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Polynomial and Transformation Composition Rings

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Abstract. The base of an arbitrary composition ring is defined. This is used, amongst others, to identify polynomial type composition rings and to describe the maximal ideals of certain types of composition rings.

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1. Introduction

Polynomials are one of the earliest entities studied in mathematics. As an algebraic entity, they have been studied for more than a century, and then mainly within the framework of a ring with respect to the usual operations of addition and multiplication of polynomials. But with any polynomial ring is associated a polynomial near-ring with respect to the operation of composition. These polynomial near-rings have also been studied, albeit much less extensively and only during the last few decades. But not much attention has been given to their natural algebraic home, that of a ring together with the additional operation of composition. Sporadically this has been done under the more general guise of a composition ring (or composition near-ring). But this approach is mostly too general to capture their significant properties for an abstract investigation. For example, if R is any ring, then the set $M(R)$ of all mappings from R to R, together with pointwise addition, pointwise multiplication and composition is a composition ring. Moreover, such a composition ring is the prototype of all composition rings, since any composition ring can be embedded in an $M(R)$ for a suitable R.

The study of the composition rings $R[x], R_0[x]$ and $R_N[[x]]$ has inevitably led to the relationship between the properties of the composition ring and those of the underlying base

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ring R. In the case where constants are allowed (e.g. $R[x]$), the base ring is easily identified as all those elements c for which $c \circ 0 = c$. The same procedure captures R in the composition ring $M(R)$. But for $R_0[x]$ and $R_N[[x]]$ things are more problematic. For these two cases, a different approach is required and one is led to consider $D(c) \circ 0$ where $D(c)$ denotes the formal derivation of c (which takes one outside the considered composition ring). But for the composition ring $M_0(R)$ (zero preserving functions from R to R) there may be no realistic derivation in sight to enable us to capture R.

It is these two problems that will be addressed here. We will define the base of an arbitrary composition ring. It will be seen that the base for each of the composition rings $R[x], R_0[x]$, $R_N[[x]]$, $M_0(R)$ and $M(R)$ is a ring isomorphic to R. As can be expected, the relationship between the composition ring and its base will not be the same for all composition rings, but at least the homological structure will be the same: The base B of a composition ring C is a subring of C (with respect to the composition) for which there is (at least) a group homomorphism $\beta : (C, +) \to (B, +)$.

To realize the existence of the mapping β and to obtain more structure on the composition rings, we will define and investigate mainly two types of composition rings. These are the composition p-rings (p standing for polynomial, power series or related structures) and composition t-rings (t standing for transformation). This will enable us, amongst others, to determine the maximal ideals of a composition p-ring in terms of those of the underlying base ring (irrespective of whether constants are allowed or not).

We should point out that Kautschitsch and Mlitz [2] have given a general method to describe the maximal ideals of $R_0[x], R[x]$ and $R_N[[x]]$. This was done within the framework of composition subrings of $R[[x]]$. The results presented here will be more general and really only serve as a guideline on how to study polynomials and power series under this more general setting.

In the sequel, we use C to denote the composition ring $(C, +, \cdot, \circ)$, i.e. $C_1 := (C, +, \cdot)$ is a ring, $C_2 := (C, +, \circ)$ is a near-ring and $ab \circ c = a \circ c \cdot\ b \circ c$ for all $a, b, c \in C$. Unless indicated otherwise by brackets, juxtaposition (which represents multiplication) has a higher priority than composition (in the order of executing the operations) and composition has a higher priority than multiplication when the latter is denoted by "$\cdot$". For example, $ab \circ c = a \circ c \cdot b \circ c$ means $(ab) \circ c = (a \circ c) \cdot (b \circ c)$.

The *foundation of* C, denoted by F, is the constant part of the near-ring C_2, i.e. $F = Found(C) = \{c \in C \mid c \circ 0 = c\}$. F is a subcomposition ring of C, but usually when we refer to the foundation, we mean the ring $F = (F, +, \cdot)$. C_0 denotes the 0-symmetric part of of the near-ring C_2, i.e. $C_0 = \{c \in C \mid c \circ 0 = 0\}$. It can be verified that C_0 is an ideal of the ring C_1 and a left ideal of the near-ring C_2.

The near-ring C_2 may have an identity which we will denote by $x \in C_2$. Often it is required that x be a *commuting composition identity*, i.e. $c \circ x = c = x \circ c$ and $xc = cx$ for all $c \in C$. The ring C_1 may have an identity i (i.e. $ci = c = ic$ for all $c \in C$), denoted by $i \in C_1$. In case $i \in F$, it will be called a *constant multiplicative identity*. The ring $(C_0, +, \cdot)$ may also have an identity, denoted by $e \in C_0$ (which may differ from i) and is called a 0-*symmetric multiplicative identity*.

For $c \in C$ and $k \in \mathbb{N}$, c^k denotes $c^k = ccc \cdots c$ (k factors) and $c^{(k)} = c \circ c \circ c \circ \cdots \circ c$ (k times). For a subset $A \subseteq C$, $A^k = \{a_1 a_2 \cdots a_k \mid a_i \in A\}$ and $A^{(k)} = \{a_1 \circ a_2 \circ \cdots \circ a_k \mid a_i \in A\}$.

When the ring R is used in the context of $R[x], R_0[x]$ or $R_N[[x]]$, it will be assumed to be commutative and to have an identity $1 \in R$. As usual, $R[x]$ is the composition ring of all polynomials over R, $R_0[x]$ is the 0-symmetric part of $R[x]$ and $R_N[[x]]$ is the composition ring of all formal power series over R (i.e. infinite sums of the form $\sum_{n=0}^{\infty} a_n x^n$) with constant term a_0 from N where N is a fixed nil ideal of R.

2. The base of a composition ring

Let C be a composition ring. The *base of* C, denoted by B, is defined by $B = Base(C) = \{v \in C \mid v$ is left distributive over $(C,+)$ and $v \circ ab = (v \circ a)b$ for all $a, b \in C\}$. Let $\overline{FB} = \{\sum n_i v_i \mid n_i \in F$ and $v_i \in B\}$ and $\overline{C_0 B} = \{\sum u_i v_i \mid u_i \in C_0$ and $v_i \in B\}$ be the subgroups of $(C,+)$ generated by FB and C_0B respectively.

2.1. Proposition. (1) $(B,+,\circ)$ *is a subring of the near-ring* $(C_0,+,\circ)$
(2) $FB \subseteq B$ *and* $FB \circ B + B \circ FB \subseteq FB$
(3) $\overline{FB} \lhd (B,+,\circ)$
(4) *If* $nc = cn$ *for all* $n \in F$ *and* $c \in C$, *then* $BF \subseteq B$.
(5) *If* C *has a constant multiplicative identity* i, *then* $FB = B$.
(6) *Any composition left identity of* C *is contained in* B.
(7) $(0:C)_{C_2} := \{d \in C \mid d \circ c = 0$ *for all* $c \in C\} \subseteq B$.

Proof. (1) $B \subseteq C_0$ since every element of B is left distributive over $(C,+)$. Let $v_1, v_2 \in B$ and $a, b \in C$. Then:

$$\begin{aligned}
(v_1 - v_2) \circ (a+b) &= v_1 \circ a + v_1 \circ b - v_2 \circ a - v_2 \circ b \\
&= (v_1 - v_2) \circ a + (v_1 - v_2) \circ b \\
(v_1 - v_2) \circ ab &= v_1 \circ ab - v_2 \circ ab \\
&= (v_1 \circ a)b - (v_2 \circ a)b = ((v_1 - v_2) \circ a)b \\
(v_1 \circ v_2) \circ (a+b) &= v_1 \circ (v_2 \circ a + v_2 \circ b) = (v_1 \circ v_2) \circ a + (v_1 \circ v_2) \circ b \text{ and} \\
(v_1 \circ v_2) \circ ab &= v_1 \circ (v_2 \circ ab) = v_1 \circ ((v_2 \circ a)b) = (v_1 \circ v_2 \circ a)b.
\end{aligned}$$

Thus B is a subnear-ring of $(C_0,+,\circ)$. Clearly B is a ring.

(2) Let $v, u \in B$, $n \in F$. Then $nv \circ (a+b) = n \cdot (v \circ (a+b)) = n \cdot v \circ a + n \cdot v \circ b = nv \circ a + nv \circ b$ and $nv \circ ab = n \cdot v \circ ab = n \cdot (v \circ a)b = (nv \circ a)b$. Thus $FB \subseteq B$. Also, $nv \circ u = n \cdot v \circ u \in FB$ and $u \circ nv = (u \circ n)v \in FB$.

(3) Follows from (2) above.

(4) By the assumption, $BF = FB$.

(5), (6) and (7) are clear. □

2.2. Proposition. *Let* $x \in C_2$ *be a commuting composition identity. Then:*
(1) $d \circ c \cdot c = c \cdot d \circ c$ *for all* $c, d \in C$. *In particular, for all* $n \in F$ *and* $c \in C, nc = cn$.
(2) $x \in B$
(3) $(v \circ c)x = vc$ *for all* $v \in B, c \in C$.
(4) $Fx = \overline{FB} = FB = BF \lhd (B,+,\circ)$. *If* B *is a simple ring, then* $Fx = B$ *or* $Fx = 0$.

(5) *For all* $k \geq 1,\ v \circ x^{k+1} = vx^k$ *for all* $v \in B$ *and* $B^{k+1} = Bx^k$.
(6) *If* $J \lhd B$, *then* $JF + FJ \subseteq J$.
(7) $v \circ nx = nx \circ v$ *for all* $n \in F, v \in B$.
(8) $C_0x = \{ux \mid u \in C_0\}$ *is a subcomposition ring of* C.

Proof. (1) $d \circ c \cdot c = d \circ c \cdot x \circ c = dx \circ c = xd \circ c = c \cdot d \circ c$. If $d = n \in F$, then $nc = cn$.

(2) follows from Proposition 2.1.

(3) $(v \circ c)x = v \circ cx = v \circ xc = (v \circ x)c = vc$.

(4) $Fx \subseteq FB$ and for $n \in F$ and $v \in B$, $n = (v \circ n)x \in Fx$ (by (3) above). Thus $Fx \subseteq FB = BF \subseteq Fx$. Then $Fx = FB = \overline{FB} \lhd B$ follows from Proposition 2.1.

(5) For $v \in B$, $v \circ x^2 = (v \circ x)x = vx$. If $v \circ x^{k+1} = vx^k$, then $v \circ x^{k+2} = (v \circ x^{k+1})x = vx^kx = vx^{k+1}$. Clearly $Bx \subseteq B^2$ and if $v, w \in B$, then $vw = (v \circ w)x \in Bx$. Thus $B^2 = Bx$. If $B^k = Bx^{k-1}$, then $B^{k+1} = BB^k = B(Bx^{k-1}) = B^2x^{k-1} = (Bx)x^{k-1} = Bx^k$.

(6) For $j \in J$ and $n \in F$, $jn = (j \circ n)x = j \circ nx \in J \circ Fx \subseteq J \circ B \subseteq J$ and $nj = jn \in J$.

(7) $v \circ nx = (v \circ n)x = vn = nv = nx \circ v$.

