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Planar Normal Sections on the Natural Embedding of a Real Flag Manifold

Cristián U. Sánchez Alicia N. García Walter Dal Lago[1]

Facultad de Matemática Astronomía y Física
Universidad Nacional de Córdoba, Ciudad Universitaria
5000, Córdoba, Argentina
e-mail: csanchez@mate.uncor.edu

Abstract. This paper contains a proof of the following result which is an extension of the main result in [5], to the general case of real flag manifolds (also called R-spaces or orbits of s-representations).

THEOREM. *Let M be a real flag manifold and let $j : M \to \mathfrak{p}$ its canonical embedding. Let $X[M] \subset RP^{n-1}$, $(n = \dim M)$ be the variety of directions of pointwise planar normal sections at a point of M and let $X_c[M] \subset CP^{n-1}$ be the natural complexification of $X[M]$. Let χ denote the Euler-Poincare characteristic then*

$$\begin{array}{lll} \text{(i)} & \chi(X[M]) = \chi(RP^{n-1}) = & \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd} \end{cases} \\ \text{(ii)} & \chi(X_c[M]) = \chi(CP^{n-1}) = & n. \end{array}$$

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1. Introduction

In [5] we introduced the variety $X[M]$ of directions of pointwise planar normal sections and its natural complexification $X_c[M]$, of a natural embedding of a real flag manifold (also

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called R-spaces or orbits of s-representations). That paper continued the pioneering work of B. Y. Chen [3] and other authors ([4], [6] etc.) on normal sections of submanifolds of Euclidean spaces. $X[M]$ is a real algebraic variety in the real projective space RP^{n-1} and $X_c[M]$ is a complex variety in CP^{n-1} where n is the dimension of M. To some extent, they measure the differences between the given manifold M and a symmetric real flag manifold.

The varieties $X[M] \subset RP^{n-1}$ and $X_c[M] \subset CP^{n-1}$ are quite different from their ambient projective spaces however, surprisingly, the main result in [5, p.224, (1.2)] is the following

Theorem 1. *Let $M^n = G/K$ be a complex flag manifold and let $j : M \to \mathfrak{g}$ be one of the natural embeddings of M in the Lie algebra $\mathfrak{g}$ of G. Let $X[M] \subset RP^{n-1}$ be the variety of directions of pointwise planar normal sections at a point $p \in M$ and $X_c[M] \subset CP^{n-1}$ the natural complexification of $X[M]$. If χ denotes the Euler characteristic then*

$$\begin{array}{lllll} \text{(i)} & \chi(X[M]) & = & \chi(RP^{n-1}) & = 0 \\ \text{(ii)} & \chi(X_c[M]) & = & \chi(CP^{n-1}) & = n \; = \dim M. \end{array}$$

The methods of [5] were not sufficiently strong to get this result for the general case of real flag manifolds (R-spaces). The objective of the present paper is to give a proof of this result in the general case.

Our result is the following

Theorem 2. *Let M^n be a general real flag manifold (also called R-space or orbit of an s-representation) and let $j : M \to \mathfrak{p}$ be its natural imbedding. Let $X[M] \subset RP^{n-1}$ the variety of directions of pointwise planar normal sections at a point $p \in M$ and $X_c[M] \subset CP^{n-1}$ the natural complexification of $X[M]$. If χ denotes the Euler characteristic then*

$$\begin{array}{lllll} \text{(i)} & \chi(X[M]) & = & \chi(RP^{n-1}) & = \begin{cases} 0 & if \ \dim M \ is \ even \\ 1 & if \ \dim M \ is \ odd \end{cases} \\ \text{(ii)} & \chi(X_c[M]) & = & \chi(CP^{n-1}) & = n = \quad \dim M. \end{array}$$

As we indicated above this result contains the previous one since every complex flag manifold is an even dimensional real flag manifold.

The paper is naturally divided into two parts considering the real and complex cases which require different arguments.

The next section contains the required notation and basic results. Section 3 contains the required arguments for the real case and Section 4 the proof of the first part of the main theorem. Sections 5 and 6 are devoted to the complex case.

2. Basic facts

In the present section we introduce some of the basic notation to be used throughout the paper. All unexplained notation will have the same meaning as in [5].

Let M be a Riemannian manifold. Let $j : M \to R^N$ be an isometric immersion and p a point in M. We may identify a neigborhood of p with its image by j and consider, in the tangent space $T_p(M)$, a unitary vector X. If $T_p(M)^\perp$ denotes the normal space to M at p, we may define an affine subspace of R^N by

$$S(p,X) = p + Span\left\{X, T_p(M)^\perp\right\}.$$

If U is a small enough neigborhood of p in M, then the intersection $U \cap S(p,X)$ can be considered the image of a C^∞ regular curve $\gamma(s)$, parametrized by arc-length, such that $\gamma(0) = p$, $\gamma'(0) = X$. This curve is called a *normal section of* M *at* p *in the direction of* X. In a strict sense, we ought to speak of the "germ" of a normal section at p determined by the unit vector X. A change in the neigborhood U will change the curve; however, this new curve will coincide with γ in the proximity of zero. Since our computations with the curve γ are done at the point p, we may take any one of these curves. We may also assume that j is an embedding.

Following B. Y. Chen, we say that the normal section γ of M at p in the direction of X is *pointwise planar* at p if its first three derivatives $\gamma'(0), \gamma''(0)$ and $\gamma'''(0)$ are linearly dependent, i.e. if $\gamma'(0) \wedge \gamma''(0) \wedge \gamma'''(0) = 0$.

In [5] we studied the pointwise planar normal sections of an orbit of an s-representation (i.e. of the natural embedding of an R-space or real flag manifold). In order to mention one of the results obtained there, we need to recall some strictly necessary notation.

Let $j: M \to R^N$ be the natural embedding of an R-space and let ∇ denote the Riemannian connection associated to the metric induced from the Euclidean metric $\langle .,. \rangle$. Let ∇^c denote the canonical connection associated to the "usual" reductive decomposition of the Lie algebra of the compact Lie group defining M. Let $D = \nabla - \nabla^c$ denote the difference tensor and α be the second fundamental form of the embedding j. The indicated result is the following.

