Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 41 (2000), No. 2, 335-344.

# A Geometric Proof of Some Inequalities Involving Mixed Volumes

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**Abstract.** A new, "spherical harmonics free" proof of mixed-volume inequalities due to Schneider and to Goodey and Groemer, is presented.

## Introduction

The Alexandrov-Fenchel inequality implies that if $K_1$ and $K_2$ are two convex bodies in $\mathbb{R}^n$, and if for $i, j = 1, 2$, $V_{ij} = V(K_i, K_j, B, \dots, B)$ is the mixed volume of $K_i$, $K_j$ with $n-2$ copies of the Euclidean unit ball $B$, then $V_{12}^2 \geq V_{11}V_{22}$. Moreover, it was proved by Schneider and by Goodey and Groemer that one can control the difference $V_{12}^2 - V_{11}V_{22}$ in the following way :

(1) If $K_2 = B$, $V_{12} = V_1 = V(K_1, B, \dots, B)$, $B_1$ is the Steiner ball of $K_1$ (see the definitions below) and $v_n$ is the volume of $B$ then

$$V_1^2 - v_n V_{11} \geq \frac{n+1}{n(n-1)} v_n \int_{S_{n-1}} \left(h_{K_1} - h_{B_1}\right)^2 d\sigma,$$

where $\sigma$ denotes the surface measure on the sphere $S_{n-1}$, and for a convex body $K$, $h_K$ denotes its support function.

(2) If the Steiner balls of $K_1, K_2$ are the same, then

$$V_{12}^2 - V_{11}V_{22} \geq \frac{n+1}{n(n-1)} v_n \int_{S_{n-1}} \left(h_{K_1} - h_{K_2}\right)^2 d\sigma.$$

The results (1) and (2) may be interpreted as stability results in specific cases of the Alexandrov-Fenchel inequality. Using them one can derive further inequalities between intrinsic volumes of different orders of a given convex body.

The proofs in [7] and [1] make use of spherical harmonics and of a representation of mixed volumes which involves the action of differential operators on support functions of convex bodies. We present here a new proof which is "spherical harmonics free" and which has a more geometric flavor. This proof is based on a variational argument involving Santaló's inequality. We believe that this variational method may prove useful for the treatment of other problems as well.

In fact we prove the following more general result (which also admits a proof using spherical harmonics):

(3) Let $K_1, \dots, K_p$ be convex bodies in $\mathbb{R}^n$ and for $1 \le i \le p$, let $V_i = V(K_i, B, \dots, B)$, $h_i = h_{K_i}$ and let $B_i$ be the Steiner ball of $K_i$. Then the quadratic form $q : \mathbb{R}^p \mapsto \mathbb{R}$ defined by $q(s_1, \dots, s_p) = \sum_{i,j=1}^p a_{ij} s_i s_j$, where

$$a_{ij} = V_i V_j - v_n V_{ij} - \frac{n+1}{n(n-1)} v_n \int_{S_{n-1}} (h_i - h_{B_i})(h_j - h_{B_j})\, d\sigma,$$

is non-negative.

The paper is organized as follows: after stating some lemmas, we give a new proof of (1), and then extend it to prove (3), of which (2) is an easy consequence.

**Notations.** Let $K$ be a convex body in $\mathbb{R}^n$, endowed with its canonical scalar product $\langle\ ,\ \rangle$. We denote by $h_K$ the support function of $K$: for all $x \in \mathbb{R}^n$, $h_K(x) = \sup_{y \in K} \langle x, y \rangle$. If $C$ is a Borel subset of $\mathbb{R}^n$, we denote by $|C|$ its Lebesgue measure (its volume). We denote by $S_{n-1}$ the unit sphere of the Euclidean space $\mathbb{R}^n$, endowed with its surface measure $\sigma$. If $x_0 \in K$, the *polar body* $K^{*x_0}$ of $K$ with respect to $x_0$ is defined by

$$K^{*x_0} = \{y \in \mathbb{R}^n; \langle y - x_0, x - x_0 \rangle \le 1 \text{ for every } x \in K\}.$$