(8) For $ux, vx \in C_0x$, $ux - vx = (u - v)x \in C_0x$; $ux \cdot vx = (uxv)x \in C_0x$ and $ux \circ vx = u \circ vx \cdot vx = (u \circ vx \cdot v)x \in C_0x$. □

2.3. Proposition. *Let* $x \in C_2$ *be a commuting composition identity and let* $i \in C_1$ *be a multiplicative identity. Then:*
(1) $B = \{v \in C \mid v \circ ab = (v \circ a)b$ *for all* $a, b \in C\}$.
(2) $(0 : x)_{C_1} \cap F = 0$ *(i.e.* $nx = 0, n \in F$, *implies* $n = 0$*).*
(3) *If* $(0 : x)_{C_1} = 0$ *(i.e.,* $cx = 0, c \in C$, *implies* $c = 0$*), then* $B = \{v \in C \mid (v \circ a)x = va$ *for all* $a \in C\}$
(4) *If* i *is a constant multiplicative identity, then* $(B, +, \circ) = (Fx, +, \circ) \cong (F, +, \cdot)$.

Proof. (1) Suppose $v \in C$ is such that $v \circ ab = (v \circ a)b$ for all $a, b \in C$. We show v is left distributive. For, $a, b \in C, v \circ (a + b) = v \circ (i(a + b)) = (v \circ i)(a + b) = (v \circ i)a + (v \circ i)b = v \circ ia + v \circ ib = v \circ a + v \circ b$; hence $v \in B$.

(2) If $nx = 0, n \in F$, then $n = ni = nx \circ i = 0$.

(3) Suppose $v \circ ab = (v \circ a)b$ for all $a, b \in C$. For any $c \in C, (v \circ c)x = v \circ cx = v \circ xc = (v \circ x)c = vc$. Conversely, suppose $(v \circ c)x = vc$ for all $c \in C$. For any $a, b \in C, (v \circ ab)x = v(ab) = (va)b = ((v \circ a)x)b = (v \circ a)bx$. Thus $v \circ ab - (v \circ a)b \in (0 : x)_{C_1} = 0$; hence $v \circ ab = (v \circ a)b$. Then (3) follows from (1).

(4) By Proposition 2.2(4) we know $Fx \subseteq B$. Conversely, for $v \in B, v = vi = (v \circ i)x \in Fx$ since by assumption $i \in F$. Thus $B = Fx$. Define $\psi : (F, +, \cdot) \longrightarrow (Fx, +, \circ)$ by $\psi(n) := nx$. Then it is straightforward to verify that ψ is a surjective (ring) homomorphism. If $nx = mx$, then $n = m$ follows from (2) above. Thus ψ is an isomorphism. □

We note that (1) above holds without the assumption of a composition identity.

2.4. Proposition. *Let* $x \in C_2$ *be a commuting composition identity. Suppose there is an* $s \in C_0$ *such that* $x = sx$. *Then* $B \subseteq C_0x$.

Proof. Let $v \in B$. Then $v = v \circ x = v \circ sx = (v \circ s)x \in C_0x$. □

2.5. Proposition. *Let C and D be composition rings with $\theta : C \longrightarrow D$ a surjective composition ring homomorphism and let S a subcomposition ring of C. Then*
(1) $\theta(Base(C)) \subseteq Base(D)$,
(2) $Base(C) \cap S \subseteq Base(S)$.

Proof. Straightforward. □

2.6. Examples. Below we will give several examples of composition rings and describe the base of each. In most cases we will give additional properties that will be referred to in the sequel.

(1) For each of $R[x], R_0[x]$ and $R_N[[x]]$, x is a commuting composition identity. In $R[x]$, $i = 1 \in R$ is a constant multiplicative identity and thus $Base(R[x]) = Fx = (Rx, +, \circ) \cong (R, +, \cdot)$ (Proposition 2.3(4)). Also, for each of $R_0[x]$ and $R_N[[x]]$, the base is $(Rx, +, \circ) \cong (R, +, \cdot)$. We will verify this, for example, for $R_N[[x]]$. Let $r \in R$ and $a, b \in R_N[[x]]$. Clearly rx is left distributive over $+$ and $rx \circ ab = r(ab) = (ra)b = (rx \circ a)b$. Conversely, suppose $v = v_1x + v_2x^2 + v_3x^3 + \cdots \in Base(R_N[[x]]), v_i \in R$. Then $v \circ x^2 = (v \circ x)x = vx$, i.e.

$$v_1x^2 + v_2x^4 + v_3x^6 + \cdots = v_1x^2 + v_2x^3 + v_3x^4 + \cdots .$$

The left hand side has only even powers of x; hence the coefficients of all the odd powers of x on the right hand side are 0, i.e. $v_{2k} = 0$ for all $k \geq 1$. Thus the term containing x^{4k} on the left hand side is 0, which means that the corresponding term on the right hand side is zero which yields $v_{4k-1} = 0$ (for all $k \geq 1$). Since $v_{4k-1} = 0$, the term containing $x^{2(4k-1)}$ on the left hand side is 0, which makes $v_{2(4k-1)-1} = v_{8k-3} = 0$. Continuing in this way, we get $v_m = 0$ where $m = 2^n k - (2^{n-1} - 1)$ for all $n, k \geq 1$. But any $m \geq 2$ is of the form $2^n k - (2^{n-1} - 1)$ for some n and k. Indeed, if m is even, take $n = 1$. Then $m = 2k$ for a suitable k. If m is odd and greater than 1, say $m = 2m' + 1$ ($m' \geq 1$), choose $n \in \mathbb{N}$ maximal such that $2^{n-2} \mid m'$ (if m' is odd, then $n = 2$). Hence $m' = 2^{n-2}p$ for some odd p, say $p = 2k - 1, k \in \mathbb{N}$. Then $m' = 2^{n-2}(2k - 1)$ from which $m = 2m' + 1 = 2^n k - (2^{n-1} - 1)$ follows. Thus $0 = v_2 = v_3 = v_4 \cdots$ and so $v = v_1x,\ v_1 \in R$.

(2) For any ring R, we will use the following notation for certain maps in the composition ring $M(R)$ (or $M_0(R)$ if applicable). Let $a \in R$. Then $\overline{a}, \hat{a}$ and $\tilde{a}$ denotes the following self maps of R:

$$\begin{aligned} &\overline{a}(t) = at \text{ for all } t \in R \\ &\hat{a}(t) = a \text{ for all } t \in R \text{ and} \\ &\tilde{a}(t) = \begin{cases} a & \text{if} \quad t \neq 0 \\ 0 & \text{if} \quad t = 0. \end{cases} \end{aligned}$$

Let R be a commutative ring with identity $1 \in R$. Both $M_0(R)$ and $M(R)$ have commutative composition identity $x = 1_R$, $M(R)$ has constant multiplicative identity $\hat{1}$ and $M_0(R)$ has a 0-symmetric multiplicative identity $\tilde{1}$. By Proposition 2.3(4), $B = Base(M(R)) = Fx = Rx = \{\overline{a} \mid a \in R\} = \{c \mid c$ is an R - endomorphism of $(R, +)\}$, for the latter we regard $(R, +)$ as a right R -module. But also $Base(M_0(R)) = \{\overline{a} \mid a \in R\} = B$. Indeed, for any $a \in R$ it is easy to see that $\overline{a} \in Base(M_0(R))$. Conversely, let $v \in Base(M_0(R))$. Then

$a := v(1) \in R$ and from $(v \circ \tilde{1})x = v \circ \tilde{1}x$ we get $v = \overline{a}$. For both cases ($M(R)$ and $M_0(R)$), we have $(B, +, \circ) \cong (R, +, \cdot)$ and since $x = \tilde{1}x$, we have $B \subseteq M_0(R)x$ (cf. Proposition 2.4).

More generally, let R and S be rings with $\alpha : S \to R$ a fixed function. Let $C = M(R, S, \alpha)$ be the sandwich composition ring determined by R, S and α. This means C consists of all the functions from R to S, the addition and multiplication are componentwise and the composition, here denoted by $*$, is defined by: $f * g = f \circ \alpha \circ g$. Then it can be shown that $B = Base(C) = \{f \in C \mid f \circ \alpha \in End_S(S, +)\}$.

(3) For rings R without identity, the base of $M_0(R)$, or $M(R)$, need not be isomorphic to R. Let $R = 2\mathbb{Z}$ and let $C = M_0(R)$. In this case, $Base(M_0(R)) = \{\overline{a} \mid a \in \mathbb{Z}\} \cong \mathbb{Z} \ncong R$: Clearly any $\overline{a}, a \in \mathbb{Z}$, is in $B = Base(C)$. Let $v \in B$ and suppose $v(2) = 2a$, $a \in \mathbb{Z}$. From $(v \circ \tilde{2})x = (v \circ \tilde{2})1_R = v \circ (\tilde{2}1_R) = v \circ (1_R\tilde{2}) = (v \circ 1_R)\tilde{2}$, we get $v(2)t = v(t)2$ for all $t \in R$. Then $2\overline{a}(t) = 2at = v(2)t = v(t)2$; hence $\overline{a} = v$. The isomorphism $(B, +, \circ) \cong (\mathbb{Z}, +, \cdot)$ is clear. Note that here we have $B \cap C_0x = \{\overline{a} \mid a \in 2\mathbb{Z}\}$. Indeed, if $\overline{a} = ux \in B \cap C_0x$ ($a \in \mathbb{Z}, u \in C_0$), then $2a = u(2) \cdot 2$. Thus $a = u(2) \in 2\mathbb{Z}$. The converse inclusion is clear.

(4) Let R be a commutative ring with $S \neq R$ a subring of R with identity $1_S \in S$. Let $C = \{f \in M(R) \mid f(R) \subseteq S\}$. Then C is a subcomposition ring of $M(R)$. Note that C does not have a composition identity. However, it has many left identities: Let $s_0 \in S$ and let
$x(t) = \begin{cases} t & \text{if } t \in S \\ s_0 & \text{if } t \in R \backslash S. \end{cases}$
Then x is a left identity. Here we have $Found(C) = S$ and $B = Base(C) = \{a_g \mid a \in S$ *and* $g \in C\}$ where $a_g : R \longrightarrow R$ is defined by $a_g(t) = \begin{cases} at & \text{if } t \in S \\ g(t) & \text{if } t \in R \backslash S. \end{cases}$

Indeed, every a_g is in C and also in B. Let $v \in B$ and let $a := v(1_s)$. We show $v = a_v \in B$. By definition of C, $a \in S$. Define $b, c : R \longrightarrow R$ by $b(t) = 1_s$ for all $t \in R$ and
$c(t) = \begin{cases} t & \text{if } t \in S \\ 1_s & \text{if } t \in R \backslash S. \end{cases}$

Then $b, c \in C$ and from $v \circ bc = (v \circ b)c$, we get $v(t) = (v \circ bc)(t) = ((v \circ b)c)(t) = at = a_v(t)$ for any $t \in S$. Thus $v = a_v$.

(5) Let $C = \overline{R[x]}$ where $\overline{R[x]}$ denotes all the polynomial functions. Then C is a subcomposition ring of $M(R)$ and $B = Base(C) = \{rx \mid r \in R\}$. Note that here we may have $B \cap C_0x \neq 0$. For example, if $R = \mathbb{Z}_4$, then $0 \neq 2x = 2x^2 \in B \cap C_0x$.

(6) Let $C = (R, +, \cdot, \circ)$ where $(R, +, \cdot)$ is a ring and $(R, +, \circ)$ is any near-ring.
(a) If $(R, +, \cdot)$ is a zero ring (i.e. $R^2 = 0$) then $Base(C) = \{c \in C \mid c \text{ is left distributive }\}$. If $(R, +, \circ)$ is constant, then $Base(C) = 0$.
(b) If $(R, +, \circ)$ is a zero ring, then $Base(C) = (C, +, \circ)$.

