion $R \hookrightarrow S$ does not split it is enough to show that $JS \cap R \neq J$, where J denotes the ideal $J = (\overline{x_{11}}, \overline{x_{21}}, \overline{x_{31}})R$ (as it is shown in [12] if $R \hookrightarrow S$ splits every ideal J of R is equal to the contraction of its own expansion to S). Clearly, $z = x_{12}y_1 + x_{11}t_1$ is contained in $JS \cap R$ since $z = x_{11}u_1 + x_{21}u_2 + x_{31}u_3$. On the other hand, let ψ denote the homomorphism

$$\psi : R/J \to \mathbb{Z}_2[X_1, X_2, X_3]/C,$$

where

$$C = (X_1X_2,\ X_1X_3,\ X_2X_3,\ X_1^2 - t_1X_1,\ X_2^2 - t_2X_2,\ X_3^2 - t_3X_3)$$

that sends $\psi(\overline{x_{i2}}) = \overline{X_i}$, $\psi(\overline{y_i}) = \overline{X_i}$, $\psi(\overline{t_i}) = 0$, $i = 1, 2, 3$. Then one immediately sees that $\psi(z) = \overline{X}_1^2 \neq 0$ and therefore $z \notin J$. □

2.2. Second counterexample

Now we see that if we specialize some of the variables of R in the previous example we obtain a counterexample of smaller Krull dimension. Actually, we will see that the Krull dimension of this counterexample is two, which makes it the smallest counterexample as far as we know. In the ring R_0 constructed above we set $x_{i1} = t_i$, and replace each variable x_{i2} by x_i. This ring is just the Stanley-Reisner ring $R_0 = \mathbb{Z}_2[x_i, y_i, z_i]/I_\Delta$ associated to the simplicial complex Δ with maximal faces the simplexes

$$\Delta_1 = \{y_1, x_1, t_2, t_3\},\ \Delta_2 = \{y_2, x_2, t_1, t_3\},\ \Delta_3 = \{y_3, x_3, t_2, t_1\}.$$

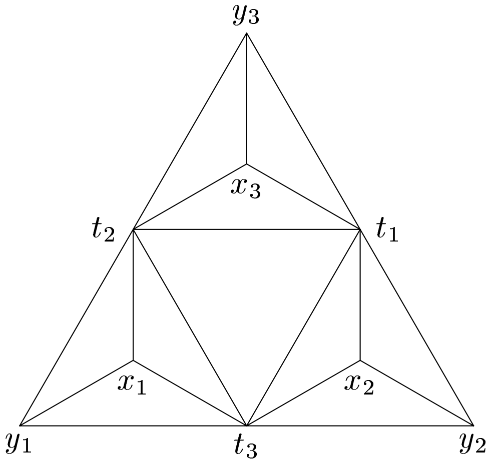

Figure 2

Let R be the quotient of R_0 by the ideal generated by the homogeneous elements

$$h_1 = t_1^2 - x_3y_3 - x_2y_2,\ h_2 = t_2^2 - x_1y_1 - x_3y_3,\ h_3 = t_3^2 - x_1y_1 - x_2y_2.$$

Clearly,

$$R = \mathbb{Z}_2[x_i, y_i, z_i]/H$$

where

$$H = (x_ix_j,\ x_it_i,\ x_iy_j,\ y_iy_j,\ y_it_i,\ t_1t_2t_3,\ h_i), \quad 1 \leq i, j \leq 3, i \neq j.$$

Now we check that by passing to this new ring we preserve all the good properties of the previous construction.

Lemma 2.5. *A primary decomposition of H is given by*

$$H = q_1 \cap q_2 \cap q_3$$

where

$$\begin{aligned} q_1 &= (t_1,\ x_2,\ x_3,\ y_2,\ y_3,\ t_2^2 - x_1y_1,\ t_3^2 - x_1y_1), \\ q_2 &= (t_2,\ x_1,\ x_3,\ y_1,\ y_3,\ t_1^2 - x_2y_2,\ t_3^2 - x_2y_2), \\ q_3 &= (t_3,\ x_1,\ x_2,\ y_1,\ y_2,\ t_1^2 - x_3y_3,\ t_2^2 - x_3y_3) \end{aligned}$$

are primary to the prime ideals

$$\begin{aligned} Q_1 &= (t_1,\ x_2,\ x_3,\ y_2,\ y_3,\ t_2^2 - x_1y_1,\ t_2 + t_3), \\ Q_2 &= (t_2,\ x_1,\ x_3,\ y_1,\ y_3,\ t_1^2 - x_2y_2,\ t_1 + t_3), \\ Q_3 &= (t_3,\ x_1,\ x_2,\ y_1,\ y_2,\ t_1^2 - x_3y_3,\ t_1 + t_2). \end{aligned}$$