Theorem 3. [5, (2.5)] *If* $j: M \to R^N$ *is the natural embedding of an R-space and* p *is a point in* M, *then the normal section* γ *with* $\gamma(0) = p$ *and* $\gamma'(0) = X$ *is pointwise planar at* p *if and only if the unit tangent vector* X *at* p *satisfies the equation*

$$\alpha(X, D(X,X)) = 0.$$

Given a point p in the R-space M we may consider, in the sphere of radius 1 in $T_p(M)$, the subset

$$\widehat{X}_p[M] = \{X \in T_p(M) : \|X\| = 1, \quad \alpha(X, D(X,X)) = 0\}. \tag{1}$$

Since $X \in \widehat{X}_p[M]$ clearly implies $-X \in \widehat{X}_p[M]$, we may take the image $X_p[M]$ of this set in the real projective space RP^{n-1}. Since M is an orbit of a group of isometries of the ambient space R^N, it is clear that $X_p[M]$ does not depend on the point p and we may denote it by $X[M]$. It is not hard to prove [5, p.227, (2.9)] that $X[M]$ is a real algebraic variety of RP^{n-1} defined by homogeneous polynomials of degree 3. Then $X_c[M]$, the natural complexification of $X[M]$, is a complex algebraic variety of CP^{n-1}.

The necessary ingredients to construct the arbitrary R-space M and its canonical embedding are the following. Let $\mathfrak{g}$ be a real semisimple Lie algebra without compact factors, $\mathfrak{k}$ a compactly imbedded subalgebra of $\mathfrak{g}$ and $\mathfrak{g}=\mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition of $\mathfrak{g}$ relative to $\mathfrak{k}$. Let B denote the Killing form of $\mathfrak{g}$; then $\mathfrak{p}$ can be considered a Euclidean space with the inner product defined by the restriction of B to $\mathfrak{p}$. Let $G = Int(\mathfrak{g})$ be the group of inner automorphisms of $\mathfrak{g}$. The Lie algebra $ad(\mathfrak{g})$ of G may be identified with $\mathfrak{g}$. This has the effect of identifying the adjoint action of G on its Lie algebra $ad(\mathfrak{g})$ with the natural action of G on $\mathfrak{g}$. Let K be the analytic subgroup of G corresponding to $\mathfrak{k}$; K is compact and acts on $\mathfrak{p}$ as a group of isometries. The R-space M is, by definition, the orbit of a nonzero vector $E \in \mathfrak{p}$ i.e., $M = K.E$. This defines also the *natural embedding* $j : M \to \mathfrak{p}$ of the R-space M into the Euclidean space $(\mathfrak{p}, B)$. We take on M the Riemannian metric induced by this embedding. Furthermore we will assume that the natural embedding is full, i.e. $j(M)$ is not contained in any affine hyperplane of $\mathfrak{p}$.

Let us denote by K_E the isotropy subgroup of the point E and then, as a homogeneous space $M = K/K_E$. In general the group K_E is not connected and we denote by $[K_E]_e$ its connected component of the identity. Let $\mathfrak{k}_E$ be the Lie subalgebra corresponding to $[K_E]_e$ in $\mathfrak{k}$. Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{k}_E$ with respect to the restriction of B to $\mathfrak{k}$ (it is negative definite on $\mathfrak{k}$). Then $\mathfrak{k}=\mathfrak{k}_E\oplus\mathfrak{m}$ is a reductive decomposition, i.e. $Ad(K_E)\mathfrak{m} \subset \mathfrak{m}$. Furthermore we have

$$T_E(M) = [\mathfrak{m}, E] = [\mathfrak{k}, E]$$

and also $ad(E)$ is one to one on $\mathfrak{m}$ and onto $T_E(M)$.

One has the following fundamental existence theorem.

Theorem 4. [9, p. 43] *Let K/K_E be a reductive homogeneous space with a fixed decomposition of the Lie algebra $\mathfrak{k} = \mathfrak{k}_E\oplus\mathfrak{m}$, $Ad(K_E)\,\mathfrak{m} \subset \mathfrak{m}$. There exists a one-to-one correspondence between the set of all invariant affine connections ∇^c on K/K_E and the set of all bilinear functions $\omega : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ which satisfy*

$$Ad(h)\,\omega(X, Y) = \omega(Ad(h)\,X, Ad(h)\,Y)$$

for each $X,\ Y \in \mathfrak{m}$ and $h \in K_E$. The correspondence is given by

$$\omega(X, Y) = \left(\nabla^c_{X^*} Y^*\right)_E.$$

Let $\pi : K \longrightarrow K/K_E$ denote the natural projection and assume that we have an invariant affine connection ∇^c on K/K_E. We want that the connection ∇^c may have the following properties.

(A1) Let $\exp(tX)$ be the one parameter subgroup of K generated by $X \in \mathfrak{m}$. Then $\pi(\exp(tX)) = (\exp(tX))\,E$ is a regular curve such that its family of tangent vectors is parallel along the curve itself.

(A2) Let us consider the curve $\pi(\exp(tX)) = (\exp(tX))\,E$ in K/K_E. Let $Y \in \mathfrak{m}$; then the parallel displacement of the vector Y, tangent at E, along this curve coincides with the translation of Y by the one parameter subgroup $\exp(tX)$.

It is clear that if the affine connection ∇^c has the property (A2) then it also satisfies (A1).

Proposition 5. [11, p. 49] *The invariant affine connection defined by the function ω satisfies* (A2) *if and only if $\omega(X, Y) = 0$ for each $X,\ Y \in \mathfrak{m}$. Then on a reductive homogeneous space there exists one and only one affine connection which satisfies* (A2). *It is defined by the connection function which is identically zero on $\mathfrak{m} \times \mathfrak{m}$.*

This invariant affine connection is called the *canonical affine connection of the second kind* on K/K_E with respect to a fixed decomposition of the Lie algebra $\mathfrak{k} = \mathfrak{k}_E \oplus \mathfrak{m}$.

3. Lemmas for the real case

Let $T \subset [K_E]_e$ be a maximal torus which we shall keep fixed. The torus T acts on $\mathfrak{p}$ and on M by isometries and we may consider in both spaces the respective sets of fixed points, namely $F = F(T, \mathfrak{p})$ and $F(T, M)$. Of course, F is a subspace which contains the vector $E \in \mathfrak{p}$ and $F(T, M)$ may have several components (all of them compact, totally geodesic submanifolds of M). Let $M_E = [F(T, M)]_E$ denote the connected component containing the point E. It is clear that $M_E = [M \cap F]_E$.