(7) Let R be a commutative ring with identity $1 \in R$. Let $(R[x, y], +, \cdot)$ be the ring of all polynomials in the two commuting indeterminates x and y. As is well-known, we can define a near-ring multiplication $\circ$ on $(R[x, y], +)$ by: $f(x, y) \circ g(x, y) := f(g(x, y), g(x, y))$. Then $C = (R[x, y], +, \cdot, \circ)$ is a composition ring with constant multiplicative identity 1. But C has no composition identity. Both x and y are composition left identities; in fact so are $c + x - c \circ x$ for all $c \in C$. These all are in $B = Base(C)$ and also $0 \neq (0 : C)_{C_2} \subseteq B$. It can be shown that $B = \{v \in C \mid v = \sum_{i=1}^{k} \sum_{j=0}^{i} c_{ij} x^{i-j} y^j$ for some $k \geq 1$ and $c_{ij} \in R$ for all i, j where $\sum_{j=0}^{i} c_{ij} = 0$ for all $i = 2, 3, ..., k\}$.

(8) With $(R[x,y],+,\cdot)$ as in (7) above, define $\circ$ by $f(x,y)\circ g(x,y)=f(g(x,x),g(y,y))$. Then $C=(R[x,y],+,\cdot,\circ)$ is a composition ring with right identities x and y and $Found(C)=R$. In this case, it can be shown that $B=Base(C)=0$. Indeed, this follows by considering $(v\circ 1)x=v\circ x=v$ and $(v\circ 1)y=v\circ y=v$ for $v\in B$.

(9) Let $(R[x,y],+,\cdot)$ be as in (7) above. On the direct sum of two copies of this ring with itself, define $\circ$ by: $(f(x,y),g(x,y))\circ(h(x,y),k(x,y)):=(f(h(x,y),k(x,y)),g(h(x,y),k(x,y)))$.

Then $C:=(R[x,y]\times R[x,y],+,\cdot,\circ)$ is a composition ring with constant multiplicative identity $(1,1)$ and commuting composition identity (x,y). Here $Found(C)=R\times R$ and $Base(C)=(R\times R)(x,y)=Rx\times Ry$.

2.7. Remark. Let C be a composition ring with $F=Found(C)$ and $B=Base(C)$. Let $S=F+B$. Then $(S,+,\circ)$ is a subnear-ring of $(C,+,\circ)$ and it can easily be verified that S is an abstract affine near-ring.

3. Composition p-rings

A composition ring C is called a *composition p-ring* if it has a commuting composition identity x and $(C_0,+)=(B,+)\oplus(C_0x,+)$.

As is well-known, $(C,+)=(C_0,+)\oplus(F,+)$; hence $(C,+)=(F,+)\oplus(B,+)\oplus(C_0x,+)$. Thus every $c\in C$ has a unique representation as $c=n+v+ux$ with $n\in F$, $v\in B$ and $u\in C_0$. This give rise to a surjective group homomorphism $\beta:(C,+)\longrightarrow(B,+)$ defined by $\beta(c)=v$ with $ker\beta=F+C_0x$. For $c=n_c+v_c+u_cx$ and $d=n_d+v_d+u_dx\in C$, $cd=n_cn_d+(v_cn_d+n_cv_d)+w$ where $w=n_cu_dx+v_cv_d+v_cu_dx+u_cxn_d+u_cxv_d+u_cxu_dx\in C_0x$ since $C_0\lhd C_1$, $B^2=Bx\subseteq C_0x$ and x is a commuting composition identity. From $FB=BF\subseteq Fx$ we get $\beta(cd)=v_cn_d+n_cv_d\in Fx$. Note that if $c,d\in C_0$, then $\beta(cd)=0$. Furthermore,

$$c\circ d=n_c+v_c\circ n_d+v_c\circ v_d+v_c\circ u_dx+u_cx\circ d$$

where $n_c+v_c\circ n_d\in F$, $v_c\circ v_d\in B$ and $v_c\circ u_dx=(v_c\circ u_d)x\in C_0x$. Let $a=u_c\circ d$. Then $u_cx\circ d=u_c\circ d\cdot d=ad$. Hence

$$\beta(c\circ d)=\beta(c)\circ\beta(d)+\beta(ad),\ \text{i.e.}$$
$$\beta(c\circ d)-\beta(c)\circ\beta(d)=\beta(ad)\in Fx \text{ from the above.}$$

If $d\in C_0$, then $a\in C_0$ and $\beta(c\circ d)=\beta(c)\circ\beta(d)$. We record these properties of β in the next number. But before we do so, note that $(B,+,\circ)$ can be made into a composition ring $(B,+,*,\circ)$ where $v*w=0$ for all $v,w\in B$.

3.1. Proposition. *Let C be a composition p-ring. Then $\beta:(C,+)\longrightarrow(B,+)$, as defined above, is a surjective group homomorphism with $ker\beta=F+C_0x$ which satisfies:*

(1) $\beta(v)=v$ *for all* $v\in B$.
(2) $\beta(cd)=\beta(c)n_d+n_c\beta(d)\in Fx$ *for all* $c,d\in C$.
(3) $\beta(c\circ d)=\beta(c)\circ\beta(d)+\beta(ad)$ *for all* $c,d\in C$ *where* $a=u_c\circ d$.

If β_0 denotes the restriction of β to C_0, then $\beta_0:(C,+,\cdot,\circ)\longrightarrow(B,+,*,\circ)$ is a surjective composition ring homomorphism with $ker\beta_0=C_0x$. □

From this result we know that C_0x is an ideal of $(C_0, +, \cdot, \circ)$. But one can say more:

3.2. Proposition. *Let C be a composition p-ring. Then for all $k \in \mathbb{N}$*
(1) $C_0^{k+1} = C_0x^k$,
(2) $C_0x^k \lhd (C_0, +, \cdot, \circ)$.

Proof. (1) (by induction on k). For $k = 1$, $C_0x \subseteq C_0^2$. If $w_1, w_2 \in C_0$, say $w_i = v_i + u_ix \in B + C_0x, i = 1, 2$, then $w_1w_2 = v_1v_2 + v_1u_2x + u_1xv_2 + u_1xu_2x \in C_0x$ since $v_1v_2 \in B^2 = Bx \subseteq C_0x$ (cf. Proposition 2.2.(5)). Thus $C_0^2 = C_0x$. If $C_0^k = C_0x^{k-1}$, then $C_0^{k+1} = C_0 \cdot C_0^k = C_0C_0x^{k-1} = C_0xx^{k-1} = C_0x^k$.
(2) Let $v, w, u \in C_0$. Then $vx^k - ux^k = (v - u)x^k \in C_0x^k, (vx^k)u = (vu)x^k \in C_0x^k,\ v(ux^k) = (vu)x^k \in C_0x^k$ and $vx^k \circ u = v \circ u \cdot u^k \in C_0C_0^k = C_0^{k+1} = C_0x^k$. Finally we have to show that $v \circ (w + ux^k) - v \circ w \in C_0x^k$. Suppose $v = v_1 + u_1x \in B + C_0x$. Then $u_1 \in C_0 = B + C_0x$ implies $v = v_1 + v_2x + u_2x^2$ with $v_2 \in B$ and $u_2 \in C_0$. Continuing in this way, we get

$$v = v_1 + v_2x + v_3x^2 + \cdots + v_kx^{k-1} + u_kx^k,\ v_i \in B,\ u_k \in C_0.$$

Substitute this expression for v in $v \circ (w + ux^k) - v \circ w$ and after simplification, we will get a sum with terms of the form

$$\begin{gathered} v_1 \circ (w + ux^k) - v_1 \circ w \\ v_{i+1}x^i \circ (w + ux^k) - v_{i+1}x^i \circ w, \qquad i = 1, 2, \cdots, k-1,\ \text{and} \\ u_kx^k \circ (w + ux^k) - u_kx^k \circ w. \end{gathered}$$

We consider these, each in turn:

$$\begin{aligned} v_1 \circ (w + ux^k) - v_1 \circ w &= v_1 \circ w + v_1 \circ ux^k - v_1 \circ w \\ &= (v_1 \circ u)x^k \in C_0x^k; \end{aligned}$$

$$\begin{aligned} &v_{i+1}x^i \circ (w + ux^k) - v_{i+1}x^i \circ w \\ &\qquad = v_{i+1} \circ (w + ux^k) \cdot (w + ux^k)^i - v_{i+1} \circ w \cdot w^i \\ &\qquad = (v_{i+1} \circ w + v_{i+1} \circ ux^k)(w^i + d_ix^k) - v_{i+1} \circ w \cdot w^i \text{ for some } d_i \in C_0 \\ &\qquad = v_{i+1} \circ w \cdot d_ix^k + (v_{i+1} \circ ux^k)w^i + (v_{i+1} \circ ux^k)d_ix^k \\ &\qquad = (v_{i+1} \circ w \cdot d_i)x^k + (v_{i+1} \circ u)x^kw^i + (v_{i+1} \circ ux^k)d_ix^k \in C_0x^k \end{aligned}$$

and finally,

$$\begin{aligned} &u_kx^k \circ (w + ux^k) - u_kx^k \circ w \\ &= u_k \circ (w + ux^k)^k \cdot (w + ux^k) - u_k \circ w \cdot w^k \in C_0^{k+1} = C_0x^k. \end{aligned}$$ □

There is much more to say about and to do with composition p-rings. To make them even more like polynomial and power series composition rings, one could add the requirement $(0 : x)_{C_1} = 0$. We have refrained from doing so, since this requirement was not generally required in what we want to present here. Futhermore, if $c \in C$ we may define the degree of c by using the decomposition $c = n + v_0 + v_1x + \cdots + v_k + u_kx^k$, $n \in F$, $v_i \in B, u_k \in C_0$. For example, if $v_0 = v_1 = \cdots = v_k = u_k = 0$, $\deg(c) = 0$, if $v_0 \neq 0$ and $v_1 = \cdots = v_k = u_k = 0$, $\deg(c) = 1$ and if $u_k \neq 0$ for all k, then $\deg(c) = \infty$, etc. One may also investigate the composition rings $C_0/C_0x^k, k \geq 2$. But these issues are not our main concern here and we will not pursue them any further. We much rather want to proceed with some examples of composition p-rings.

3.3. Examples. (1) Any one of the composition rings $R[x], R_0[x]$ and $R_N[[x]]$ is a composition p-ring. For the associated composition rings of polynomial functions $\overline{R[x]}$ and $\overline{R_0[x]}$ things are not so clear. For example, if R is an infinite field, then $\overline{R[x]} \cong R[x]$ is a composition p-ring, but if R is a finite field, then $\overline{R[x]} = M(R)$ which is not a composition p-ring (if $C = M(R)$, then $0 \neq Base(C) \subseteq C_0x$; cf Example 2.6(2)). In fact, we have:
$C = \overline{R[x]}, R$ a commutative ring with $1 \in R$, is not a composition p-ring if and only if there exists an $f = f_1x + f_2x^2 + \cdots + f_kx^k$ in $R[x]$ with $f_1 \neq 0$ and $f(t) = 0$ for all $t \in R$. Indeed, if such an f exists, then $(-f_1)t = f_2t^2 + \cdots + f_kt^k$ for all $t \in R$ and so $0 \neq (-f_1)x = f_2x^2 + \cdots + f_kx^k \in B \cap C_0x$. Conversely, if $B \cap C_0x \neq 0$, then there are $f_1, f_2, \ldots,$ $f_k \in R$ with $f_1 \neq 0$ and $f_1x = f_2x^2 + \cdots + f_kx^k \in B \cap C_0x$. Then $f := (-f_1)x + f_2x^2 + \cdots f_kx^k$ is the desired element of $R[x]$.
As examples of such elements, one may consider $2x + 2x^2$ in $\mathbb{Z}_4[x]$ or $3x + 3x^2$ in $\mathbb{Z}_6$.