Proof. It is clear that $R/Q_1 \simeq \mathbb{Z}_2[x_1, y_1, t_2]/(t_2^2 - x_1y_1)$ is a domain and therefore Q_1 is a prime ideal. In a similar way we show that Q_2 and Q_3 are prime. These primes are precisely the minimal primes of H by the same argument as in the previous counterexample.

Now, let us see that the expansions of H and q_1 coincide in any localization of R at any of the vertices of the tetrahedron $\Delta_1 = \{y_1, x_1, t_2, t_3\}$. To see this, notice that if we invert any of the vertices of Δ_1 the monomial generators of H become the variables t_1, x_2, x_3, y_2, y_3, up to units. Thus, the polynomial h_1 becomes superfluous and h_2, h_3 can be replaced with $t_2^2 - x_1y_1, t_3^2 - x_1y_1$. Consequently H coincides with q_1 after localization. From this we see that $HR_{Q_1} = q_1R_{Q_1}$ and therefore q_1 must be the Q_1-primary component of H. If we localize at any prime Q different from the homogeneous maximal ideal m of R, and containing Q_1, the expansions of H and q_1 also coincide because any such prime Q cannot contain all the vertices of Δ_1 (since this would imply that it contains all the nine variables of R_0 and this in turn would imply $Q = m$). Thus, H cannot have any embedded component with radical a prime Q containing Q_1.

By symmetry, localizing at any of the vertices of Δ_2 (respectively Δ_3), we see that q_2 and q_3 are the Q_2, respectively Q_3, primary components of H and no other components exist that contain any of these two primes. $\square$

From this lemma it follows that

Corollary 2.6. *The Krull dimension of R is equal to two.*

Lemma 2.7. *Let C be the cokernel of the matrix*

$$\mathrm{X} = \begin{bmatrix} t_1 & x_1 \\ t_2 & x_2 \\ t_3 & x_3 \end{bmatrix}.$$

Then C has projective dimension one over R.

Proof. As in the Lemma 2.5 we check that the element

$$z = t_1 \begin{vmatrix} t_1 & x_1 \\ t_2 & x_2 \end{vmatrix} + t_2 \begin{vmatrix} t_2 & x_2 \\ t_3 & x_3 \end{vmatrix} + t_3 \begin{vmatrix} t_1 & x_1 \\ t_3 & x_3 \end{vmatrix}$$

is not in any of the minimal primes of H. But in R/Q_1 (recall that)

$$Q_1 = (t_2, x_1, x_3, y_1, y_3, t_1 + t_3, t_3^2 + x_2y_2),$$

we see that $\overline{z} = t_1^2x_2$ is not zero. In a similar way we show that z is not contained in any of the other two primes Q_2, Q_3. $\square$

With R and C constructed as above we can construct as in the previous theorem a smaller counterexample to Koh's conjecture.

Theorem 2.8. *Let $R = \mathbb{Z}_2[x_i, y_i, t_i]/H$, $i = 1, 2, 3$, and $f \in \mathrm{Hom}_R(R^2, R)$ be the homomorphism that sends e_1, e_2, the standard basis of R^2, to the elements $t_1^2 + x_1y_1$ and 0, respectively. Then the R-module S constructed as a pushout, as in Theorem 2.4, is a module-finite extension of R with $\mathrm{pd}_R(S) = 1$, and the natural inclusion of R in S does not split.*

Proof. The fact that S is a module-finite extension algebra of R follows from the fact that R is a homomorphic image of the original R constructed in the first counterexample, and S, as an R-module, is of finite projective dimension by the previous lemma. On the other hand, the inclusion of R in S does not split since the element $w = x_1y_1 + t_1^2$ is in $JS \cap R$, where J denotes the ideal $J = (t_1, t_2, t_3)R$, (because $w = t_1u_1 + t_2u_2 + t_3u_3$), and it is not contained in J; to prove this, we use the same homomorphism ψ constructed above to show that $\psi(w) = X_1^2 \neq 0$. □