If 0 is in the interior of $K$, we write $K^* = K^{*0}$. We define (see [8]) the *Steiner point* $z_A$ of a convex body $A$ in $\mathbb{R}^n$ by the identity

$$\int_{S_{n-1}} \langle x, u \rangle\, h_A(x)\, d\sigma(x) = v_n \langle z_A, u \rangle$$

for every $u \in \mathbb{R}^n$. The *Steiner ball* $B(A)$ of the convex body $A$ is

$$B(A) = z_A + \frac{w(A)}{2} B,$$

where $w(A)$, the *mean width* of $A$, is defined by

$$w(A) = \frac{2}{n v_n} \int_{S_{n-1}} h_A(x)\, d\sigma(x).$$

The *Santaló point* of $K$ is the unique point $x_0 \in K$ with the property that $x_0$ is the center of mass of $K^{*x_0}$. It is well-known that this point is characterized by the fact that

$$|K^{*x_0}| = \min_{x \in K} |K^{*x}| .$$

To say that $x_0 = 0$ means that for every $u \in \mathbb{R}^n$, one has

$$\int_{K^*} \langle x, u \rangle \, dx = 0.$$

We define now the *volume product* $\mathcal{P}(K)$ of $K$ by

$$\mathcal{P}(K) = \min_{x \in \mathbb{R}^n} |K| \cdot |K^{*x}| = |K||K^{*x_0}|.$$

By a celebrated theorem of Santaló (see [6], [4]), $\mathcal{P}(K) \leq \mathcal{P}(B)$ for all $K$.

**Lemma 1.** *Let $K$, $A$ be two convex bodies with 0 in their interiors. Let*

$$C = \{(y,t) \in \mathbb{R}^n \times \mathbb{R} \ ; \ 1 + th_A(x) - \langle x, y \rangle \ > 0 \text{ for all } x \in K^*\},$$

*and*

$$D = \{t \in \mathbb{R} \ ; \ (y,t) \in C \text{ for some } y \in \mathbb{R}^n\} .$$

*Define $F : C \mapsto \mathbb{R}$ by*

$$F(y,t) = \int_{K^*} \frac{1}{(1 + th_A(x) - \langle x, y \rangle)^{n+1}} \, dx.$$

*Then there exists $a > 0$ satisfying $]-a, +\infty[ \subset D$ and such that for each $t > -a$, the equation $F(y,t) = \min_{w;(w,t) \in C} F(w,t)$ has a unique solution $y(t)$. Moreover, the mapping $t \mapsto y(t)$ is $C^\infty$ on $]-a, +\infty[$.*

*Proof.* Observe first that $C$ is an open convex subset of $\mathbb{R}^n \times \mathbb{R}$, that $0 \in D$ and that $F$ is $C^\infty$ and strictly convex on $C$ (because $u \mapsto (1+u)^{-n-1}$ is strictly convex on $]-1, +\infty[$ ). By the proof of Lemma 2 below, for $t \geq 0$ and $(y,t) \in C$, one has $F(y,t) = |(K+tA)^{*y}|$. By [6] (see also Theorem 9 in [5] for a more precise statement and a complete proof), $F(y,t) \to +\infty$ when $(y,t) \to \partial C$ with fixed $t \geq 0$. Thus by the strict convexity of $F$, for every $t \geq 0$ there exists a unique $y(t)$ such that $(y(t),t) \in C$ and $F(y(t),t) = \min_{y;(y,t) \in C} F(y,t)$. Thus $(\nabla_y F)(y(t),t) = 0$. Define $G : C \mapsto \mathbb{R}^n$ by $G(y,t) = (\nabla_y F)(y,t)$, and let $D_y G$ be the differential of $G$ at $y \in \mathbb{R}^n$. In the canonical basis of $\mathbb{R}^n$, the matrix of $D_y G(y,t)$ is $H_y F(y,t)$, the Hessian matrix of $F$ as a function of $y$ at $(y,t)$. For any $(y,t) \in C$, one has

$$\langle w, H_y F(y,t) w \rangle = (n+1)(n+2) \int_{K^*} \frac{\langle x, w \rangle^2}{(1 + th_A(x) - \langle x, y \rangle)^{n+3}} \, dx > 0 \text{ for } w \in \mathbb{R}^n \setminus \{0\},$$

which implies $\det(H_y F(y,t)) > 0$. Hence from the implicit functions theorem it follows that for $t > 0$, the unique solution $y(t)$ of the equation $G(y,t) = 0$ is $C^\infty$ on $]0, +\infty[$.