At the point E we have the tangent space $T_E(M)$ and the subspace $V = T_E(M_E) \subset T_E(M)$. It is clear that

$$V = \{x \in T_E(M) : \ Ad(g)x = x \quad \forall g \in T\} \subset F$$

and we can determine a subspace $\mathfrak{u} \subset \mathfrak{m}$ such that

$$V = [\mathfrak{u}, E] .$$

The subspace F is, of course, determined by our choice of the torus $T \subset [K_E]_e$. However the subspace F is a proper subspace of $\mathfrak{p}$ for any torus T, as the next lemma shows.

Lemma 6. *For every maximal torus* $T \subset [K_E]_e$, $F = F(T, \mathfrak{p}) \subsetneq \mathfrak{p}$.

Proof. Let us see first that there exists at least one maximal torus in $[K_E]_e$ with the required property.

If for every maximal torus $T \subset [K_E]_e$ the subspace F coincides with $\mathfrak{p}$ then the subgroup $[K_E]_e$ acts trivially on $\mathfrak{p}$. This implies that the group $[K_E]_e$ which is normal in K_E is also normal in K.

Now according to [7, p.252, 1.1,(iii)] we may write every element in G as $(\exp X)\, g$ for some $X \in \mathfrak{p}$, $g \in K$ and then

$$(\exp X)\, g\, [K_E]_e\, g^{-1} (\exp(-X)) = (\exp X)\, [K_E]_e\, (\exp(-X)) .$$

Now for each $h \in [K_E]_e$ we have

$$\begin{aligned}(\exp X)\, h\, (\exp(-X)) &= (\exp X)\, h\, (\exp(-X))\, h^{-1} h = \\ &= (\exp X)\, (\exp Ad(h)\, (-X))\, h = \\ &= (\exp X)\, (\exp(-X))\, h = h,\end{aligned}$$

and this proves that the subalgebra $\mathfrak{k}_E = Lie\,([K_E]_e)$ is a compact ideal in the semisimple Lie algebra $\mathfrak{g}$ which is supposed to contain no compact factors. This contradiction proves that there is at least one maximal torus T such that $F = F(T, \mathfrak{p}) \subsetneq \mathfrak{p}$.

Now any other maximal torus in $[K_E]_e$ is of the form gTg^{-1} for some $g \in [K_E]_e$. Let $F_g = F(gTg^{-1}, \mathfrak{p})$, then $F_g \supset Ad(g)\, F$ because, if $Y \in Ad(g)\, F$ then $Y = Ad(g)\, X$ for some $X \in F$ and then for $h \in T$

$$Ad\left(ghg^{-1}\right) Y = Ad(g)\, X = Y.$$

If $Z \in F_g$ then $Ad(h)\,Ad(g^{-1})\,Z = Ad(g^{-1})\,Z$ and so $Ad(g^{-1})\,Z \in F$ which shows that $Z \in Ad(g)\,F$ and proves that $F_g \subset Ad(g)\,F$.

Then $F_g = Ad(g)\,F$ and this shows that if F is a proper subspace of $\mathfrak{p}$ then so is F_g and the lemma follows. □

Corollary 7. *For any maximal torus $T \subset [K_E]_e$, $[F(T,M)]_E \subsetneq M$.*

Proof. By the previous lemma, for any maximal torus $T \subset [K_E]_e$, F is a proper subspace of $\mathfrak{p}$. If for this torus we have $[F(T,M)]_E = M$ then $M \subset F$ and this contradicts our assumption that the embedding is full. □

Since ∇^c is the canonical affine connection of the second kind in M the ∇^c-geodesics through E in M are of the form $Ad(\exp(tY))\,E$ for $Y \in \mathfrak{m}$.

Lemma 8. *For $X \in \mathfrak{u}$, the ∇^c-geodesic in M $Ad(\exp(tX))\,E$ is contained in M_E for every $t \in R$.*

Proof. Let g be any element of T. We have

$$\begin{aligned} Ad(g)\,Ad(\exp(tX))\,E &= Ad(g)\,Ad(\exp(tX))\,Ad\left(g^{-1}\right)E = \\ &= Ad(\exp(tAd(g)\,X))\,E = Ad(\exp(tX))\,E. \end{aligned}$$

This means that $Ad(\exp(tX))\,E \in F$ and hence $Ad(\exp(tX))\,E \in [M \cap F]_E = M_E$. □

Let us denote by $C_K(T)$ the centralizer of the torus $T \subset [K_E]_e$ in the group K.

Lemma 9. *The group $C_K(T)$ acts transitively on M_E.*

Proof. If $X \in \mathfrak{u}$, then the monoparametric subgroup $\{\exp(tX) : t \in R\}$ is contained in the group $C_K(T)$ because all its elements commute with T. It is well known that the group $C_K(T)$ is connected and therefore the orbit $Ad(C_K(T))\,E$ is connected and it is contained in M_E. Furthermore this orbit is clearly closed in M_E by compactness.

But it is also open in M_E because

$$T_E(Ad(C_K(T))\,E) \supseteq [\mathfrak{u}, E] = V = T_E(M_E)$$

and therefore

$$T_E(Ad(C_K(T))\,E) = T_E(M_E).$$

This proves the lemma. □

Let $\mathfrak{c} \subset \mathfrak{k}$ be the Lie subalgebra corresponding to the subgroup $C_K(T)$. Then $\mathfrak{u} \subset \mathfrak{c}$. Let now

$$\mathfrak{c}_E = \{X \in \mathfrak{c} : [X, E] = 0\}$$

we have the following

Lemma 10. $\mathfrak{c} = \mathfrak{c}_E \oplus \mathfrak{u}$ *and this decomposition is reductive.*

Proof. It is clear from the previous lemma that $\mathfrak{c} = \mathfrak{c}_E + \mathfrak{u}$ and since $ad\,(E)$ is injective on $\mathfrak{u}$,this is a direct sum.

According to the definition [9, p. 41] we need to check that $Ad\,(g)\,\mathfrak{u} \subset \mathfrak{u}$, $\forall g \in C_K\,(T)$ such that $Ad(g)E = E$.

Since

$$\mathfrak{u} = \{X \in \mathfrak{m} : Ad(h)X = X,\ \forall h \in T\}\,.$$

$$Ad(g)\mathfrak{u} = \{Ad(g)X \in \mathfrak{m} : Ad(h)X = X,\ \forall h \in T\}$$

but, for $X \in \mathfrak{u}$,

$$Ad(g)X = Ad(g)Ad(h)X = Ad(h)Ad(g)X \qquad \forall h \in T$$

and then

$$Ad\,(g)\,\mathfrak{u} \subset \mathfrak{u}$$

and the lemma is proved. □

Let us recall that a submanifold M_E of a manifold M with an affine connection ∇^c is said to be *autoparallel* [11, III, p. 32] if ∇^c-parallel translation in M along a curve in M_E always takes vectors tangent to M_E into vectors tangent to M_E.