(2) Let $I \lhd R$, R a commutative ring with $1 \in R$. Then $C := \{c \in R[x] \mid c \circ 0 \in I\}$ is a subcomposition ring of $R[x]$. It contains x and it is a composition p-ring.

(3) Let R be a commutative ring with $1 \in R$ and $char\, R \neq 2$. Suppose further that R has an ideal I which satisfies $I^2 \neq 0, I^3 = 0$ and for $a, b \in R$, if $ai = bi^2$ for all $i \in I$, then $b \in I$ (below we will give an example of such a ring).
Let $C = \{f \in M(I) \mid f$ is of the form $f = f_0 + f_1x + f_2x^2$ where $f_0 \in I$ and $f_1, f_2 \in R\}$. Here, as usual $x = 1_I$ (identity function on I) and $f(t) = f_0 + f_1t + f_2t^2 \in I$ for all $t \in I$. It can be verified that C is a subcomposition ring of $M(I)$ and $x = 0 + 1x + 0x^2 \in C$ is the commuting composition identity. We show $B = Rx$. The inclusion $Rx \subseteq B$ is clear, so let $v = v_1x + v_2x^2 \in B$. For any $a, b \in C$, $v \circ (a + b) = v \circ a + v \circ b$ implies $2v_2ab = 0$. Thus $v_2ab = 0$. Let $i \in I$ and let $a(t) = b(t) = i$ for all $t \in I$. Then $a, b \in C$ and $v_2i^2 = v_2a(t)b(t) = 0 = 0i$. By our assumption on R, we get $v_2 \in I$. This means $v_2x^2 = 0$ (since $(v_2x^2)(i) = v_2i^2 \in I^3 = 0$) and so $v = v_1x \in Rx$. Thus $B = Rx$. Note also, since $I^3 = 0$, $C_0x = Rx^2$. Next we show $B \cap C_0x = 0$. Let $ax = bx^2 \in B \cap C_0x$, $a, b \in R$. For any $i \in I$, $ai = bi^2$ from which $b \in I$ follows. Thus $bx^2 = 0$ and so $B \cap C_0x = 0$. We conclude that $(C_0, +) = (B, +) \oplus (C_0x, +)$ and so C is a composition p-ring.
We note that $(0 : x)_{C_1} \neq 0$. For example, choose $0 \neq b \in I^2$. Then $bx = 0$ but $b \neq 0$.

Finally we show the existence of a ring R with the above claimed properties. Let $\mathbb{Z}$ be the ring of integers and let I be the commutative ring of all 3 x 3 matrices of the form $\begin{bmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix}$ where $a, b \in \mathbb{Z}$.

Then $I^2 \neq 0$ but $I^3 = 0$ and $char\, I \neq 2$. Let R be the Dorroh extension of I (i.e. the standard unital extension). This means $(R, +) = (I, +) \oplus (\mathbb{Z}, +)$ and $(a, n)(b, m) = (ab + nb + ma, nm)$ and $I \cong (I, 0) \lhd R$. R is a commutative ring with identity $(0, 1)$ and

$char\, R = 0$. Finally, let $(a,n),(b,m) \in R$ such that $(a,n)(i,0) = (b,m)(i,0)^2$ for all $i \in I$. From this we get $ai + ni = bi^2 = mi^2$ for all $i \in I$ (where $a, b \in I$, $n, m \in \mathbb{Z}$). Suppose

$$a = \begin{bmatrix} 0 & \alpha & \alpha' \\ 0 & 0 & \alpha \\ 0 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & \beta & \beta' \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{bmatrix} \text{ and } i = \begin{bmatrix} 0 & j & j' \\ 0 & 0 & j \\ 0 & 0 & 0 \end{bmatrix} \text{ where } \alpha, \alpha', \beta, \beta', j, j' \in \mathbb{Z}.$$

From $ai + ni = bi^2 + mi^2$ we get $\alpha j + nj' = mj^2$ and $nj = 0$ for all $j, j' \in \mathbb{Z}$. For $j = 1$ we get $n = 0$ and so $\alpha j = mj^2$ for all $j \in \mathbb{Z}$. From this we get $m = \alpha = 0$ and we conclude with $(b,m) = (b,0) \in I$.

(4) Let $C = (R, +, \cdot, \circ)$ where $(R, +, \cdot)$ is a zero ring and $(R, +, \circ)$ is a ring with identity x. Then $F = Found(C) = 0$, $B = Base(C) = C$ and $C_0x = Cx = 0$. Hence C is (trivially) a composition p-ring.

(5) Let $(R[x,y], +, \cdot)$ be the ring as in Example 2.6 (7). On $R[x,y] \times R[x,y]$ define $\circ$ by: $(f(x,y), g(x,y)) \circ (h(x,y), k(x,y)) = (f(h(x,y), y), g(x, k(x,y)))$. Then $C = (R[x,y] \times R[x,y], +, \cdot, \circ)$ is a composition ring with commuting composition identity (x,y) and constant multiplicative identity $(1,1)$. Here $F = Found(C) = \{(f(x,y), g(x,y)) \in C \mid f(x,y)$ is a polynomial in y and $g(x,y)$ is a polynomial in x, i.e. $f(x,y) = f(r,y)$ and $g(x,y) = g(x,r)$ for all $r \in R\}$. By Proposition 2.3(4) we get $B = Base(C) = \{(f(x,y)x, g(x,y)y) \mid (f(x,y), g(x,y)) \in F\}$. We note that $C_0 = \{(f(x,y), g(x,y)) \mid$ every term of $f(x,y)$ contains at least one x, and every term of $g(x,y)$ contains at least one $y\}$. Then it can be verified that $(C_0, +) = (B, +) \oplus (C_0(x,y), +)$, hence C is a composition p-ring.

We should point out that the composition ring on $R[x,y] \times R[x,y]$, as defined in Example 2.6(9), is not a composition p-ring. Even though it may have some other useful applications (in connection to abstract affine near-rings), the above shows that the composition on $R[x,y] \times R[x,y]$ as defined in the present example, is more 'polynomial like'.

It should be clear that this example can be generalized to any finite number of commuting indeterminates $x_1, x_2, x_3, ..., x_n$: On n copies of the ring $R[x_1, x_2, x_3, ..., x_n]$, define the composition by:

$$(f_1, f_2, \ldots, f_n) \circ (g_1, g_2, \ldots, g_n) =$$
$$(f_1(g_1, x_2, x_3, \ldots, x_n),\ f_2(x_1, g_2, x_3, ..., x_n), \ldots, f_n(x_1, x_2, x_3, ..., g_n))$$

where $f_i = f_i(x_1, x_2, x_3, ..., x_n)$ and $g_i = g_i(x_1, x_2, x_3, ..., x_n)$ are from $R[x_1, x_2, x_3, ..., x_n]$.

3.4. Remark. In our definition of a composition p-ring, we have opted for clarity of presentation following our motivating examples as closely as possible. If more generality is required, one need not require the existence of the commuting composition identity. In such a case, one could replace all occurences of Fx and C_0x with $\overline{FB}$ and $\overline{C_0B}$ respectively. A composition ring C can then be called a *composition p-ring* if $BF = FB$ and $(C_0, +) = (B, +) \oplus (\overline{C_0B}, +)$. In this case, the definition of β stays the same, but then one can show that $\beta(cd) \in \overline{FB}$ and with slightly more effort that $\beta(c \circ d) - \beta(c) \circ \beta(d) \in \overline{FB}$. In case C does have a commuting composition identity x, then we know from Proposition 2.2(4) that $\overline{BF} = \overline{FB} = Fx$. But also $\overline{C_0B} = C_0x$: Let $u \in C_0$ and $v \in B$. Since $C_0 = B + \overline{C_0B}$,

$u = w + \sum u_i v_i$ for some $u_i \in C_0$ and $w, v_i \in B$. Using the fact that $B^2 = Bx \subseteq C_0x$, we will get $uv \in C_0x$. Thus $C_0x \subseteq C_0B \subseteq C_0x$ and so $\overline{C_0B} = C_0x$.

3.5. Example. Let R be the ring of even integers, i.e. $R = (2\mathbb{Z}, +, \cdot)$. Then $C := (R[x], +, \cdot, \circ)$ is a composition ring which does not have a multiplicative identity nor a composition identity. Here $F = Found(C) = R$ and $B = Fx = Rx$ with $BF = FB = \overline{FB} \subset Fx$ (for example, $2x \in Fx$, but $2x \notin FB$). Futhermore, $B \cap \overline{C_0B} = B \cap C_0B = 0$ and $(C_0, +) = (B, +) \oplus (\overline{C_0B}, +)$.

4. Semi-constants

We start with

4.1. Proposition. *Let C be a composition ring with $x \in C$ a commuting composition identity. Let $s \in C$ be such that*

(1) $c \circ s \circ a \cdot a = c \circ s \cdot a$ *for all* $c, a \in C$.
(2) $c \circ s \cdot s = c \circ s$ *for all* $c \in C$.
(3) $(0 : s)_{C_1} \cap B = 0$ *(i.e.* $vs = 0$, $v \in B$, *implies* $v = 0$ *).*

Then $(C, +) = (B, +) \oplus (D_s, +)$ *where* $D_s = \{c \in C \mid (c \circ s)x = 0\} \lhd C_1$.

Proof. For any $c \in C$, $c = (c \circ s)x + (c - (c \circ s)x)$. We show $(c \circ s)x \in B$. For any $a, b \in C$ we have:

$$\begin{aligned}(c \circ s)x \circ (a + b) &= c \circ s \circ (a + b) \cdot (a + b)\\ &= c \circ s \cdot (a + b) \qquad \text{by (1)}\\ &= (c \circ s)a + (c \circ s)b\\ &= c \circ s \circ a \cdot a + c \circ s \circ b \cdot b \qquad \text{by (1)}\\ &= (c \circ s)x \circ a + (c \circ s)x \circ b\end{aligned}$$

and

$$(c \circ s)x \circ ab = c \circ s \circ ab \cdot ab = c \circ s \cdot ab = ((c \circ s)a)b = (c \circ s \circ a \cdot a)b = ((c \circ s)x \circ a)b.$$

Next we show $c - (c \circ s)x \in D_s$:

$$\begin{aligned}&((c - (c \circ s)x) \circ s)x\\ &= (c \circ s)x - (c \circ s \circ s \cdot s)x\\ &= (c \circ s)x - (c \circ s \cdot s)x \qquad \text{by (1)}\\ &= (c \circ s)x - (c \circ s)x \qquad \text{by (2)}\\ &= 0.\end{aligned}$$

If $v \in B \cap D_s$, then $0 = (v \circ s)x = vs$. Hence $v \in (0 : s)_{C_1} \cap B = 0$ by (3) and we conclude that $(C, +) = (B, +) \oplus (D_s, +)$.