We apply the same theorem in a neighbourhood of $(y(0),0)$ to get that for some $a>0$ and some neighbourhood $U$ of $y(0)$ such that $U\times]-a,a[\subset C$, there exists a $C^\infty$ function $z:]-a,a[\mapsto U$ such that $G(z(t),t)=0$. Since $y(t)$ is the unique solution of $F(y(t),t)=\min_{y;(y,t)\in C}F(y,t)$ for $t\geq 0$, it follows that $y(t)=z(t)$ for $t\geq 0$. Setting $y(t)=z(t)$ for $t\in]-a,0[$, $y(t)$ becomes a $C^\infty$ function on $]-a,+\infty[$. Moreover, again by the strict convexity of $F$ on $C$, the equality $G(y(t),t)=0$ implies that one has also $F(y(t),t))=\min_{y;(y,t)\in C}F(y,t)$ for $t\in]-a,0[$. □

**Lemma 2.** *Let $K$, $A$ be two convex bodies with 0 in their interiors, let*

$$E=\{t\in\mathbb{R};\ h_K+th_A \text{ is convex and positive on } \mathbb{R}^n\setminus\{0\}\},$$

*and suppose that 0 is the Santaló point of $K$. For $t\in E$, denote by $K+tA$ the convex body with support function $h_K+th_A$. Then, there exist $y_1,y_2\in\mathbb{R}^n$ such that for all $t\in E$, the Santaló point $y(t)$ of $K+tA$ satisfies*

$$y(t)=ty_1+t^2y_2+t^2\varepsilon(t) \text{ where } \varepsilon(t)\to 0 \text{ when } t\to 0, t\in E.$$

*Moreover $y_1$ is defined by the equality*

$$\int_{K^*}\langle x,u\rangle h_A(x)\,dx=\int_{K^*}\langle x,u\rangle\langle x,y_1\rangle\,dx \quad \text{for all } u\in\mathbb{R}^n.$$

*Proof.* With the notations of Lemma 1, one has $E\subset D$. For $t\in E$ and $(y,t)\in C$ one has

$$|(K+tA)^{*y}|=\int_{K^*}\frac{1}{(1+th_A(x)-\langle x,y\rangle)^{n+1}}\,dx.$$

As a matter of fact, using polar coordinates,

$$\int_{K^*}\frac{1}{(1+th_A(x)-\langle x,y\rangle)^{n+1}}\,dx=\int_{S_{n-1}}\Big(\int_0^{\frac{1}{h_K(\theta)}}\frac{r^{n-1}}{(1+r(th_A(\theta)-\langle\theta,y\rangle))^{n+1}}dr\Big)\,d\theta$$

$$=\frac{1}{n}\int_{S_{n-1}}\Big[\Big(\frac{r}{1+r(th_A(\theta)-\langle\theta,y\rangle)}\Big)^n\Big]_0^{\frac{1}{h_K(\theta)}}\,d\theta=\frac{1}{n}\int_{S_{n-1}}\frac{1}{(h_K(\theta)+th_A(\theta)-\langle\theta,y\rangle)^n}\,d\theta$$

$$=\frac{1}{n}\int_{S_{n-1}}\frac{1}{(h_{(K+tA-y)}(\theta))^n}\,d\theta=|(K+tA)^{*y}|.$$

By Lemma 1, $y(t)$ is $C^\infty$ on a neighbourhood of 0 (and actually also on $E$), hence Taylor's formula of order 2 ensures the existence of the points $y_1$ and $y_2$ and of the function $\varepsilon(t)$. Now, $y(t)$ is characterized by the identity:

$$\int_{K^*}\frac{\langle x,u\rangle}{(1+th_A(x)-\langle x,y(t)\rangle)^{n+2}}\,dx=0 \text{ for every } u\in\mathbb{R}^n$$

