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On Parametric Density of Finite Circle Packings

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Abstract. The parametric density of a finite circle packing $X_n + D$ is defined as ratio of areas $n \cdot A(D)/A\left(\operatorname{conv}(X_n + \rho D)\right)$. We show that densest finite lattice packings are attained in a critical lattice. Further, we give an upper bound for the density with parameters $\rho \leq 3.232\ldots$ which is attained by Groemerpackings. Moreover, we show that the given bound holds for arbitrary finite circle packings and parameters less than 1 as a consequence of known results by H. Groemer [5] and G. Wegner [11].

1. Introduction

We call a non-overlapping arrangement of a finite number of congruent discs a *finite circle packing*. As an example, we may think of the cross-section of a multi-core (e.g. optical fibers) cable, enclosed by some type of insulation (cf. Figure 1). Problem: How can one pack a fixed number of cores to minimize the area of such a cross-section; or: what is the maximal parametric density of such a circle packing?

The answer depends on the parameter defined thickness of the used insulation. With infinitely small insulation the cross-section is the convex hull of the packing, which has been investigated in many papers (cf. [3]). The best known results were given by Groemer [5] (Theorem 3.1) and Wegner [11] (Theorem 3.2). Their bounds hold with equality only for Groemerpackings (see Theorem 3.1). In these packings the disc-centers belong to a 'critical lattice'.

We show that for any parameter the highest parametric density of a lattice packing is attained if and only if the lattice is critical (Theorem 3.3). It turns out that 'extremal

Groemerpackings' (see Theorem 3.2) which minimize the convex hull are best possible lattice packings for all parameters in $\left[\frac{\sqrt{3}}{2}, \frac{3}{2}+\sqrt{3}\right]$ (Theorem 3.4). This shows that cable-cross-sections become minimal when the thickness of the insulation is less than $\left(\frac{1}{2}+\sqrt{3}\right)$-times the disc-radii.

2. Notation and simplification

Let $\mathbb{E}^2$ denote the Euclidian plane with norm $\|\cdot\|$, inner product $\langle\cdot,\cdot\rangle$ and unit disc D. A finite set $X_n = \{x_1,\ldots,x_n\} \subset \mathbb{E}^2$ defines a finite circle packing $X_n + D$ when $\|x - x'\| \geq 2$ holds for all pairs of distinct points $x, x' \in X$. The (parametric) density of the packing (cf. [1]) with respect to a parameter $\rho > 0$ is given by

$$\delta\left(D, X_n, \rho\right) = \frac{n \cdot \pi}{A\left(\operatorname{conv}(X_n + \rho D)\right)}. \tag{1}$$

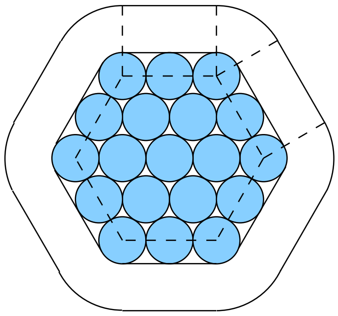

Figure 1.

Here, A denotes the area function, and $\operatorname{conv}(\cdot)$ denotes the convex hull. In order to maximize the density of n discs, we have to minimize the denominator. With Steiner's Formula (cf. [6]) we get

$$A\left(\operatorname{conv}(X_n + \rho D)\right) = A\left(\operatorname{conv}(X_n)\right) + \rho \cdot C\left(\operatorname{conv}(X_n)\right) + \rho^2 \cdot \pi,$$

where C evaluates the circumference of the center-polygon $P = \operatorname{conv}(X_n)$. Figure 1 shows a geometrical interpretation of the three summands in Steiner's Formula. Since we want to find densest packings for a fixed parameter ρ, we have to minimize the linear function

$$F_P(\rho) = A(P) + \rho \cdot C(P) \;\text{ over }\; \mathcal{P}(n) = \{P \mid P = \operatorname{conv}(X_n), X_n + D\,\text{packing}\}.$$

Hence, any lower bound for $F_P(\rho)$ yields an upper bound for the parametric density of the densest packing with n translates of D.