Finally we show $D_s \lhd C_1$. Let $c_1, c_2 \in D_s$ and $c \in C$. Then $((c_1 - c_2) \circ s)x = (c_1 \circ s)x - (c_2 \circ s)x = 0$, $(c_1 c \circ s)x = c_1 \circ s \cdot c \circ s \cdot x = ((c_1 \circ s)x)c \circ s = 0$ and $(cc_1 \circ s)x = c \circ s \cdot (c_1 \circ s)x = 0$. $\square$

4.2. Corollary. *Let $x \in C$ be a commuting composition identity and let $s \in C$ be such that:*
(1) $c \circ s \circ a \cdot a = c \circ s \cdot a$ *for all $c, a \in C$ and*
(2) $x = sx$.
Then $(0 : s)_{C_1} \cap B = 0$ and $c \circ s \cdot s = c \circ s$ for all $c \in C$; hence $(C, +) = (B, +) \oplus (D_s, +)$ and $D_s \lhd C_1$.

Proof. We only have to verify that conditions (2) and (3) of Proposition 4.1 are fulfilled. For any $c \in C$, $c \circ s = x \circ (c \circ s) = sx \circ (c \circ s) = s \circ c \circ s \cdot c \circ s$

$$\begin{aligned} &= x \circ s \circ (c \circ s) \cdot c \circ s = x \circ s \cdot c \circ s && \text{by (1)} \\ &= s \cdot c \circ s \\ &= c \circ s \cdot s && \text{by Proposition 2.2(1)} \end{aligned}$$

If $v \in B$, then $vs = (v \circ s)x = v \circ sx = v \circ x = v$; hence $vs = 0$ if and only if $v = 0$. Thus $(0 : s)_{C_1} \cap B = 0$. □

4.3. Definition. *Let C be a composition ring. Then $s \in C$ is called a semi-constant if:*
(1) $s \circ s = s$,
(2) $(c \circ s)b \circ a = c \circ s \cdot b \circ a$ *for all $c \in C$ and $a, b \in C_0$.*
A minimal semi-constant s is a semi-constant which satisfies: $c \circ s \circ t \cdot a = c \circ s \cdot a$ for all $c \in C, a \in C_0$ and non-zero semi-constants t.

Any constant s (i.e. $s \in F$) is a minimal semi-constant. We now describe all the minimal semi-constants of $M_0(R)$ and $M(R)$. In case C has a commuting composition identity x and $s \in C$ is a semi-constant, then $c \circ s \circ a \cdot a = c \circ s \cdot a$ for all $c \in C$ and $a \in C_0$ (just let $b = x$).

4.4. Proposition. *Let R be a commutative ring which satisfies $Ry = 0$ implies $y = 0$ (e.g., if $1 \in R$). Let $C = M_0(R)$. Then $s \in M_0(R)$ is a semi-constant if and only if $s = \tilde{r}$ for some $r \in R$ (for the definition of $\tilde{r}$, see* Example 2.6(2)).

Proof. If S is of the form $s = \tilde{r}$ for some $r \in R$, then it is straightforward to verify that S is a semi-constant. Conversely, suppose $s \in C$ is a semi-constant. If $s = 0$ we are done, so suppose $s(t_0) = r_0 \neq 0$ for some $0 \neq t_0 \in R$. Choose $y \in R$ with $yt_0 \neq 0$. We firstly show that $t \in R$, $t \neq 0$ implies $s(t) \neq 0$. To the contrary, if $s(t) = 0$ for some $t \neq 0$, let

$$a(r) = \begin{cases} t_0 & \text{if} \quad r = t \\ 0 & \text{if} \quad r \neq t. \end{cases}$$

Then $a \in C = C_0$ and from $(\tilde{y} \circ s)b \circ a = \tilde{y} \circ s \cdot b \circ a$, with $b = 1_R$, we get $0 \neq yt_0 = \tilde{y}(r_0)t_0 = \hat{y}(s(t_0)) \cdot t_0 = \tilde{y}(s(a(t))a(t)) = \tilde{y}(s(t)) \cdot a(t) = \tilde{y}(0) \cdot t_0 = 0$; a contradiction. Thus $t \neq 0$ implies $s(t) \neq 0$.
Suppose for $t_1 \in R$, $t_1 \neq 0$, $t_1 \neq t_0$, we have $s(t_1) = r_1 \neq r_0$. Then $r_1 \neq 0$. Define functions d and b by

$$d(t) = \begin{cases} y & \text{if} \quad t = r_0 \\ 0 & \text{if} \quad t \neq r_0 \end{cases} \qquad \text{and} \qquad b(t) = \begin{cases} t_0 & \text{if} \quad t = t_1 \\ 0 & \text{if} \quad t \neq t_1 \end{cases}$$

Then $d \in C$, $b \in C = C_0$ and from $(d \circ s)1_R \circ b = d \circ s \cdot 1_R \circ b$ we get $0 \neq yt_0 = d(r_0)b(t_1) = d(s(t_0)) \cdot b(t_1) = d(s(b(t_1)))b(t_1) = d(s(t_1)b(t_1) = d(r_1)b(t_1) = 0$; a contradiction. Hence $r_1 = r_0$ and we conclude that $s = \tilde{r}_0$. □

4.5. Corollary. *Let R be a commutative ring which satisfies $Ry = 0$ implies $y = 0$. Let $C = M_0(R)$. Then the following are equivalent:*
(1) $s \in C$ is a semi-constant.
(2) $s = \tilde{r}$ for some $r \in R$.
(3) s is a minimal semi-constant. □

There is a similar characterization of the semi-constants in $M(R)$ but, since the mappings are not necessarily 0-symmetric, the proof is slightly different.

4.6. Proposition. *Let R be a commutative ring which satisfies $Ry = 0$ implies $y = 0$. Let $C = M(R)$. Then the following are equivalent:*
(1) $s \in C$ is a semi-constant.
(2) There exists an $r \in R$ such that $s = \hat{r}$ (i.e. s is constant) or $s = \tilde{r}$.
(3) s is a minimal semi-constant.

Proof. We will only provide the verification of (1) $\Rightarrow$ (2); the other implications being straightforward. Suppose $s \in C$ is a semi-constant. If s is not a constant, we are done, so suppose s is a constant. Then there are $t_1 \neq t_2$ in R such that $s(t_1) = r_1 \neq r_2 = s(t_2)$. At least one of t_1 or t_2 is non-zero, say $t_2 \neq 0$. We will show that $t_1 = 0$. Suppose this is not the case, i.e. $t_1 \neq 0$. We also know that at least one of r_1 or r_2 must be non-zero. We will consider both these cases.
Suppose $r_1 \neq 0$ and choose $y \in R$ with $yt_1 \neq 0$. Define a and c in C by:

$$a(t) = \begin{cases} t_1 & \text{if } t = t_2 \\ 0 & \text{if } t \neq t_2 \end{cases} \qquad \text{and} \qquad c(t) = \begin{cases} y & \text{if } t = r_1 \\ 0 & \text{if } t \neq r_1 \end{cases}$$

Then $a \in C_0$ and from $(c \circ s)1_R \circ a = c \circ s \cdot 1_R \circ a$ we get $0 \neq yt_1 = c(r_1)t_1 = c(s(t_1))t_1 = c(s(a(t_2)))a(t_2) = c(s(t_2))a(t_2) = c(r_2)a(t_2) = 0$; a contradiction.

If $r_2 \neq 0$, choose $z \in R$ such that $zt_2 \neq 0$ and let

$$b(t) = \begin{cases} t_2 & \text{if } t = t_1 \\ 0 & \text{if } t \neq t_1 \end{cases} \qquad \text{and} \qquad d(t) = \begin{cases} z & \text{if } t = t_2 \\ 0 & \text{if } t \neq t_2. \end{cases}$$

Then $d \in C, b \in C_0$ and from $d \circ s \circ b \cdot b = d \circ s \cdot b$ we will get the contradiction $zt_2 = 0$. Thus we must have $t_1 = 0$. But this means that s is of the form $s(t) = \begin{cases} r_2 & \text{if } t \neq 0 \\ r_1 & \text{if } t = 0. \end{cases}$
Since $s \circ s = s$, for any $0 \neq t \in R$, we then have $r_2 = s(t) = s(s(t)) = s(r_2)$. This means $r_2 \neq 0$ and $r_1 = s(0) = s(s(0)) = s(r_1)$. Thus $r_1 = 0$ and so $s = \tilde{r}_2$. □

4.7. Example. In $C = \overline{\mathbb{Z}_4[x]}$, $s = x + x^3$ is a non-zero semi-constant (which is not a constant). □

Let $0 \neq s \in C$ be a minimal semi-constant, C any composition ring. Let

$$k = \begin{cases} s & \text{if} \quad F = 0 \\ s \circ n & \text{for} \quad 0 \neq n \in F. \end{cases}$$

We note that in general, if $F \neq 0$, then k is not uniquely defined. One would be tempted to argue if $x \in C_2$ is a commuting composition identity and if $i \in C_1$ is a multiplicative identity, then $s \circ n = s$ since $s \circ n = x \circ s \circ n \cdot i = x \circ s \cdot i = si = s$ (using the fact that s is a minimal semi-constant and $n \neq 0$ is a semi-constant). But this is not a valid argument since $i \in C_0$ need not hold (as is required in the definition of a minimal semi-constant). If $i \in C_0$, then $C = C_0$ and $F = 0$ in which case $s \circ n$ is of no concern. In any case, k itself is a minimal semi-constant.

Let $v \in B$. Define a mapping $\beta : (C, +, \cdot) \to (B, +, \circ)$ by $\beta(c) = (c \circ k)v$ for all $c \in C$. This mapping depends only on s and v (and not on the n chosen) since s is a minimal semi-constant and $v \in B \subseteq C_0$. It is easy to see that β is a group homomorphism, but first priority should be to confirm that it is well-defined, i.e. $(c \circ k)v \in B$ for all $c \in C$. We distinguish two cases.

If $F = 0$, then for all $a, b \in C = C_0$: $(c \circ k)v \circ (a+b) = (c \circ s)v \circ (a+b) = c \circ s \cdot v \circ (a+b) = c \circ s \cdot (v \circ a + v \circ b) = c \circ s \cdot v \circ a + c \circ s \cdot v \circ b = (c \circ s)v \circ a + (c \circ s)v \circ b$. Futhermore, $(c \circ k)v \circ ab = (c \circ s)v \circ ab = c \circ s \circ ab \cdot v \circ ab = c \circ s \cdot v \circ ab = c \circ s \cdot (v \circ a)b = (c \circ s \circ a \cdot v \circ a)b = ((c \circ s)v \circ a)b = ((c \circ k)v \circ a)b$. Thus if $F = 0$, then $(c \circ k)v \in B$ for all $c \in C$. If $F \neq 0$, then $(c \circ k)v \in Fx \subseteq B$. Thus $\beta(c) \in B$ for all $c \in C$. We record this (and more), for the special case $v = x$, in

4.8. Proposition. *Let C be a composition ring with commuting composition identity x and non-zero semi-constant s. Let $k = s$ if $F = 0$ and $k = s \circ n$ for $0 \neq n \in F$. Then $\beta_s : (C, +, \cdot) \to (B, +, \circ)$ defined by $\beta_s(c) = (c \circ k)x = (c \circ s)x$ is a ring homomorphism. If $F \neq 0$, then $\beta_s(C) \subseteq Fx$. The homomorphism β_s is surjective if and only if there is a $d \in C$ with $d \circ s \cdot x = x$. If β_s is surjective and $F \neq 0$, then $(C \circ k)x = \beta_s(C) = B = Fx$.*

Proof. We start by showing that β_s is a ring homomorphism. For $c, d \in C$, and since k is a semi-constant with $(d \circ k)x \in C_0$, we get

$$\begin{aligned} \beta_s(cd) = (cd \circ k)x = c \circ k \cdot d \circ k \cdot x &= c \circ k \circ (d \circ k)x \cdot (d \circ k)x \\ &= (c \circ k)x \circ ((d \circ k)x) = \beta_s(c) \circ \beta_s(d). \end{aligned}$$

If β_s is surjective, then $x \in B = \beta_s(C)$ implies there is a $d \in C$ with $x = (d \circ k)x = d \circ s \cdot x$. Conversely, if such a $d \in C$ exists, then for any $v \in B$, $v \circ d \in C$ and $\beta_s(v \circ d) = (v \circ d \circ k)x = (v \circ d \circ s)x = (v \circ (d \circ s))x = v \circ (d \circ s)x = v \circ x = v$. Finally, if β_s is surjective and $F \neq 0$, then $Fx \subseteq B = \beta_s(C) = (C \circ k)x \subseteq Fx$. □

For two non-zero minimal semi-constants s_1 and s_2 in C, with associated ring homomorphisms β_1 and β_2 respectively (i.e. $\beta_i(c) = (c \circ s_i)x$), we always have $\beta_1(C) = \beta_2(C)$, but β_1 need not coincide with β_2. Indeed, for any $c \in C$, $\beta_1(c) = (c \circ s_1)x = (c \circ s_1 \circ s_2)x \in (C \circ s_2)x = \beta_2(C)$ and $\beta_2(C) \subseteq \beta_1(C)$ follows similarly. The next example will show that in general $\beta_1 \neq \beta_2$ and in Proposition 5.1(9) we will show under which circumstances they do coincide.