3. Lifting these counterexamples to mixed characteristic

These counterexamples can be lifted to counterexamples in mixed characteristic following the same procedure as in [6]. The ring R_0 is defined as in any of the two examples before, except that we know replace the field $\mathbb{Z}_2$ by the ring of integers. Let us denote this ring by R_0'. Notice now that in both counterexamples above the ideal H_0 is already defined over the integers so it makes sense to define $R' = R_0'/H_0'$, where H_0' denotes ideal H_0 (respectively H), as in counterexamples 1 and 2, respectively, but viewed as ideals with integer coefficients. Let S' be a ring constructed over R' using the same pushout construction as before. Clearly $R_0 = R_0'/2R_0'$, and we have the obvious surjection $h : R' \to R$. If m' denotes the contraction of m, the homogeneous maximal ideal of R to R', then h extends to a map $h : R'_{m'} \to R_m$. Now, we can show that the inclusion

$$R'_{m'} \hookrightarrow S'_{m'}$$

does not split using the same argument as above: the elements z and w constructed in counterexamples 1 and 2 are contained in $J'S'_{m'} \cap R'_{m'}$ but not in $J'R'_{m'}$ since otherwise they would be contained in JR_m, a contradiction since $R_m \hookrightarrow S_m$ does not split.

Alternatively, one can see that one could also modify the two counterexamples above by changing the coefficient field $\mathbb{Z}_2$ by the ring $\mathbb{Z}_{(2)}$ of integers localized at the prime (2), without changing the projective dimension of S (as before, one can verify that the sum of minors of X is not a zero divisor). These new extensions do not split either since one can easily check, using the same arguments as above, that the contraction of the expansion of J to S is strictly bigger than J.

4. Some remarks on the Direct Summand Conjecture

The Direct Summand Conjecture D.S.C. asserts that every module-finite extension $R \hookrightarrow S$ of a regular ring R splits as a map of R-modules. This conjecture has been proved when the ring R contains a field and for rings of dimension less than three, in mixed characteristic [11]. In [6] it is shown that the problem of determining whether an extension $R \hookrightarrow S$ splits can be reformulated in the following way. First, as it is proved in [11] one can easily reduce to the case where R is a local regular ring (R, m). Since S is a finite R-module one can represent S in the form T/J, with T denoting the free R-algebra

$$T = R[Z_1, \dots, Z_n]/(f_1(Z_1), \dots, f_n(Z_n)),$$

where $f_i(Z_i)$ is a monic polynomial on the variable Z_i alone and $J \subset T$ is an ideal of T such that $J \cap R = (0)$ (so that $R \hookrightarrow S = T/J$ is injective). If d_i denotes the degree of f_i then T is a free R-module of rank m, with $m = d_1 d_2 \cdots d_n$, and with free R-basis consisting of the product monomials $u_1^{\alpha_1} u_2^{\alpha_2} \cdots u_n^{\alpha_n}$, $0 \leq \alpha_j < d_j$, where u_i denotes the class of Z_i in T. In this set up the splitting of $R \hookrightarrow S$ is equivalent to the fact that $\operatorname{Ann}_T(J) \not\subseteq mT$ [6].

In this section we will show that the problem of determining whether

$$\operatorname{Ann}_T(J) \not\subseteq mT$$

can be reduced to the case where J is a principal ideal. We will show that this is the case if the projective dimension of J, as an R-module, is less than two, or equivalently, in the case where $\operatorname{pd}_R(S) < 3$.

Lemma 4.1. *Let (R_0, m_0) be a regular local ring and let T_0 denote the ring*

$$T_0 = R_0[Z_1, \ldots, Z_n]/(f_1(Z_1), \ldots, f_n(Z_n)).$$

Let $J = (g_1, \ldots, g_s)$ be an ideal of T_0 such that $J \cap R = (0)$, so that $R_0 \hookrightarrow S_0 = T_0/J$ is a module-finite extension algebra. Let $R = R_0[X_1, \ldots, X_s]$ be the polynomial ring in the variables $X_1, \ldots, X_s$ over R_0, and let T be the algebra

$$T = T_0 \otimes_{R_0} R \simeq R[Z_1, \ldots, Z_n]/(f_1(Z_1), \ldots, f_n(Z_n))^e.$$

If g denotes the element $g = X_1 g_1 + \cdots + X_s g_s$ in T and S denotes the quotient ring (T/gT) then