A lattice Λ is called a *packing lattice* when $\Lambda + D$ is a packing. Λ is called *critical* if and only if every disc in $\Lambda + D$ touches six others. We call $P = \operatorname{conv}(X_n)$ a Λ-*polygon* if X_n is a subset of Λ, and a *critical lattice polygon*, if Λ is critical. The number of lattice points covered by P is denoted by $G_\Lambda(P)$. A segment $[AB] \subset \operatorname{bd}(P)$ is called an *edge* of P when it satisfies $[AB] \cap X_n = \{A, B\}$. Its length is given by $|AB|$. The number of edges is denoted by $E_\Lambda(P)$, where edges are counted twice for degenerated polygons ('sausages'). Fundamental for the work with lattice polygons is Pick's Identity (cf. [4])

$$G_\Lambda(P) = \frac{A(P)}{\det(\Lambda)} + \frac{E_\Lambda(P)}{2} + 1,$$

where $\det(\Lambda)$ denotes the determinant (area of a fundamental cell) of Λ (cf. [6]).

We only investigate lattice polygons in order to find densest lattice packings. This is sufficient, since a densest lattice packing, has to be saturated ($P \cap \Lambda = X_n$). Otherwise we could construct a new and denser packing by replacing a vertex of P in X_n with another lattice point.

3. Results

First we are going to use the introduced notation to state the known results by H. Groemer and G. Wegner. We then put those results into a wider context.

Theorem 3.1. (Groemer) *For $n \in \mathbb{N}$, $\rho = \frac{\sqrt{3}}{2}$ and $P \in \mathcal{P}(n)$ holds*

$$F_P(\rho) \geq (n-1)\, 2\sqrt{3}.$$

Equality holds if and only if P is a Groemerpolygon. That is a (possibly degenerated) critical lattice polygon with all edges having length 2. The corresponding saturated circle packing is called a Groemerpacking.

An elegant proof was given by Graham, Witsenhausen and Zassenhaus [2], who generalized a result by Oler [7]. This proof also takes arbitrary centrally symmetric discs into consideration. The other classical result on finite circle packings is given by

Theorem 3.2. (Wegner) *For $n \in \mathbb{N}$, $\rho = 1$ and $P \in \mathcal{P}(n)$ holds*

$$F_P(\rho) \geq (n-1)\, 2\sqrt{3} + \left(2\rho - \sqrt{3}\right) p_0(n),$$

with $p_0(n) = \left\lceil \sqrt{12n-3}\, \right\rceil - 3$.[1] *Equality holds if and only if P is a so-called extremal Groemerpolygon. That is a Groemerpolygon satisfying $E_\Lambda(P) = p_0\left(G_\Lambda(P)\right)$ with respect to a critical lattice Λ.*

The number $p_0(n)$ gives a lower bound for the number of edges of a Groemerpolygon (cf. [11]).[2] The Theorem holds with equality if and only if the bound is attained, and therefore, an extremal Groemerpolygon exists. The exceptions ($n = 121$ is the smallest) were studied by Wegner [10] in a second paper. Their exact determination and the determination of packings with minimal convex hull for those n stays still open.

Now, extremal Groemerpolygons are the only polygons with equality in both Theorems. Since $F_P(\rho)$ is linear we instantly get

$$F_P(\rho) \geq (n-1)\, 2\sqrt{3} + \left(2\rho - \sqrt{3}\right) p_0(n),$$

for $n \in \mathbb{N}$, $\rho \in \left(\frac{\sqrt{3}}{2}, 1\right]$ and $P \in \mathcal{P}(n)$. Equality holds if and only if P is an extremal Groemerpolygon. With the same argument we get $F_P(\rho) \geq (n-1)4\rho$ for $\rho \in \left[0, \frac{\sqrt{3}}{2}\right)$ with equality for sausages, the intermediate case $\rho = \frac{\sqrt{3}}{2}$ being covered by Theorem 3.1, with equality for any Groemerpolygon.

In order to see how densest circle packings may look like for parameters bigger than 1, we investigate the problem restricted on lattice packings. First we show for all parameters

[1] $\lceil \cdot \rceil$ is known as ceiling function ($\lceil x \rceil$ being the least integer k with $k \geq x$).

[2] An inductive proof for Wegner's remark is easy verified using the geometrical definition for p_0 (Theorem 4.3) and the lemma in Section 6.