(this is exactly the fact that $y(t)$ is the center of mass of $(K+tA)^{*y(t)}$). Using the expansion of $y(t)$, we get for every $u \in \mathbb{R}^n$,

$$0 = \int_{K^*} \frac{\langle x, u\rangle}{(1 + th_A(x) - \langle x, ty_1 + t^2 y_2 + t^2\varepsilon(t)\rangle)^{n+2}}\, dx$$

$$= \int_{K^*} \langle x, u\rangle\, dx - (n+2)t \int_{K^*} \langle x, u\rangle (h_A(x) - \langle x, y_1\rangle)\, dx$$

$$+ t^2(n+2) \int_{K^*} \big(\langle y_2, x\rangle + \frac{n+3}{2}(h_A(x) - \langle x, y_1\rangle)^2\big)\langle x, u\rangle\, dx + t^2\varepsilon(t).$$

Thus $y_1$ satisfies

$$\int_{K^*} \langle x, u\rangle h_A(x)\, dx = \int_{K^*} \langle x, u\rangle\langle x, y_1\rangle\, dx \quad \text{for all } u \in \mathbb{R}^n$$

(clearly, only one point satisfies this equality). □

**Lemma 3.** *With the assumptions and the notations of Lemma 2, let*

$$|K + tA| = |K| + nV_{n-1,1}(K, A)\, t + \frac{n(n-1)}{2} V_{n-2,2}(K, A)t^2 + t^2\eta(t)$$

*be the mixed volume expansion of* $|K + tA|$, *where* $\eta(t) \to 0$ *when* $t \to 0$, $t \geq 0$. *Then*

$$\mathcal{P}(K + tA) = \mathcal{P}(K) + c_1 t + c_2 t^2 + t^2\varepsilon(t)$$

*with* $\varepsilon(t) \to 0$ *when* $t \to 0$, $t \geq 0$ *and*

$$c_1 = n|K^*|V_{n-1,1}(K, A) - (n+1)|K| \int_{K^*} h_A(x)\, dx,$$

$$c_2 = \frac{n(n-1)}{2}|K^*|V_{n-2,2}(K, A) + \frac{(n+1)(n+2)}{2}|K| \int_{K^*} (h_A(x) - \langle x, y_1\rangle)^2 dx$$

$$- n(n+1)V_{n-1,1}(K, A) \int_{K^*} h_A(x)\, dx.$$

*Proof.* With the notations of Lemmas 1 and 2, for some functions $\varepsilon$ such that $\varepsilon(t) \to 0$ when $t \to 0$, $t \geq 0$, one has

$$|(K + tA)^{*y(t)}| = \int_{K^*} \frac{1}{(1 + th_A(x) - \langle x, y(t)\rangle)^{n+1}}\, dx$$

$$= \int_{K^*} \frac{1}{\big(1 + (h_A(x) - \langle x, y_1\rangle)t - \langle x, y_2\rangle t^2 + t^2\varepsilon(t)\big)^{n+1}}\, dx$$

$$= |K^*| - (n+1)t \int_{K^*} (h_A(x) - \langle x, y_1 \rangle)\, dx$$

$$- (n+1)t^2 \int_{K^*} \Big( \langle x, y_2 \rangle - \frac{n+2}{2} (h_A(x) - \langle x, y_1 \rangle)^2 \Big)\, dx + t^2 \varepsilon(t).$$

Since by hypothesis $\int_{K^*} \langle x, u \rangle\, dx = 0$ for every $u \in \mathbb{R}^n$, we get

$$|(K+tA)^{*y(t)}| = |K^*| - (n+1)t \int_{K^*} h_A(x)\, dx$$

$$+ \frac{(n+1)(n+2)}{2} t^2 \int_{K^*} (h_A(x) - \langle x, y_1 \rangle)^2 dx + t^2 \varepsilon(t).$$

The result follows. □

The proof of the following lemma is easy.