1. *$gT \cap R = (0)$ and therefore $R_m \to S_m = (T/gT)_m$ is a module-finite extension of R, where R_m and S_m denote the localization of R and S at the maximal ideal $m = m_0 R + (X_1, \ldots, X_s)R$.*
2. *If the inclusion $R_m \to S_m$ splits then the original inclusion $R_0 \hookrightarrow S_0 = T_0/J$ also splits.*

Proof. First, since $R_0 \to R$ is faithfully flat the inclusion $R_0 \hookrightarrow T_0/J$ stays injective after tensoring with R hence $R \to (T_0 \otimes_{R_0} R)/J^e = T/JT$ is also injective. Since $(g)T \subset JT$ we get $(g)T \cap R \subset JT \cap R = (0)$ thus $R \to T/gT$ is also injective and this is preserved after localizing at m. This proves the first part.

Now, for part 2, let us assume that $R_m \hookrightarrow S_m$ splits. This implies that there is an element

$$h = \sum_{\alpha} \frac{a_\alpha(X)}{s_\alpha(X)} u^\alpha, \quad \frac{a_\alpha(X)}{s_\alpha(X)} \in R_m,$$

which is contained in the annihilator of g in $T_m = R_m[u_1, \ldots, u_n]$ but not in mT_m, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ denotes a multiindex, and u^α the monomial $u^{\alpha_1} \cdots u^{\alpha_n}$. This means that at least one of the coefficients $\frac{a_\alpha(X)}{s_\alpha(X)}$ is not contained in mR_m. After multiplying by the product $s = \prod s_\alpha(X)$ of the denominators we may assume that h has the form $h = \sum_\alpha c_\alpha(X) u^\alpha$, with all its coefficients in R. Since the $s_\alpha(X)$ are all units of R_m, we may still assume that at least one of the coefficients $c_\alpha(X)$ is not in mR. Now, since the coefficients of h and g, viewed as polynomials in the u_j, are all contained in $R = R_0[X_1, \dots, X_s]$ the equation $hg = 0$ in $R_m[u_1, \dots, u_n]$ also holds in $R[u_1, \dots, u_n]$:

$$hg = (\sum_\alpha c_\alpha(X)u^\alpha) \cdot (X_1 g_1 + \cdots + X_s g_s) = 0 \text{ in } R[u_1, \dots, u_n]. \tag{4.1}$$

We may identify $R[u_1, \dots, u_n] = R_0[X_1, \dots, X_s][Z_1, \dots, Z_n]/(f_1(Z_1), \dots, f_n(Z_n))$ with

$$R_0[Z_1, \dots, Z_n]/(f_1(Z_1), \dots, f_n(Z_n))[X_1, \dots, X_n] = T[X_1, \dots, X_s].$$

We thus may view (4.1) as an equation in $T[X_1, \dots, X_n]$. Let us write $c_\alpha(X)$ as

$$c_\alpha(X) = \sum_\beta r_{\alpha\beta} X^\beta,$$

where $\beta = (\beta_1, \dots, \beta_s)$ denotes a multiindex, and $r_{\alpha\beta}$ denote coefficients in R_0. By changing the order of summation we may write h as

$$h = \sum_\beta \sum_\alpha r_{\alpha\beta} u^\alpha X^\beta = \sum_\alpha r_{\alpha 0} u^\alpha + \sum_{\beta \neq 0} \sum_\alpha r_{\alpha\beta} u^\alpha X^\beta,$$

and then the equation (4.1) takes the form

$$hg = (\sum_\alpha r_{\alpha 0} u^\alpha + \sum_{\beta \neq 0} \sum_\alpha r_{\alpha\beta} u^\alpha X^\beta)(g_1 X_1 + \cdots + g_s X_s) = 0.$$

In this product the coefficient of X_i is given by

$$h_0 g_i = (\sum_\alpha r_{\alpha 0} u^\alpha) g_i = 0,$$

for $i = 1, \dots, s$, where h_0 denotes the polynomial $h_0 = \sum_\alpha r_{\alpha 0} u^\alpha$. Thus h_0 is an element in the annihilator of J in T. On the other hand, since not all the $c_\alpha(X)$ are contained in mR there must exist one element $r_{\alpha 0}$, for some α, which is not contained in m_0 (clearly all the other coefficients of u^α in $c_\alpha(X)$ are multiples of the X_i) and consequently h_0 is not contained in $m_0 T_0$. Thus $R_0 \hookrightarrow T_0/J$ splits as we wanted to show. □