Theorem 3.3. *$F_P(\rho)$ becomes minimal for a lattice polygon P and $\rho \geq 0$ if and only if P is a critical lattice polygon.*

Besides Theorems 3.2 and 3.1, many arguments show that Theorem 3.3 holds for arbitrary circle packings and parameters less than $\frac{3}{2} + \sqrt{3} = 3.232\ldots$.[3] For lattice packings we have

Theorem 3.4. *Theorem 3.2 holds for $\rho \in \left(\frac{\sqrt{3}}{2}, \frac{3}{2} + \sqrt{3}\right)$ and all lattice polygons $P \in \mathcal{P}(n)$. Moreover, the theorem is true for $\rho = \frac{3}{2} + \sqrt{3}$, if we modify the case of equality to critical lattice polygons with edges having length 2 or $2\sqrt{3}$ and with the (uniquely defined) smallest circumscribed Groemerpolygon having $p_0(n)$ edges.*

Six is the smallest number of circles where the second case of equality holds. The corresponding packing, its center-polygon and the smallest circumscribed Groemerpolygon is shown in Figure 2. Unfortunately, it is impossible to give a general lower bound for $F_P(\rho)$ with $\rho > \frac{3}{2}+\sqrt{3}$. Nevertheless, we are able to determine the asymptotic shape ($n \to \infty$) of the best possible polygons. This will be done in a wider context in another paper [8].

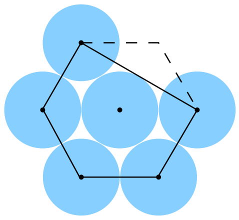

Figure 2.

4. Proofs

Proof of Theorem 3.3. The assertion is true for $\rho \leq \frac{\sqrt{3}}{2}$ (Theorem 3.1) and $G_\Lambda(P) \leq 2$. Therefore, we may assume that P has a non-empty interior. This follows from Theorem 3.1 because non-sausage Groemerpolygons define better packings for $\rho > \frac{\sqrt{3}}{2}$, since their volume is positive.

The remaining proof is constructive: a non-critical lattice Λ will be transformed by regular linear mappings into a critical lattice. The Λ-polygon P becomes a critical lattice polygon P_0 with the same number of edges and the same number of covered lattice points. The area changes because of Pick's Identity in dependency of the lattice determinant, which is minimal for the critical lattice (cf. [6]). Hence $A(P_0) < A(P)$. We show that the lattice transformation can be chosen that $C(P_0) < C(P)$ becomes valid. This proves the Theorem.

To begin with, we choose a reduced basis (cf. [9]) a_1, a_2 for the lattice Λ. A necessary and sufficient condition for being reduced is $0 \leq 2\langle a_1, a_2\rangle \leq \|a_1\|^2 \leq \|a_2\|^2$, if we assume that $\|a_1\|$ is smaller than $\|a_2\|$. Now the circumference $C(P)$ can be written as the sum $\sum |AB|$ over all edges $[AB]$ of P. With fitting numbers $m, n \in \mathbb{Z}$ we have

$$|AB| = \|ma_1 + na_2\| = \sqrt{\|a_1\|^2 m^2 + 2\langle a_1, a_2\rangle mn + \|a_2\|^2 n^2}.$$

Now, we first map a_1 into $a_1' = \frac{2a_1}{\|a_1\|}$ and a_2 into a_2' with $\|a_2'\| = 2$ and $\langle a_1', a_2'\rangle = \frac{4}{\|a_1\|^2}\langle a_1, a_2\rangle$. The new basis is reduced because

$$0 \leq 2\langle a_1', a_2'\rangle = \frac{8}{\|a_1\|^2}\langle a_1, a_2\rangle \leq 4 = \|a_1'\|^2 = \|a_2'\|^2.$$

[3]This would imply a proof for a presumption of Wegner (cf. [10]), which covers the exceptions of Theorem 3.2.

Therefore, the distance from the origin to a lattice point $na_1' + ma_2'$, $n, m \in \mathbb{Z}\setminus\{0\}$, is at least as long as the one to a_1' and a_2'. Hence, the new lattice is a packing lattice for D. An edge $[AB]$ of P is transformed into an edge $[A'B']$ with

$$\begin{aligned} |A'B'| &= \sqrt{4m^2 + 2\langle a_1', a_2'\rangle mn + 4n^2} \\ &\le \frac{2}{\|a_1\|}\sqrt{\|a_1\|^2 m^2 + 2\langle a_1, a_2\rangle mn + \|a_2\|^2 n^2} \le |AB|. \end{aligned}$$

Equality holds if and only if $\|a_1\| = \|a_2\| = 2$. This shows that the new polygon has a smaller circumference if a_2 is longer than 2.