It is possible to "induce" an affine connection into the autoparallel submanifold called the *induced connection.*

Lemma 11. *M_E is ∇^c-autoparallel in M and the induced connection coincides with the canonical connection in M_E associated to the reductive decomposition $\mathfrak{c} = \mathfrak{c}_E \oplus \mathfrak{u}$.*

Proof. According to [11, p. 32, 14] M_E is autoparallel if and only if for each pair of fields X and Y tangent to M_E, $\nabla^c_X Y$ is also tangent to M_E.

Let a be a point in M_E. Given two fields X and Y tangent to M_E we need to prove that $\nabla^c_X Y|_a \in T_a\,(M_E)$.

We know that the connection ∇^c is invariant by the action of the group K on M i.e. for each $g \in K$

$$dAd\,(g)\,\nabla^c_X Y = \nabla^c_{dAd(g)X}\, dAd(g)Y.$$

But $dAd(g) = Ad(g)$ and therefore the invariance means

$$Ad\,(g)\,\nabla^c_X Y = \nabla^c_{Ad(g)X}\, Ad(g)Y.$$

Then, for every $g \in T$, we have

$$Ad\,(g)\,\nabla^c_X Y = \nabla^c_X Y$$

which shows, in particular, that

$$\nabla^c_X Y|_a \in F.$$

Since this belongs also to $T_a\,(M)$ we have

$$\nabla^c_X Y|_a \in F \cap T_a\,(M) = T_a\,(M_E)\,.$$

We want to prove now that the induced connection coincides with the canonical connection of the reductive decomposition $\mathfrak{c} = \mathfrak{c}_E \oplus \mathfrak{u}$. Let us denote by ∇^{c1} this last connection.

It is clear now that the induced connection ∇^c satisfies the axiom (A2) because the ∇^c-geodesics in M generated by vectors in V remain in M_E (see Lemma 8) and the ∇^c-parallel translation along these geodesics is the same as in M.

Then by Proposition 5 we have

$$\nabla^{c1} \equiv \nabla^c.$$

This completes the proof of the lemma. □

Let us set the notation $L = C_K(T)$ and $\mathfrak{l} = Lie\,(C_K(T))$. Let $Q \subset F$ be the subspace generated by M_E. This subspace is clearly invariant by the group L and $V \subset Q$.

Proposition 12. *We have in Q a representation ρ of the group L (with induced representation $\rho_* : \mathfrak{l} \to \mathfrak{gl}(Q)$) and M_E is an orbit of this representation. This defines a full isometric embedding of M_E in Q.*

The second fundamental form of this embedding is parallel with respect to the canonical connection ∇^c on M_E.

Proof. Clearly the representation $\rho : L \to GL(Q)$ is just the restriction to L of the adjoint representation of K on $\mathfrak{p}$, i.e. $\rho(g)U = Ad(g)U \;\; \forall g \in L,\; U \in Q$.

By the definition of Q, the orbit $M_E = Ad(L)E$ clearly defines a full isometric embedding of M_E into Q.

In order to compute the canonical covariant derivative of the second fundamental form α of M_E in Q we resort to [8, p. 359, Lemma 12] which we reproduce, with our present notation, for the benefit of the reader.

Lemma 13. *For $x,\; y,\; z$ arbitrary tangent vectors in $T_E(M)$ the following formulas hold:*

(i) $\alpha_E\,(y,z) =\perp [\rho_*(Y)\rho_*(Z)E]$

(ii) $D_E\,(y,z) = \top\,[\rho_*(Y)\rho_*(Z)E]$

(iii) $(\nabla^c_x\alpha)\,(y,z) =\perp [\rho_*(X)\alpha_E\,(y,z)]$

where $X,\; Y,\; Z \in \mathfrak{u}$ are such that $x = \rho_*(X)E,\;\; y = \rho_*(Y)E$ and $z = \rho_*(Z)E$ ($x,\; y,\; z$ are in $T_E(M)$, $\top$ and $\perp$ indicate tangent and normal component respectively).

Now we compute the canonical covariant derivative of the second fundamental form α of M_E in Q

$$(\nabla^c_x\alpha)\,(y,z) =\perp [\rho_*(X)\alpha_E\,(y,z)] \quad \forall\; X,\; Y,\; Z \in \mathfrak{u}.$$

To compute $\rho_*(X)\alpha_E\,(y,z)$, we use the structure of our R-space M. Since

$$\alpha_E\,(y,z) =\perp [\rho_*(Y)\rho_*(Z)E]$$

where $y = \rho_*(Y)E$ and $z = \rho_*(Z)E$, if ξ is any normal vector to M_E at the point E, we have

$$\begin{aligned} B\,(A_\xi y, z) &= B\,(\xi, \alpha_E\,(y,z)) = B\,(\xi, \perp [\rho_*(Y)\rho_*(Z)E]) \\ &= B\,(\xi, \rho_*(Y)\rho_*(Z)E) = B\,(-\rho_*(Y)\xi, \rho_*(Z)E) \\ &= B\,(-\rho_*(Y)\xi, z)\,. \end{aligned}$$

Then

$$A_{\xi} y = -\rho_*(Y)\xi$$

and we obtain

$$\rho_*(X)\alpha_E\,(y,z) = -A_{\alpha_E(y,z)}x.$$

This clearly shows that

$$(\nabla^c_x \alpha)\,(y,z) = 0 \quad \forall\ x,\ y,\ z \in T_E\,(M_E)$$

and completes the proof of the proposition. □

Lemma 14. *The dimensions of the manifolds M and M_E are either both even or both odd and* $\dim M_E < \dim M$.

Proof. Let us assume that the dimension of M is even and let S be the unit sphere in $T_E(M)$. Then $\dim S$ is odd and this implies that its Euler-Poincare characteristic vanishes $\chi\,(S) = 0$. Let S_E be the unit sphere in $T_E\,(M_E)$. The sphere S supports the action of the torus $T \subset [K_E]_e \subset K_E$ (by the isotropy representation) and since $S_E = F\,(T, S)$ we have $\chi\,(S_E) = 0$. This obviously indicates that $\dim S_E$ is odd and in turn implies that $\dim M_E$ is even as we wanted to prove. The proof in the other case is similar. The inequality follows from orollary 7. □

Lemma 15. $X\,[M_E] = F\,(T, X\,[M])$.