Theorem 4.2. (Notation as above) *Let (R, m) be a regular local ring of mixed characteristic and of Krull dimension n. Suppose $R \hookrightarrow S$ is a module-finite extension ring and write S as T/J with $J = (\zeta)$ a principal ideal such that* $\mathrm{pd}_R(J) < 2$. *Then the inclusion of R in S splits as a map of R-modules.*

Proof. We notice first that the module $J = (\zeta)$ is generated as an R-module by the m elements $v_{(i)} = \zeta u_1^{i_1} u_2^{i_2} \cdots u_t^{i_t}$, $0 \leq i_j < d_j$. If $\alpha : R^{\mathrm{m}} \to J \to 0$ denotes the map that sends the canonical base $e_{(i)} \to v_{(i)}$, then the kernel of this map is precisely the annihilator of J in T since it is the set of all m-tuples of the form $(r_{(i)}) \subset R^{\mathrm{m}}$ such that

$\sum r_{(i)}\zeta u_1^{i_1}u_2^{i_2}\cdots u_t^{i_t} = 0 \iff \zeta(\sum r_{(i)}u_1^{i_1}u_2^{i_2}\cdots u_t^{i_t}) = 0$. Now, to say that $\mathrm{Ann}_T(J) \nsubseteq mT$ is exactly the same as saying that among the linear relations

$$\sum r_{(i)}u_1^{i_1}u_2^{i_2}\cdots u_t^{i_t} = 0$$

there is at least one with some coefficient $r_{(i)}$ not in m. By Nakayama's lemma this is equivalent to the fact that J is minimally generated as an R-module by less than m elements, or equivalently, that there is at least one entry which is a unit in the matrix of the map β of the resolution:

$$0 \to R^k \to \cdots \to R^b \xrightarrow{\beta} R^{\mathrm{m}} \to J \to 0, \tag{4.2}$$

where $0 \to R^k \to \cdots \to R^b \xrightarrow{\beta} \mathrm{Ker}(\alpha) \to 0$ is a minimal resolution of the Kernel of α. In the case where J is free as an R-module we have $\beta = 0$ and therefore J must be minimally generated by m elements otherwise the sequence $0 \to (J \simeq R^{\mathrm{m}}) \to (T \simeq R^{\mathrm{m}}) \to T/J \to 0$ would be exact and therefore the torsion free rank of T/J would be zero, a contradiction.

Now, let us assume that $\mathrm{pd}_R(J) = 1$. In this case $\beta \neq 0$, and to prove that at least one entry of the matrix representing β is a unit we can proceed by descending induction on the dimension of R. Without loss of generality we may assume that $\dim(R) \geq 3$ since the D.S.C is true for ring of Krull dimension less that 3, [11]. At the top, where $\dim(R) = n$, we have $\mathrm{depth}_m(J) = n-1$ since $\mathrm{pd}_R(J) = 1$. Hence by avoiding the union of the associated primes of T/J and the square of the maximal ideal in R we can choose $x \in R$ such that after tensoring the inclusion $R \hookrightarrow S = T/J$ with R/x we get to the situation where

$$R/x \to R/x \otimes T/(\zeta) \simeq \overline{T}/(\overline{\zeta})$$

is also injective, depth of J/xJ in the maximal ideal of R/x equals $\mathrm{depth}_m(J) - 1 = \dim(R/x) - 1$, and where R/x is regular of dimension $n-1$. (The injectivity of the map follows from the fact that $\mathrm{depth}(J) = n-1$ implies $\mathrm{depth}(S = T/J) = n-2 \Longrightarrow \mathrm{depth}(S/R) > 0$ otherwise, since we are assuming $n \geq 3$, $\mathrm{depth}(S/R) = 0 \Longrightarrow \mathrm{depth}(R) = 1$, a contradiction.) Finally, tensoring the exact sequence

$$0 \to R^k \to \cdots \to R^b \xrightarrow{\beta} R^{\mathrm{m}} \to J \hookrightarrow T = R^{\mathrm{m}} \to T/J \to 0$$

with R/x we get an exact sequence over R/x in which the matrix of $\overline{\beta}$ has its entries in m if and only the same is true for β, and thus we can use the inductive hypothesis to conclude the argument. □

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