4.9. Example. Let $R = \mathbb{Z}_4$ and let $C = M_0(R)$. Then $s_1 = \tilde{1}$ and $s_2 = \tilde{2}$ are two non-zero minimal semi-constants in C. Let β_1 and β_2 be the associated ring homomorphism respectively. Then $\beta_1(x) = (x \circ s_1)x = s_1 x = \tilde{1}$ while $\beta_2(x) = (x \circ s_2)x = s_2 x = \tilde{2} \neq \tilde{1}$. □

Worthwhile to point out is that if β_s is surjective for any one non-zero minimal semi-constant s, then $\beta_{s'}$ is surjective for all non-zero minimal semi-constants s'. Indeed, since β_s is surjective, there is a $d \in C$ with $d \circ s \cdot x = x$ (by Proposition 4.8). Since s' is a non-zero semi-constant and s is minimal $(d \circ s) \circ s' \cdot x = d \circ s \circ s' \cdot x = d \circ s \cdot x = x$; hence $\beta_{s'}$ is surjective.

5. Composition *t*-rings

A composition ring C is called a *composition t-ring* if C has a commuting composition identity x and a non-zero minimal semi-constant s with $x = sx$.

If we use this s to define k as in the previous section, there are many useful properties that we list in:

5.1. Proposition. *Let C be a composition t-ring and let*
$k = \begin{cases} s & \text{if} \quad F = 0 \\ s \circ n & \text{for} \quad 0 \neq n \in F \end{cases}$ *where $0 \neq s$ is a minimal semi-constant in C with $x = sx$.*
Let $\beta_s : C \to B$ be the mapping defined by $\beta_s(c) = (c \circ s)x$. Then:

(1) *k is a non-zero minimal semi-constant and $kx = x = xk$.*
(2) *$c \circ k \circ a \cdot a = c \circ k \cdot a$ for all $a, c \in C$.*
(3) *$a = ka$ for all $a \in C$ and $vk = v$ for all $v \in B$.*
(4) *If $F \neq 0$, then $k \in F$ is an identity for the ring $(F, +, \cdot)$ and $B = Fx$.*
(5) *$k \circ k = k$, $k \circ s = k = s \circ k$ and $kk = k$.*
(6) *$(0 : x)_{C_1} \cap F = 0$*
(7) *$c \circ k = k \cdot c \circ k = c \circ k \cdot k$*
(8) *If s' is any other non-zero minimal semi-constant in C and k' is defined analogously to k, then $k = k'$ and $(c \circ k)x = (c \circ k')x$ for all $c \in C$ (i.e. $\beta_s = \beta_{s'}$)*
(9) *β_s is surjective.*

Proof. (1) If $k = s \circ n = 0$ $(0 \neq n \in F)$, then $n = x \circ n = sx \circ n = s \circ n \cdot n = 0$; a contradiction. That k is a semi-constant is clear, since either $k = s$ or $k \in F$. If $k = s$, then $kx = x = xk$. If $k = s \circ n$, then $kx = (x \circ s \circ n)x = (x \circ s)x = sx = x$ since s is a minimal semi-constant.

(2) We distinguish two cases:
If $F = 0$, then $c \circ k \circ a \cdot a = c \circ s \circ a \cdot a = c \circ s \cdot a = c \circ k \cdot a$ for all $c \in C, a \in C = C_0$. If $F \neq 0$, then $c \circ k \circ a \cdot a = c \circ k \cdot a$ for all $c, a \in C$ (since $k \in F$).

(3) For any $a \in C$, $a = x \circ a = kx \circ a = k \circ a \cdot a = x \circ k \circ a \cdot a = x \circ k \cdot a = ka$ (by (2) above) and $v = v \circ x = v \circ xk = (v \circ x)k = vk$.

(4) If $F \neq 0$, then $k \in F$ and the result follows from (3) above and Proposition 2.2(1). For any $v \in B$, $v = v \circ x = v \circ kx = (v \circ k)x \in Fx$.

(5) Straightforward.

(6) If $F = 0$ we are done. If $F \neq 0$ and $m \in F$ with $mx = 0$, then $m = mk = mx \circ k = 0$.

(7) By (3) above and Proposition 2.2(1).

(8) Let $k' = s'$ if $F = 0$ and let $k' = s' \circ n$ for $0 \neq n \in F$. Then k' has all the properties that k does. From (4) above we get $k' = k$.

(9) β_s surjective follows from Proposition 4.8 since $x = kx = (x \circ k)x$. □

From Propositions 5.1, 4.8 and 4.1 we get most of :

5.2. Proposition. *Let C be a composition t-ring. Then there exists a non-zero minimal semi-constant k such that $(C, +) = (B, +) \oplus (D, +)$ where $D = \{c \in C \mid (c \circ k)x = 0\} \lhd C_1$ and $\beta : (C, +, \cdot) \to (B, +, \circ)$ defined by $\beta(c) = (c \circ k)x$ for all $c \in C$ is a surjective ring homomorphism with $ker\beta = D$ and $\beta(v) = v$ for all $v \in B$. The mapping β is independent of which minimal semi-constant $k \neq 0$ is chosen. If $F \neq 0$, then $B = Fx$.* □

We note that if C is a composition ring with $F = 0$, then it cannot be both a composition p-ring as well as a composition t-ring. The reason being that any 0-symmetric composition t-ring has $0 \neq x \in B \subseteq C_0x$ (cf. Proposition 2.4). But if $F \neq 0$, a composition ring can be of both types. For example, let $C = R[x]$. Then C is a composition p-ring as well as a composition t-ring ($s = 1 \in F$ is a non-zero minimal (semi-) constant with $x = sx$). We note further that the two associated mappings, say β_p and β_t respectively, are not the same. For $c = c_0 + c_1x + c_2x^2 + \cdots + c_kx^k \in C$, $\beta_p(c) = c_1x$ and $\beta_t(c) = (c_0 + c_1 + \cdots + c_k)x$. Of course, both have image $B = Fx = Rx$, but $ker\beta_p = \{c \in C \mid c_1 = 0\}$ and $\ker\beta_t = \{c \in C \mid c \circ 1 = 0\}$ are not comparable: For any $0 \neq r \in R$, $-r + rx \in ker\beta_t$ but $-r + rx \notin ker\beta_p$ and for $a, b \in R$ with $a + b \neq 0$, $a + bx^2 \in ker\beta_p$ but $a + bx^2 \notin \ker\beta_t$.

5.3. Examples. (1) Let R be a commutative ring with $1 \in R$ and let C be a subcomposition ring of $M(R)$ such that $x = 1_R \in C$. If $Found(C) \neq 0$, then C is a composition t-ring if and only if $\hat{1} \in C$. Indeed, if $\hat{1} \in C$, then $\hat{1}$ is a non-zero minimal (semi-) constant in C with $x = \hat{1}x$. Conversely, if C is a composition t-ring, let $0 \neq k \in Found(C)$ be such that $x = kx$. Then $t = k(t)t$ for all $t \in R$ and so $k(1) = 1$. Since k is constant, we conclude that $\hat{1} = k \in C$.
If $Found(C) = 0$ and $\tilde{1} = k \in C$, then C is a composition t-ring. If R has no non-zero zero divisors, then also the converse is true, i.e. if C is a composition t-ring, then $\tilde{1} \in C$.

(2) $\overline{R[x]}$ is always a composition t-ring since $\hat{1} \in \overline{R[x]}$. $\overline{R_0[x]}$ need not be a composition t-ring. For example, suppose R has a non-zero nilpotent element b, say $b^n = 0$ but $b^{n-1} \neq 0$. Then $\overline{R_0[x]}$ is not a composition t-ring. To the contrary, if it is, then $x = sx$ for some $0 \neq s \in \overline{R_0[x]}$. Thus, if $s = s_1x + s_2x^2 + \cdots + s_kx^k$, we get $t = s(t)t = s_1t^2 + s_2t^3 + \cdots + s_kt^{k+1}$ for all $t \in R$. Hence $t^{n-1} = s_1t^{n-1} + s_2t^{n+2} + \cdots s_kt^{n+k-1}$ for all $t \in R$. For $t = b$, we get $0 \neq b^{n-1} = s_1b^{n+1} + s_2b^{n+2} + \cdots + s_kb^{n+k-1} = 0$, a contradiction.

6. Ideals in composition rings

At first we will consider the relationship between the ideals of C and those of the foundation F of C. Most of these are known results (often with their origins in the more general composition near-rings) and are, of course, only interesting if $F \neq 0$.

Let $\psi : C \to M(F)$ be the composition ring homomorphism defined by $\psi(c) = \psi_c : F \to F$ ($F = Found(C)$), $\psi_c(n) = c \circ n$ for all $n \in F, c \in C$. $Ker\psi = (0 : F)_{C_2} = \{c \in C \mid c \circ F = 0\} \lhd C$ and if $Ker\psi = 0$, then C can be regarded as a subcomposition ring of $M(F)$. In particular, if C is simple and $F \neq 0$, then C is a subcomposition ring of $M(F)$. But one can say more:

6.1. Proposition. (Adler [1]) *Let R be a ring. If every subcomposition ring C of $M(R)$ with Found $(C) = R$ is simple, then R is simple. Conversely, if R is a simple ring with identity, then every subcomposition ring C of $M(R)$ with $Found(C) = R$ is simple.* □

The foundation F can always be considered as a C-module in the following sense: Let C be a composition ring and let $S = (S, +, \cdot)$ be a ring. Then S is called a *C-module* if there is a mapping $C \times S \to S, (c, s) \longmapsto cs$ which satisfies $(c + d)s = cs + ds$, $(cd)s = (cs)(ds)$ and $(c \circ d)s = c(ds)$ for all $c, d \in C, s \in S$. A subset J of S is an *C-ideal* of S if J is an ideal of S and $c(s + j) - cs \in J$ for all $c \in C, s \in S$ and $j \in J$. S is *C-simple* if it has no non-trivial proper C-ideals. Clearly, if S is simple, then it is C-simple. The foundation F of C is always an C-module in the cannonical sense: $(c, n) := c \circ n$.