Proof. We work at the point E. We know that M_E is a totally geodesic submanifold of M with respect to the Riemannian connection and ∇^c-autoparallel. This implies that we have the following identities where the upper index M or M_E indicates that the object corresponds to the manifold M or to M_E.

If $x,\ y,\ z \in T_E\,(M_E)$ and y^* is any tangent vector field in M_E such that $y^*\,(E) = y$ we have

(i) $\nabla^{cM_E}_x y^* = \nabla^{cM}_x y^*$,
(ii) $\nabla^{M_E}_x y^* = \nabla^{M}_x y^*$,
(iii) $D^{M_E}\,(y,z) = D^M\,(y,z)\,,$
(iv) $\alpha^{M_E}\,(y,z) = \alpha^M\,(y,z)$.

These last two follow from (i) and (ii), and then we get

(v) $\alpha^{M_E}\big(x, D^{M_E}\,(y,z)\big) = \alpha^M\big(x, D^M\,(y,z)\big)$.

Let now $x \in \widehat{X}\,[M_E]$ (for definition see (1) or [5, p.227]) then $Ad\,(g)\,x = x \quad \forall g \in T$ and

$$\alpha^{M_E}\big(x, D^{M_E}\,(x,x)\big) = 0.$$

By (v) we have now

$$\alpha^{M}\big(x, D^{M}\,(x,x)\big) = 0.$$

which shows that $x \in F(T, \widehat{X}\,[M])$.

If, on the other hand we have $x \in F(T, \widehat{X}\,[M])$ then $Ad\,(g)\,x = x \quad \forall g \in T$ which implies $x \in V$ and since $\alpha^M\big(x, D^M\,(x,x)\big) = 0$ we have by (v) $\alpha^{M_E}\big(x, D^{M_E}\,(x,x)\big) = 0$ and then $x \in \widehat{X}\,[M_E]$. This yields the equality of the lemma. □

Corollary 16. $\chi\,(X\,[M_E]) = \chi\,(X\,[M])$.

Proof. This follows from the previous lemma and [5, p.234, (3.11)]. □

4. First part of the proof of the main theorem

We can give now a proof of (i) of Theorem 2.

Let M be our real flag manifold and $j : M \to \mathfrak{p}$ its natural embedding. By Corollary 7 we may construct a proper submanifold $M_1 \subset M$ $(M_1 = M_E = [F(T,M)]_E)$ which, by Lemma 11, is ∇^c-autoparallel in M and its "induced" connection coincides with the canonical connection associated to the reductive decomposition $\mathfrak{c} = \mathfrak{c}_E \oplus \mathfrak{u}$.

Furthermore there is a subspace $\mathfrak{q} \subset \mathfrak{p}$ such that $j(M_1) \subset \mathfrak{q}$ and this is an embedding with parallel second fundamental form with respect to the canonical connection that we have in M_1 ($\mathfrak{q} = Q$ from Proposition 12). By the main result in [10] this is equivalent to the fact that M_1 is a real flag manifold and $j\,|M_1$ is its canonical embedding (this is full by the way in which we constructed the subspace $\mathfrak{q}$). Clearly this process has lowered the dimension of M into that of M_1. By Lemma 14 the dimension of M_1 is even or odd in accordance with the dimension of M.

We may start now all over again with M_1 instead of M and construct a new real flag manifold M_2 and continuing in this manner we may obtain a sequence of proper submanifolds

$$M \supsetneqq M_1 \supsetneqq M_2 \supsetneqq \ldots \supsetneqq M_k,$$

all of which are real flag manifolds and having even or odd dimension according to that of M itself.

This process must stop and we need to find out where. To that end let us analyze what are the possibilities when we want to construct M_{s+1} from M_s. Let us first remark that since $j\,|M_s$ is in fact the full canonical embedding of the real flag manifold M_s, Corollary 7 implies that if T_s is a maximal torus in the connected component of the identity of the isotropy group of E in M_s then

$$[F(T_s, M_s)]_E \subsetneqq M_s.$$

Clearly we have now two possibilities, namely

$$\begin{array}{lrcl} 1) & [F(T_s, M_s)]_E & = & \{E\} \\ 2) & [F(T_s, M_s)]_E & \supsetneqq & \{E\} \end{array}$$

and in any case, we define $M_{s+1} = [F(T_s, M_s)]_E$.

In the first case (which happens only when the dimension of M is even) we have $M_{s+1} = \{E\}$, $X[M_{s+1}] = \phi$ and $\chi(X[M_{s+1}]) = 0$. From Corollary 16 it follows that

$$\chi(X[M]) = 0.$$

Let us consider now the situation (2).

If the dimension of M is even our process of construction $M_{s+1}, \ldots, M_{s+t}$ may continue until we reach a t such that $\dim M_{s+t} = 0$ and then we are back to case (1) above.

If on the other hand $\dim M$ is odd, since $\dim M_s - \dim M_{s+1} = 2r > 0$, we could not end with $\dim M_{s+t} = 0$ so our last possibility is $\dim M_{s+t} = 1$ for some t. We have then a one-dimensional real flag manifold M_{s+t} and $j\,|M_{s+t}$ is its full canonical embedding. It is known that for this embedding the second fundamental form is onto ([10]) and therefore the normal space of the embedding must be one-dimensional too. This means that the subspace $\mathfrak{q} \subset \mathfrak{p}$ such that $j(M_{s+t}) \subset \mathfrak{q}$, has dimension two and so M_{s+t} is just a curve contained in a plane. Then there is only one normal section which is obviously planar and in turn $X[M_{s+t}]$ consists of a single point. Therefore $\chi(X[M_{s+t}]) = 1$ and this yields $\chi(X[M]) = 1$. This completes the proof of (i) of Theorem 2.

5. Lemmas for the complex case

Let $V^{\perp}$ be the orthogonal complement of $V = T_E(M_E)$ in $T_E(M)$. It is clear that $V^{\perp}$ is T-invariant.

Lemma 17. $F(T, CP(T_E(M)^c)) = CP(T_E(M_E)^c) \cup F\left(T, CP\left(V^{\perp c}\right)\right)$.

Proof. For $v \in T_E(M)^c$, $[v]$ means the equivalence class of v in the projective space $CP(T_E(M)^c)$. We have $[v] \in F(T, CP(T_E(M)^c))$ if and only if Cv is a subspace in $T_E(M)^c$ invariant by the action of T.