The next transformation fixes a_1' and maps a_2' into a_2'' with $\|a_2''\| = 2$. The inner product $\langle a_1', a_2'\rangle$ has a fixed value $2x_0$ with $x_0 \in [0,1]$. An edge $[A'B']$ has the length $|A'B'| = 2\sqrt{m^2 + x_0 mn + n^2}$. Then the inner product $\langle a_1', a_2''\rangle$ may be $2x$ with $x \in [-1,1]$. Consequently, the packing lattice Λ_x with basis a_1', a_2'' is critical only for $x \in \{-1, 1\}$.

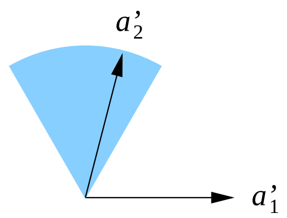

Figure 3.

The length of the corresponding edges $[A_xB_x]$ add up to the circumference $c(x)$ of a Λ_x-polygon. The function $c(x)$ is two times differentiable and the length of an edge $[A_xB_x]$ has a second derivation

$$\frac{d^2}{dx^2}|A_xB_x| = -\frac{m^2n^2}{2\left(n^2 + xmn + m^2\right)^{\frac{3}{2}}}.$$

Hence, $\frac{d^2}{dx^2}c(x) \le 0$. Therefore, $c(x)$ attains its minimum either for $x = -1$ or $x = 1$, which proves the theorem. □

Proof of Theorem 3.4. The theorem is true for $\rho \le 1$ as a consequence of Theorem 3.2. In order to prove the rest of the assertion it is sufficient to prove it for $\rho = \frac{3}{2} + \sqrt{3} = \frac{\sqrt{3}}{4-2\sqrt{3}}$ because $F_P(\rho)$ is a linear function. Because of Theorem 3.3 we may further assume that P is a critical lattice polygon with respect to Λ.

The smallest circumscribed Groemerpolygon P_G of P is uniquely defined, since every Groemerpolygon is an intersection of six fixed half-spaces (all bordered by edges of length 2) and vice versa. P_G has at least $p_0(n)$ edges because p_0 is monotonically increasing, and because p_0 gives a lower bound for the number of edges of a Groemerpolygon (cf. [11]).

Now, let $e_p(l)$ denote the number of edges with length l of P. With Pick's identity we get

$$\begin{aligned} F_P(\rho) &= (n-1)2\sqrt{3} - \sqrt{3}\cdot E_\Lambda(P) + \rho\cdot C(P) \\ &= (n-1)2\sqrt{3} + \sum_l \left(l\cdot\rho - \sqrt{3}\right) e_P(l), \end{aligned}$$

where the sum is taken over all possible l in Λ. For $l \ge 2\sqrt{3}$ we get with fitting natural numbers $m_l \ge n_l$

$$l = 2\sqrt{m_l^2 + m_l n_l + n_l^2} = 2\sqrt{\frac{3}{4}(m_l+n_l)^2 + \frac{1}{4}(m_l-n_l)^2} \ge \sqrt{3}\,(m_l+n_l)$$

$$
\begin{aligned}
&= \left(2\sqrt{3}-2\right)(m_l+n_l)+\left(2-\sqrt{3}\right)(m_l+n_l)\\
&\geq \left(2-\left(4-2\sqrt{3}\right)\right)(m_l+n_l)+\left(4-2\sqrt{3}\right).
\end{aligned}
$$

Equality holds if and only if $m_l = n_l = 1$. Hence, $F_P(\rho)$ for $\rho = \frac{\sqrt{3}}{4-2\sqrt{3}}$ evaluates to

$$
\begin{aligned}
F_P(\rho) &= (n-1)2\sqrt{3}+\left(2\cdot\rho-\sqrt{3}\right)e_P(2)+\sum_{l\geq 2\sqrt{3}}\left(l\cdot\rho-\sqrt{3}\right)e_P(l)\\
&\geq (n-1)2\sqrt{3}+\left(2\cdot\rho-\sqrt{3}\right)e_P(2)+\sum_{l\geq 2\sqrt{3}}\left(2\cdot\rho-\sqrt{3}\right)(m_l+n_l)\,e_P(l)\\
&= (n-1)2\sqrt{3}+\left(2\cdot\rho-\sqrt{3}\right)\sum_{l}(m_l+n_l)\,e_P(l).
\end{aligned}
$$

This proves the theorem because of $E_\Lambda(P_G) = \sum_l (m_l+n_l)\, e_P(l) \geq p_0(n)$. □

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