The composition rings C for which the notions "C-ideal" and "ideal" of F coincide, are called *compatible*. Any composition ring with $F = 0$ is trivially compatible. Not all composition p-rings are compatible. For example, let R be a commutative ring with $1 \in R$ which contains a nil ideal N and suppose N has a non-zero ideal J which is not an ideal in R (an example of such a ring R is given below). Then there are elements $j \in J$ and $r \in R$ such that $rj \notin J$. Let $C = R_N[[x]]$. Then $Found(C) = N$ and J is an ideal of N which is not a C-ideal of N: $rx \circ (n + j) - rx \circ n = rj \notin J$ $(rx \in C, n \in N)$. We now give an example of the ring R with the properties as claimed. Let A be a commutative ring with $1 \in R$ and which contains an ideal $B \neq 0$ with $B^2 = 0$ (for example, take B any ring with zero multiplication and let A be any unital ideal extension of B). Let R be the subring of the full 3×3 matrix ring over A consisting of all matrices of the form $\begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}$. It can be verified that R is a commutative ring with identity (the usual matrix ring identity).

Let J be the set of all matrices of the form $\begin{bmatrix} 0 & b & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix}$ with $b \in B$ and let N be the set of all matrices of the form $\begin{bmatrix} 0 & b & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix}$ with $b \in B$ and $c \in A$. Then $J \lhd N \lhd R$, $N^2 = 0$ and J is not an ideal of R: For $0 \neq b \in B$,

$$\begin{bmatrix} 0 & b & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix} \in J, \text{ but } \begin{bmatrix} 0 & b & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & b & b \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix} \notin J.$$

6.2. Proposition. *Let C be a composition ring with commuting composition identity x. If every $c \in C_0$ is of the form $c = v_0 + v_1x + v_2x^2 + \cdots + v_kx^k$ for some $k \geq 0$, $v_i \in B$ for all $i = 0, 1, 2, \cdots, k$, then C is compatible.*

Proof. Let $j \in J \lhd F = Found(C)$, $c \in C$ with $c = m + u \in F + C_0$ and $u = \Sigma_{i=0}^{k} v_i x^i$, $k \geq$

$0, v_i \in B$. Let $n \in F$. Then $(n+j)^i = n^i + d_i j$ for some $d_i \in F$ and we get:

$$\begin{aligned} &c \circ (n+j) - c \circ n \\ &\qquad = m \circ (n+j) + \Sigma v_i x^i \circ (n+j) - m \circ n - \Sigma v_i x^i \circ n \\ &\qquad = \Sigma v_i \circ (n+j) \cdot (n+j)^i - \Sigma v_i \circ n \cdot n^i \\ &\qquad = \Sigma v_i \circ n \cdot (n+j)^i + \Sigma v_i \circ j \cdot (n+j)^i - \Sigma v_i \circ n \cdot n^i \\ &\qquad = \Sigma v_i \circ n \cdot d_i j + \Sigma v_i \circ j \cdot n^i + \Sigma v_i \circ j \cdot d_i j \\ &\qquad = \Sigma v_i \circ (n d_i j) + \Sigma v_i \circ (j n^i) + \Sigma v_i \circ (j d_i j). \end{aligned}$$

Now $v_i \circ (n d_i j) = (v_i \circ n d_i) j \in FJ \subseteq J$, $v_i \circ (j n^i) = (v_i \circ n^i) j \in FJ \subseteq J$ (cf. Proposition 2.2(1)) and $v_i \circ (j d_i j) = (v_i \circ j d_i) j \in FJ \subseteq J$. Hence $c \circ (n+j) - c \circ n \in J$. □

For a composition ring C with $F = Found(C)$ we will use the following notation: Let $A \subseteq C$ and let $S \subseteq C$ be either a subring of C_1 or a subnear-ring of C_2 or a subcomposition ring of C.

$\langle A \rangle_s$ denotes the ideal in S generated by $A \cap S$. For example, if $S = C_1$ then $\langle A \rangle_{C_1}$ denotes the ideal in C_1 generated by A. If $S = (C, +, \cdot, \circ)$, then we usually write $\langle A \rangle_C$ just as $\langle A \rangle$. We will also use

$$(A \cap F : F)_{C_2} := \{c \in C \mid c \circ F \subseteq A \cap F\}.$$

We recall two further results; the latter already part of the folklore of the theory of polynomial near-rings.

6.3. Proposition. [4] *Let $J \lhd F$, $F = Found(C)$, C a composition ring. Then the following is equivalent:*

(1) *J is an C-ideal of F.*
(2) *$(J : F)_{C_2} \lhd C$.*
(3) *$J = \langle J \rangle \cap F$.* □

6.4. Proposition. *Let $I \lhd C$. Then there is a unique C-ideal J of F (namely $J = F \cap I$) such that $\langle J \rangle \subseteq I \subseteq (J : F)_{C_2}$.* □

6.5. Proposition. *The mapping $\theta : \{I \mid I \lhd C\} \to \{J \mid J$ an C-ideal of $F\}$ defined by $\theta(I) = I \cap F$ is a surjective function which retains sums of ideals, inclusions and intersections. In general, θ need not be a bijection.*

Proof. Let $I = \Sigma_\alpha I_\alpha$, $I_\alpha \lhd C$. Then $c \in I \cap F$ if and only if $c \in F$ and $c = \sum_{i=1}^k s_{\alpha_i}, s_{\alpha_i} \in I_{\alpha_i}$. For such a c, $c = c \circ 0 = \Sigma s_{\alpha_i} \circ 0$ and $s_{\alpha_i} \circ 0 \in I_{\alpha_i} \cap F$. Thus $\theta(I) = \sum_\alpha \theta(I_\alpha)$ and so θ retains sums of ideals. θ is surjective: Let J be an C-ideal of F. Then $\langle J \rangle \subseteq (J : F)_{C_2}$ and $J \subseteq \langle J \rangle \cap F = \theta(\langle J \rangle) \subseteq \theta((J : F)_{C_2}) = (J : F)_{C_2} \cap F = J$. θ need not be a bijection: For a C-ideal J of F, $\theta(\langle J \rangle) = \theta((J : F)_{C_2}) = J$, but $\langle J \rangle$ and $(J : F)_{C_2}$ need not coincide. For example if $C = C_0$, let $J = 0$. Then $(0 : F)_{C_2} = (0 : 0)_{C_2} = C$ and $\theta(0) = 0 = \theta(C) = \theta((0 : F)_{C_2})$. The remaining assertions are clear. □

6.6. Proposition. (Peterson and Veldsman [3]) *Let* C be a composition ring with $Found(C) = F \neq 0$. Then $(0:F)_{C_2}$ is a maximal ideal of C if and only if F is C-simple and $(0:F)_{C_2} + \langle F \rangle = C$. $\square$

6.7. Corollary. (Peterson and Veldsman [3]) *Let C be a composition ring with $Found(C) = F \neq 0$ and $\langle F \rangle = C$. Then:*
(1) *F is C-simple if and only if $(0:F)_{C_2}$ is the unique maximal ideal of C.*
(2) *C is simple if and only if F is C-simple and $(0:F)_{C_2} = 0$.* $\square$

Let us mention that if C is infinite and $0 \neq Found(C) = F$ is finite, then C is not simple. Indeed, $(0:F)_{C_2}$ is non-zero (otherwise $\psi : C \to M(F)$ defined by $\psi(c) = \psi_c : F \to F$, $\psi_c(n) = c \circ n$, is an injective mapping from the infinite set C into the finite set $M(F)$) and $(0:F)_{C_2} \neq C$ (otherwise $F = 0$). Note also that if the composition ring is compatible, then all occurrences of "C-ideal" can be replaced by "ideal". We next look at the relationship between the ideals of F and those of B.

6.8. Proposition. *Let C be a composition ring with commuting composition identity x, $Found(C) = F$ and $Base(C) = B$.*
(1) *If $J \lhd F$, then $Jx \lhd (Fx, +, \circ)$.*
(2) *If J is an C-ideal of F, then $B \circ J + J \circ B \subseteq J$ and $Jx \lhd (B, +, \circ)$.*
(3) *Let $K \lhd B$. Let $J = \{n \in F \mid nx \in K \cap F\}$. Then $Jx = K \cap Fx$ and $J \lhd F$. If $B = Fx$, then $K \lhd B$ if and only if $K = Jx$ for some $J \lhd F$.*

Proof. (1) For $j \in J$ and $n \in F$, $jx \circ nx = (jn)x \in Jx$ and $nx \circ jx = (nj)x \in Jx$.
(2) $J \circ B = J$ (since $J \subseteq F$). For $v \in B$, $j \in J, v \circ j = v \circ (0 + j) - v \circ 0 \in J$ since J is an C-ideal of F. Hence $J \circ B + B \circ J \subseteq J$. Furthermore,

$$\begin{aligned} jx \circ v &= jv \\ &= vj \qquad \text{since } j \in J \subseteq F, \text{ cf. Proposition 2.2(1)} \\ &= (v \circ j)x \qquad \text{since } v \in B \\ &\in (B \circ j)x \subseteq Jx \\ \text{and } v \circ jx &= (v \circ j)x \in (B \circ J)x \subseteq Jx. \end{aligned}$$

(3) Clearly $Jx = K \cap Fx$. Let $j \in J, n \in F$. Then $(jn)x = (jx)n \in (K \cap Fx)F \subseteq K \cap Fx$ (cf. Proposition 2.2 (6)). Hence $jn \in J$ and thus also $nj = jn \in J$. $\square$

Next we will look at the transfer of ideals between C and its base B. For this we will make the following

Assumption. In the sequel we suppose that $\beta : (C, +) \to (B, +)$ is a surjective group homomorphism.

Let $A \subseteq B$. We define

$$[A] := \{c \in C \mid \beta(c) \in A + \overline{FB}\}.$$

If C is a composition p-ring, we take β to be the mapping as defined in §3 and if C is a composition t-ring we take β as defined in Proposition 5.2. For both these two cases, $\overline{FB} = Fx$. In case C is both a composition p-ring as well as a composition t-ring (cf §5), then it doesn't matter how we choose β, at least for the purpose we have in mind here. This follows, since in this case, $F \neq 0$, $B = Fx$ and hence $[A] = \{c \in C \mid \beta(c) \in A + B\} = C$.

We will at times impose further conditions on β. The first of these are:

(B_1) For all $a, b \in C, \beta(ab) \in \overline{FB}$ **or** for all $a, b \in C, \beta(ab) = \beta(a) \circ \beta(b)$.
(B_2) For all $a, b \in C, \beta(a \circ b) - \beta(a) \circ \beta(b) \in \overline{FB}$.

For a composition p-ring, both these conditions are satisfied. For a composition t-ring condition (B_1) holds and if $F \neq 0$, then also (B_2) is satisfied. More generally, any composition ring C with β surjective and $B = \overline{FB}$ satisfies (B_1) and (B_2) (trivially) but in such a case $[A] = C$ for any $A \subseteq B$.

6.9. Proposition. *Let J be a subgroup of B such that $J \circ B + B \circ J \subseteq J + \overline{FB}$ (for example, if $J \lhd B$ or if $J \subseteq \overline{FB}$). Then $\beta([J]) = J + \overline{FB}$ and:*
(1) If β satisfies (B_1), then $[J] \lhd C_1$.
(2) If β satisfies (B_1) and (B_2), then $[J] \lhd C$.

Proof. $\beta([J]) = J + \overline{FB}$ follows from the definition and since β is assumed to be surjective. Let $c, d \in [J]$ and $p, q \in C$. Then $\beta(c - d) = \beta(c) - \beta(d) \in J + \overline{FB}$.