Since

$$T_E(M)^c = T_E(M_E)^c \oplus V^{\perp c}$$

and $T_E(M_E)^c = F(T, T_E(M)^c)$, any T-invariant complex 1-dimensional subspace is contained either in $T_E(M_E)^c$ or in $V^{\perp c}$. □

In Section 2 we gave the definition of a real flag manifold as orbit of $E \in \mathfrak{p}$ by the adjoint action of K, the analytic subgroup of G corresponding to $\mathfrak{k}$. Let $\mathfrak{g}_c$ be the complexification of $\mathfrak{g}$ and σ the conjugation of $\mathfrak{g}_c$ with respect to $\mathfrak{g}$. Since $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition, there exists a compact real form $\mathfrak{g}_u$ of $\mathfrak{g}_c$ such that

$$\begin{aligned} \sigma\mathfrak{g}_u &\subset \mathfrak{g}_u, \\ \mathfrak{k} &= \mathfrak{g} \cap \mathfrak{g}_u, \\ \mathfrak{p} &= \mathfrak{g} \cap i\mathfrak{g}_u, \\ \mathfrak{g}_u &= \mathfrak{k} \oplus i\mathfrak{p}. \end{aligned}$$

Let G_c be the simply connected semisimple Lie group associated to $\mathfrak{g}_c$. Let G_1 and G_u be the analytic subgroups of G_c corresponding to the subalgebras $\mathfrak{g}$ and $\mathfrak{g}_u$ respectively. By [7, p. 152, 4 (ii)] they are both closed in G_c and by [7, p.256, 2.2 (iii)] G_u is also simply connected. Let K_1 be the analytic subgroup of G_c corresponding to $\mathfrak{k}$. The group K_1 is compact and clearly $K_1 \subset G_1 \cap G_u$. Now $Ad_{G_1} : G_1 \to G$ is a surjective analytic homomorphism and $K = Ad_{G_1}(K_1)$.

On the other hand, if we take on $i\mathfrak{p}$ the Euclidean metric induced by $-B_u$, the orbit $\widehat{M} = Ad_{G_1}(K_1)\, iE \subset i\mathfrak{p}$ is isometric to our space M. In fact, by [7, p. 180] if $X,\ Y \in \mathfrak{p}$ then

$$-B_u(iX, iY) = -B_c(iX, iY) = B_c(X, Y) = B(X, Y).$$

Let M_c be the orbit of iE by $Ad_{G_u}(G_u)$; it is obviously a complex flag manifold. Now consider on M_c the Riemannian metric induced by the inner product on $\mathfrak{g}_u$ defined by $-B_u$. It is clear that $\widehat{M}$ is isometrically embedded in M_c.

Proposition 18. *$\widehat{M}$ is totally geodesic in M_c.*

Proof. Recall that σ denotes the conjugation of $\mathfrak{g}_c$ with respect to $\mathfrak{g}$. Since σ is an automorphism of $\mathfrak{g}_u$ and G_u is simply connected, σ induces an automorphism Θ of G_u defined by

$$\Theta\left(\exp\left(sX\right)\right) = \exp\left(s\sigma X\right).$$

Then

$$Ad\left(\Theta\left(\exp\left(sX\right)\right)\right) = \exp\left(ad\left(s\sigma X\right)\right) = \sigma \circ \exp\left(ad\left(sX\right)\right) \circ \sigma$$

that is

$$Ad\left(\Theta\left(g\right)\right) = \sigma \circ Ad\left(g\right) \circ \sigma, \qquad \forall g \in G_u.$$

Since $iE \in i\mathfrak{p}$, $\sigma\left(iE\right) = -iE$ and so

$$Ad\left(\Theta\left(g\right)\right) iE = \sigma\left[Ad\left(g\right)\left(\sigma\left(iE\right)\right)\right] = -\sigma\left(Ad\left(g\right) iE\right)$$

and since $M_c = \left\{Ad\left(g\right) iE : g \in G_u\right\}$, the automorphism Θ induces a well defined map $\varepsilon : M_c \to M_c$ by

$$\varepsilon\left(Ad\left(g\right) iE\right) = Ad\left(\Theta\left(g\right)\right) iE.$$

Clearly

$$\varepsilon\left(p\right) = -\sigma\left(p\right) \qquad \forall p \in M_c$$

and hence ε is an isometry of M_c with the induced metric from $-B_u$ on $\mathfrak{g}_u$.

Since $\widehat{M} \subset i\mathfrak{p}$ it is clear that $\widehat{M}$ is contained in $\left[F\left(\varepsilon, M_c\right)\right]_{iE}$, the connected component of $F\left(\varepsilon, M_c\right)$ containing iE. It also clear that $\varepsilon_{*iE} = \varepsilon$ and

$$T_{iE}\left(F\left(\varepsilon, M_c\right)\right) = F\left(\varepsilon_{*iE}, T_{iE}\left(M_c\right)\right).$$

Now

$$T_{iE}\left(M_c\right) = \left[\mathfrak{g}_u, iE\right] = \left[\mathfrak{k}, iE\right] \oplus \left[i\mathfrak{p}, iE\right] = T_{iE}(\widehat{M}) \oplus \mathfrak{w}$$

where $T_{iE}(\widehat{M}) \subset i\mathfrak{p}$ and $\mathfrak{w} \subset \mathfrak{k}$.

Then we have

$$F\left(\varepsilon_{*iE}, T_{iE}\left(M_c\right)\right) = T_{iE}(\widehat{M})$$

because any vector in $T_{iE}\left(M_c\right)$ with non-zero component in $\mathfrak{w}$ is not fixed by ε_{*iE}.

This proves that $\widehat{M}$ is open in $\left[F\left(\varepsilon, M_c\right)\right]_{iE}$ but since it is compact, it is also closed and so we have $\widehat{M} = \left[F\left(\varepsilon, M_c\right)\right]_{iE}$.

It is a well known fact that the fixed point set of an isometry in a Riemannian manifold is totally geodesic. This proves the assertion. $\square$

Following previous notation, we use upper indexes $\widehat{M}$ or M_c to indicate that the object under consideration corresponds to $\widehat{M}$ or to M_c.

Corollary 19. i) $\alpha^{\widehat{M}} = \alpha^{M_c}\Big|\widehat{M}$, ii) $D^{\widehat{M}} = D^{M_c}\Big|\widehat{M}$.