(1) Suppose β satisfies condition (B_1). Either $\beta(cp) \in \overline{FB} \subseteq J + \overline{FB}$ or $\beta(cp) = \beta(c) \circ \beta(p) \in (J + \overline{FB}) \circ B \subseteq J + \overline{FB}$ and similarly either $\beta(pc) \in \overline{FB} \subseteq J + \overline{FB}$ or $\beta(pc) = \beta(p) \circ \beta(c) \in B \circ (J + \overline{FB}) \subseteq J + \overline{FB}$. Thus $[J] \lhd C_1$.

(2) Suppose β also satisfies condition (B_2). Then $\beta(c \circ p) = \beta(c) \circ \beta(p) + u \in (J + \overline{FB}) \circ B + \overline{FB} \subseteq J + \overline{FB}$ for some $u \in \overline{FB}$. Lastly, $\beta(p \circ (q + c) - p \circ q) = \beta(p) \circ \beta(c) + w \in B \circ (J + \overline{FB}) + \overline{FB} \subseteq J + \overline{FB}$ for some $w \in \overline{FB}$. Hence $[J] \lhd C$. □

6.10 Proposition. *Suppose β satisfies condition (B_2). Let $I \lhd C$. Then $J := \beta(I)$ is a subgroup of B which satisfies $J \circ B + B \circ J \subseteq J + \overline{FB}$ and $I \subseteq [J]$. If J' is any other subgroup of B with $I \subseteq [J']$, then $J + \overline{FB} \subseteq J' + \overline{FB}$.*

Proof. Let $a, d \in I$ and $b \in B = \beta(C)$, say $b = \beta(c), c \in C$. Then $\beta(a) - \beta(d) = \beta(a - d) \in \beta(I)$. Also, $\beta(a) \circ \beta(c) = \beta(a \circ c) + g \in \beta(I) + \overline{FB} = J + \overline{FB}$ for some $g \in \overline{FB}$ and $\beta(c) \circ \beta(a) = \beta(c \circ a) + w$ for some $w \in \overline{FB}$. If $c = k + u$ with $k \in F$ and $u \in C_0$, then $\beta(c \circ a) = \beta(u \circ a) \in \beta(I) + \overline{FB}$ since $C_0 \circ I \subseteq I$ and $\beta(k) = \beta(k \circ 0) = \beta(k) \circ \beta(0) + h = h$ for some $h \in \overline{FB}$. Thus $\beta(c) \circ \beta(a) \in \beta(I) + \overline{FB}$. Clearly $I \subseteq [J]$. Let J' be a subgroup of B such that $I \subseteq [J']$. Let $j + d \in J + \overline{FB}$, say $j = \beta(i), i \in I$. Since $I \subseteq [J']$, $\beta(i) \in J' + \overline{FB}$ and so $j + d \in J' + \overline{FB}$. □

6.11. Corollary. *Suppose β satisfies conditions (B_1) and (B_2). The mapping $\Phi : \{I \mid I \lhd C\} \longrightarrow \{J + \overline{FB} \mid J$ a subgroup of B with $J \circ B + B \circ J \subseteq J + \overline{FB}\}$ defined by $\Phi(I) = \beta(I) + \overline{FB}$ is a well-defined surjection which preserves inclusions and sums.*

Proof. Most of the proof follows from Propositions 6.9 and 6.10. The remainder is straightforward. □

6.12. Proposition. *Suppose β satisfies conditions* (B_1) *and* (B_2). *Let I be a maximal ideal of C. Then one and only one of the following three cases holds:*

(1) *$F \not\subseteq I$ and $I = (I \cap F : F)_{C_2}$ with $I \cap F$ a maximal C-ideal of F.*
(2) *$F \subseteq I$ and $I = [\beta(I)]$ with $\beta(I)$ a maximal ideal of B.*
(3) *$F \subseteq I$ and $\beta(I) + \overline{FB} = B$.*

Proof. By Proposition 6.4 and the fact that I is maximal, we have $I = (I \cap F : F)_{C_2}$ or $(I \cap F : F)_{C_2} = C$.
Suppose firstly $I = (I \cap F : F)_{C_2}$. Then $F \not\subseteq I$ for if $F \subseteq I$ then $I = (I \cap F : F)_{C_2} = (F : F)_{C_2} = C$; a contradiction. We show $I \cap F$ is a maximal C-ideal of F. Let K be a C-ideal of F with $I \cap F \subset K \subset F$. Then $I = (I \cap F : F)_{C_2} \subseteq (K : F)_{C_2}$ and so $I = (K : F)_{C_2}$ or $(K : F)_{C_2} = C$. If the latter equality holds, then $F = C \circ F = K$, a contradiction. Thus $I = (K : F)_{C_2}$ and from this we get $K \subseteq I \cap F$, also a contradiction. Hence $I \cap F$ is a maximal C-ideal of F.
Suppose now $(I \cap F : F)_{C_2} = C$. Then $F \subseteq I$. By Proposition 6.10 $J := \beta(I)$ is a subgroup of B which satisfies $J \circ B + B \circ J \subseteq J + \overline{FB}$ and $I \subseteq [J]$. From Proposition 6.9 we get $[J] \lhd C$ and so $I = [J]$ or $[J] = C$.

If $[J] = C$, then $B = \beta(C) = \beta([J]) = J + \overline{FB} = \beta(I) + \overline{FB}$. Suppose thus $[J] = I$. Then $J \lhd B$: Note firstly that $J = \beta(I) = \beta([J]) = J + \overline{FB}$; hence $J \subseteq \overline{FB}$. Thus $J \circ B + B \circ J \subseteq J + \overline{FB} = J$. Finally we show that J is maximal in B. Let $K \lhd B$ such that $J \subset K \subset B$. By Proposition 6.9 we have $[K] \lhd C$ and $I = [J] \subseteq [K]$. Thus $[J] = [K]$ or $[K] = C$. If $[J] = [K]$, then $J = J + \overline{FB} = \beta([J]) = \beta([K]) = K + \overline{FB} = K$; a contradiction. If $[K] = C$, then $B = \beta(C) = \beta([K]) = K + \overline{FB} = K$; also a contradiction. Hence $\beta(I) = J$ is a maximal ideal of B. □

The next two results are immediate special cases of this result:

6.13. Corollary. *Suppose β satisfies conditions* (B_1) *and* (B_2) *and that $Found(C) = 0$. If I is a maximal ideal of C, then only case (2) or (3) of Proposition 6.12 can occur, i.e.: $I = [\beta(I)]$ and $\beta(I)$ is a maximal ideal of B or $\beta(I) = B$.* □

6.14. Corollary. *Suppose $C = \langle F \rangle$ and β satisfies conditions* (B_1) *and* (B_2). *If I is a maximal ideal of C, then $I = (I \cap F : F)_{C_2}$ and $I \cap F$ is a maximal C-ideal of F.* □

The third case in Proposition 6.12 can occur (cf. Example 6.16 below) and in the known cases with a stronger conclusion:

6.15. Proposition. *Let C be a composition ring with commuting composition identity x, let $I \lhd C$ such that $F \subseteq I$ and $\beta(I) + Fx = B$. Suppose C and β satisfies:*

(B_3) *For all $m \geq 0$ ($m \in \mathbb{Z}$) there exists a function $\beta_m : C \to B$ with $\beta_0(c) := (c \circ 0)x$ for all $c \in C$ and $\beta_1 = \beta$, such that for all $m \geq 0$ and for all $c \in C$, there exists a $c' \in \langle c \rangle$ such that $\beta_m(c) = \beta_{m+1}(c')$.*

Then $\beta_m(I) = B$ *for all* $m \geq 1$.

Proof. We start by showing $\beta_1(I) = \beta(I) = B$, i.e. $Fx \subseteq \beta(I)$. Let $k \in F$. Then $kx = (k \circ 0)x = \beta_0(k) = \beta_1(k')$ for some $k' \in \langle k \rangle$. From $F \subseteq I$ we get $k' \in I$ and thus $kx \in \beta_1(I) = \beta(I)$. Hence $\beta(I) = B$.
We proceed by induction and suppose $B = \beta_m(I)$ for some $m \geq 1$. Let $b \in B = \beta_m(I)$, say $b = \beta_m(a), a \in I$. By (B_3) there is an $a' \in \langle a \rangle \subseteq I$ such that $b = \beta_m(a) = \beta_{m+1}(a') \in \beta_{m+1}(I)$; hence $B = \beta_{m+1}(I)$. □

6.16. Example. Let C be a composition p-ring. As we have seen before, for any $k \geq 0$ and any $c \in C$, there is a unique representation.

$$c = n_c + v_0 + v_1 x + v_2 x + \cdots + v_k x^{k-1} + u_k x^k$$

where $n_c \in F$, $v_i \in B$ and $u_k \in C_0$. Define $\beta_m : C \to B$ as follows: $\beta_0(c) = n_c x = (c \circ 0)x$; $\beta_1(c) = \beta(c) = v_0$, $\beta_2(c) = v_1, \ldots, \beta_{m+1}(c) = v_m, \ldots$. Let $c \in C$. Then $cx \in \langle c \rangle$ and since $cx = n_c x + v_0 x + v_1 x^2 + \cdots + v_k x^{k+1} + u_k x^{k+1}$ (and remember $B \cap Bx \subseteq B \cap C_0 x = 0$), we get $\beta_m(c) = v_{m-1} = \beta_{m+1}(cx)$ for $m \geq 2, \beta_0(c) = n_c x = \beta_1(cx)$ and $\beta_1(c) = v_0 = \beta_2(cx)$.

Thus $\beta_m(c) = \beta_{m+1}(cx)$ and $cx \in \langle c \rangle$ for all $m \geq 0$. □

We know exactly when only cases (1) and (2) of Proposition 6.12 can occur:

6.17. Proposition.. *Suppose* β *satisfies conditions* (B_1) *and* (B_2). *Let* I *be a maximal ideal of* C. *Then only cases (1) and (2) of 6.12 can occur if and only if there is a* $c \in C$ *such that* $\beta(c + I) \cap \overline{FB} = \phi$.

Proof. Suppose only cases (1) and (2) can occur. From the proof of Proposition 6.12 we then know that $I = [\beta(I)] \subset C$. If $\beta(c + I) \cap \overline{FB} \neq \phi$ for all $c \in C$, we will derive a contradiction. Let $c \in C$ and $i \in I$ such that $\beta(c + i) = w$ for some $w \in \overline{FB}$. Then $\beta(c) = -\beta(i) + w \in \beta(I) + \overline{FB}$. Thus $c \in [\beta(I)]$. Hence $[\beta(I)] = C$; a contradiction. Conversely, suppose $\beta(c + I) \cap \overline{FB} = \phi$ for some $c \in C$. If $[\beta(I)] = C$, then $c \in [\beta(I)]$ implies $\beta(c) = \beta(i) + u$ for some $i \in I, u \in \overline{FB}$. Thus $\beta(c - i) \in \overline{FB}$ which contradicts our assumption. Hence $\beta(I) \subset C$. □

6.18. Corollary. *Suppose* $Found(C) = 0$ *and* β *satisfies conditions* (B_1) *and* (B_2). *Let* I *be a maximal ideal of* C. *Then* $I = [\beta(I)]$ *and* $\beta(I)$ *is a maximal ideal of* B *if and only if there is a* $c \in C$ *such that* $\beta(c + i) \neq 0$ *for all* $i \in I$. □

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