Proof. (i) follows from the fact that $\widehat{M}$ is totally geodesic in M_c. (ii) is a consequence of Proposition 5. □

The above corollary yields that

$$\alpha^{\widehat{M}}(x, D^{\widehat{M}}(x,x)) = \alpha^{M_c}\left(x, D^{M_c}(x,x)\right) \qquad \forall x \in T_{iE}(\widehat{M}). \tag{2}$$

The complexifications of α and D are

$$\begin{aligned} \alpha^c(x_1+iy_1, x_2+iy_2) &= \alpha(x_1,x_2) - \alpha(y_1,y_2) + i\left[\alpha(y_1,x_2) + \alpha(x_1,y_2)\right] \\ D^c(x_1+iy_1, x_2+iy_2) &= D(x_1,x_2) - D(y_1,y_2) + i\left[D(y_1,x_2) + D(x_1,y_2)\right]. \end{aligned}$$

By definition $X_c[\widehat{M}]$ (resp. $X_c[M_c]$) is the algebraic subvariety of $CP(T_{iE}(\widehat{M})^c)$ (resp. $CP(T_{iE}(M_c)^c)$) defined by the equation

$$(\alpha^{\widehat{M}})^c(z, (D^{\widehat{M}})^c(z,z)) = 0$$

$$(\text{resp.} \qquad (\alpha^{M_c})^c\left(z, (D^{M_c})^c(z,z)\right) = 0)$$

for $z = x+iy \in T_{iE}(\widehat{M})^c$ (resp. $z = x+iy \in T_{iE}(M_c)^c$).

Now a straight forward computation shows that

Lemma 20. $(\alpha^{\widehat{M}})^c(z, (D^{\widehat{M}})^c(z,z)) = (\alpha^{M_c})^c(z, (D^{M_c})^c(z,z))$ *for every* $z = x+iy \in T_{iE}(\widehat{M})^c \subset T_{iE}(M_c)^c$.

Then we have

Lemma 21. $X_c[\widehat{M}] = CP(T_{iE}(\widehat{M})^c) \cap X_c[M_c]$.

Let us set $\widehat{M}_{iE} = [F(T,\widehat{M})]_{iE}$ and $\widehat{V} = T_{iE}(\widehat{M}_{iE})$. The reader can immediately verify that the proof of Lemma 17 translates mutatis mutandis into a proof of the following

Lemma 22. $F(T, CP(T_{iE}(\widehat{M})^c)) = CP(T_{iE}(\widehat{M}_{iE})^c) \cup F(T, CP(\widehat{V}^{\perp c}))$.

We have now

Proposition 23. $F(T, X_c[\widehat{M}]) = X_c[\widehat{M}_{iE}] \cup [F(T, CP(\widehat{V}^{\perp c})) \cap X_c[M_c]]$.

Proof. By Lemma 21

$$X_c[\widehat{M}] = CP(T_{iE}(\widehat{M})^c) \cap X_c[M_c]$$

and since $\widehat{V}^{\perp c} \subset T_{iE}(\widehat{M})^c$ we have

$$CP(\widehat{V}^{\perp c}) \subset CP(T_{iE}(\widehat{M})^c).$$

Then

$$CP(\widehat{V}^{\perp c}) \cap X_c[M_c] \subset X_c[\widehat{M}]$$

and furthermore
$$F(T, CP(\widehat{V}^{\perp c})) \cap X_c\,[M_c] \subset F(T, X_c[\widehat{M}]). \tag{3}$$

Besides, it is clear that equality (v) from the proof of Lemma 15 holds for the manifolds $\widehat{M}_{iE}$ and $\widehat{M}$ therefore
$$X_c[\widehat{M}_{iE}] \subset F(T, X_c[\widehat{M}]) \tag{4}$$
and then (3) and (4) prove one of the required inclusions.

Instead of proving the other one, we rather see that
$$F(T, X_c[\widehat{M}]) \subset X_c[\widehat{M}_{iE}] \cup F(T, CP(\widehat{V}^{\perp c})).$$

Since by definition
$$X_c[\widehat{M}] \subset CP(T_{iE}(\widehat{M})^c)$$
it is clear that
$$\begin{aligned} F(T, X_c[\widehat{M}]) &\subset F(T, CP(T_{iE}(\widehat{M})^c)) = \\ &= CP(T_{iE}(\widehat{M}_{iE})^c) \cup F(T, CP(\widehat{V}^{\perp c})) \end{aligned}$$
by Lemma 22.

Then it suffices to show that
$$F(T, X_c[\widehat{M}]) \cap CP(T_{iE}(\widehat{M}_{iE})^c) \subset X_c[\widehat{M}_{iE}]$$
and in fact it is enough to see that
$$X_c[\widehat{M}] \cap CP(T_{iE}(\widehat{M}_{iE})^c) \subset X_c[\widehat{M}_{iE}]$$
and this is clear.

Let us consider now the sets A, B and C defined as follows
$$\begin{aligned} A &= X_c[\widehat{M}_{iE}] \cup [F(T, CP(\widehat{V}^{\perp c})) \cap X_c\,[M_c]] \\ B &= F(T, X_c[\widehat{M}]) \\ C &= X_c[\widehat{M}_{iE}] \cup F(T, CP(\widehat{V}^{\perp c})). \end{aligned}$$

Then we have the inclusions
$$A \subset B \subset C$$
and intersecting with $X_c\,[M_c]$ we observe that
$$\begin{aligned} A \cap X_c\,[M_c] &= A \\ B \cap X_c\,[M_c] &= B \\ C \cap X_c\,[M_c] &= A. \end{aligned}$$

Then
$$A = B$$
and the proof of the proposition is complete. □

Since $X_c[\widehat{M}_{iE}]$ and $F(T, CP(\widehat{V}^{\perp c}))$ are clearly disjoint, it follows that

$$\chi(F(T, X_c[\widehat{M}])) = \chi(X_c[\widehat{M}_{iE}]) + \chi(F(T, CP(\widehat{V}^{\perp c})) \cap X_c[M_c]).$$

Now we have the following crucial fact

Proposition 24. $\chi(F(T, CP(\widehat{V}^{\perp c})) \cap X_c[M_c]) = \chi(F(T, CP(\widehat{V}^{\perp c})))$.

Proof. For the manifold $M_c = G_u / (G_u)_{iE}$ we may consider the center S of the isotropy group $(G_u)_{iE}$. The group S plays an essential role in the proof of the main result in [5]. It acts naturally on $T_{iE}(M_c)$ and then on $CP(T_{iE}(M_c)^c)$ (see [5, p.231–234]) . There, $T_{iE}(M_c)^c$ is denoted by $\mathfrak{M}^c$ and we shall adopt here the same simpler notation.

Since $\widehat{V}^{\perp c} \subset \mathfrak{M}^c$, we have

$$F(T, CP(\widehat{V}^{\perp c})) \subset CP(\mathfrak{M}^c).$$

Our torus T is maximal in $K_E \subset K$ but, as we have observed, $K = Ad_{G_1}(K_1)$ and $K_1 \subset G_1 \cap G_u$. There is a torus $T_1 \subset K_1$ such that $Ad_{G_1}(T_1) = T$ (in fact we may take $T_1 = [Ad_{G_1}^{-1}(T)]_e$ which is clearly compact, connected and abelian).

Now T_1 acts on $\widehat{M}$ exactly as T and then

$$F(T_1, CP(\widehat{V}^{\perp c})) = F(T, CP(\widehat{V}^{\perp c})).$$

Furthermore $F(T, CP(\widehat{V}^{\perp c}))$ is invariant by the action of S on $CP(\mathfrak{M}^c)$ because S commutes with T_1. Then it makes sense to consider the submanifold $F(S, F(T, CP(\widehat{V}^{\perp c})))$.

We have now the following obvious identity

$$F(S, F(T, CP(\widehat{V}^{\perp c}))) = F(S, F(T, CP(\widehat{V}^{\perp c}))) \cap F(S, CP(\mathfrak{M}^c)). \tag{5}$$

Now, part (ii) of Proposition (3.10) in [5, p.234] says, in our notation,

$$F(S, CP(\mathfrak{M}^c)) = F(S, X_c[M_c])$$

and hence we may replace it in (5) obtaining

$$\begin{aligned} F(S, F(T, CP(\widehat{V}^{\perp c}))) &= F(S, F(T, CP(\widehat{V}^{\perp c}))) \cap F(S, X_c[M_c]) = \\ &= F(S, [F(T, CP(\widehat{V}^{\perp c})) \cap X_c[M_c]]). \end{aligned}$$

This proves

$$\chi(F(S, F(T, CP(\widehat{V}^{\perp c})))) = \chi(F(S, [F(T, CP(\widehat{V}^{\perp c})) \cap X_c[M_c]])).$$

Now we apply Theorem (3.11) in [5, p.234] and obtain our proposition. $\square$

Now, from Propositions 23 and 24 we obtain (again by [5, p.234, (3.11)])

Corollary 25. $\chi(X_c[\widehat{M}]) = \chi(X_c[\widehat{M}_{iE}]) + \chi(F(T, CP(\widehat{V}^{\perp c})))$.

Let us consider now the diagram

$$\begin{array}{ccccc} M_E & \hookrightarrow & M & \hookrightarrow & \mathfrak{p} \\ \downarrow f & & \downarrow f & & \downarrow f \\ \widehat{M}_{iE} & \hookrightarrow & \widehat{M} & \hookrightarrow & i\mathfrak{p} \end{array}$$

where $f(X) = iX$ is clearly an isometric isomorphism from $\mathfrak{p}$ to $i\mathfrak{p}$ and therefore it induces isometries on the other levels because we are always taking the induced metric on the submanifolds. Furthermore, since f_* takes planar normal sections into planar normal sections, from our previous corollary, we get

Corollary 26. $\chi\left(X_c\left[M\right]\right) = \chi\left(X_c\left[M_E\right]\right) + \chi\left(F\left(T, CP\left(V^{\perp c}\right)\right)\right)$.

6. Second part of the proof of the main theorem

We are, finally, in position to prove part (ii) of Theorem 2. By Corollary 26 we have

$$\chi\left(X_c\left[M\right]\right) = \chi\left(X_c\left[M_E\right]\right) + \chi\left(F\left(T, CP\left(V^{\perp c}\right)\right)\right).$$

If, by inductive hypothesis, we know that

$$\chi\left(X_c\left[M_E\right]\right) = \chi\left(CP\left(T_E\left(M_E\right)^c\right)\right) \tag{6}$$

then

$$\chi\left(X_c\left[M\right]\right) = \chi\left(CP\left(T_E\left(M_E\right)^c\right)\right) + \chi\left(F\left(T, CP\left(V^{\perp c}\right)\right)\right).$$

By Lemma 17 we have now

$$\chi\left(X_c\left[M\right]\right) = \chi\left(F\left(T, CP\left(T_E\left(M\right)^c\right)\right)\right)$$

and then again by [5, p. 234, (3.11)]

$$\chi\left(X_c\left[M\right]\right) = \chi\left(CP\left(T_E\left(M\right)^c\right)\right).$$

The proof is then reduced to get identity (6) and to that end we have to proceed as in Section 4 setting $M_1 = M_E = \left[F\left(T, M\right)\right]_E$ and constructing our sequence of proper submanifolds

$$M \supsetneqq M_1 \supsetneqq M_2 \supsetneqq \ldots \supsetneqq M_k,$$

all of which are real flag manifolds and having even or odd dimension according to that of M itself.

This process must stop and, as we noticed in Section 4, if our last manifold is M_k then we have the following possibilities

$$\begin{array}{llll} 1) & M_k = & \{E\} & (\dim M \text{ even}) \\ 2) & \dim M_k & = 1 & (\dim M \text{ odd}). \end{array}$$

If we have $M_k = \{E\}$ then $X\left[M_k\right] = \phi$ and also $X_c\left[M_k\right] = \phi$ which clearly implies $\chi\left(X_c\left[M_k\right]\right) = 0$. Furthermore in this case $T_E\left(M_k\right)^c = \{0\}$ and then $CP\left(T_E\left(M_k\right)^c\right) = \phi$.

This yields $\chi\left(CP\left(T_E\left(M_k\right)^c\right)\right)=0$ and so the identity (6) holds trivially in the last step of our "reversed " induction process.

On the other hand, if we have the second possibility for M_k then, as was indicated in Section 4, we have that $X\left[M_k\right]$ consists of a single point. Since $\dim M_k=1$ we have $T_E\left(M_k\right)\simeq R$ and of course $\left(T_E\left(M_k\right)\right)^c\simeq C$. Then $CP\left(T_E\left(M_k\right)^c\right)$ consists of a single point and clearly so does $X_c\left[M_k\right]$.

Then we have

$$\chi\left(X_c\left[M_k\right]\right)=1=\chi\left(CP\left(T_E\left(M_k\right)^c\right)\right)$$

and the identity (6) also holds trivially in the last step of our "reversed" induction process if the dimension of M is odd.

This completes the proof of Theorem 2. $\square$

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