**Lemma 4.** *Let $K_1$ and $K_2$ be convex bodies in $\mathbb{R}^n$, with Steiner balls $B_1$ and $B_2$ and Steiner points $z_{K_1}$ and $z_{K_2}$. Then*

$$\int_{S_{n-1}} \big(h_{K_1}(x) - h_{B_1}(x)\big)\big(h_{K_2}(x) - h_{B_2}(x)\big)\, d\sigma(x)$$

$$= \int_{S_{n-1}} \big(h_{K_1}(x) - \langle z_{K_1}, x \rangle\big)\big(h_{K_2}(x) - \langle z_{K_2}, x \rangle\big)\, d\sigma(x)$$

$$- \frac{1}{n v_n} \Big( \int_{S_{n-1}} h_{K_1}(x)\, d\sigma(x) \Big) \Big( \int_{S_{n-1}} h_{K_2}(x)\, d\sigma(x) \Big).$$

The following result was originally proved by Schneider [7] and by Goodey and Groemer [1].

**Proposition 5.** *Let $A$ be a convex body with Steiner ball $B_A$. Then*

$$V_1(A)^2 - v_n V_2(A) \geq \frac{n+1}{n(n-1)} v_n \int_{S_{n-1}} (h_A(x) - h_{B(A)}(x))^2 d\sigma(x),$$

*where $V_1(A) = V(A, B, \ldots, B)$ and $V_2(A) = V(A, A, B, \ldots, B)$ ($B$ is the Euclidean unit ball).*

*Proof.* Let $h = h_A$, $V_1 = V_1(A)$ and $V_2 = V_2(A)$. By Santaló's inequality, we know that

$$\mathcal{P}(B + tA) \leq \mathcal{P}(B) = v_n^2 \quad \text{for every } t \geq 0.$$

Since

$$n|B|V_1 = v_n \int_{S_{n-1}} h(x)\, d\sigma(x) = (n+1)|B| \int_B h(x)\, dx,$$

it follows from Lemma 3 that

$$\frac{n(n-1)}{2}v_n V_2 + \frac{(n+1)(n+2)}{2}v_n \int_B (h(x) - \langle x, y_1\rangle)^2 dx \le n(n+1)V_1 \int_B h(x)\, dx,$$

or

$$\frac{n(n-1)}{2}v_n V_2 + \frac{n+1}{2}v_n \int_{S_{n-1}} (h(x) - \langle x, y_1\rangle)^2 d\sigma(x) \le nV_1 \int_{S_{n-1}} h(x)\, d\sigma(x).$$

Observe that when $K = B$ in Lemma 2, $y_1$ is the Steiner point of $A$.
Since $V_1 = \frac{1}{n}\int_{S_{n-1}} h(x)\, d\sigma$, the preceding inequality is equivalent to

$$n^2V_1^2 - \frac{n(n-1)}{2}v_n V_2 \ge \frac{n+1}{2}v_n \int_{S_{n-1}} (h(x) - \langle x, y_1\rangle)^2 d\sigma(x).$$

It follows from Lemma 4 that

$$n^2V_1^2 - \frac{n(n-1)}{2}v_n V_2$$

$$\ge \frac{n+1}{2}\, v_n\Big( \int_{S_{n-1}} (h(x) - h_{B(A)}(x))^2 d\sigma(x) + \frac{1}{nv_n}\big(\int_{S_{n-1}} h(x)\, d\sigma(x)\big)^2\Big)$$

or equivalently

$$\frac{n(n-1)}{2}\big(V_1^2 - v_n V_2\big) \ge \frac{n+1}{2}v_n \int_{S_{n-1}} (h(x) - h_{B(A)}(x))^2 d\sigma(x). \qquad \square$$

We want now to generalize the method we used to more than one convex body. For this we use the so-called *Minkowski differences* or *"érosions"*. By the same arguments as in the proofs of Lemmas 2 and 3, one gets :

**Lemma 6.** *Let $K_1, \dots, K_p$ be convex bodies in $\mathbb{R}^n$ with support functions $h_1, \dots, h_p$, and let*

$$E = \Big\{s = (s_1, \dots, s_p) \in \mathbb{R}^p \ ;\ h_B + \sum_{i=1}^{p} s_i h_i \text{ is convex on } \mathbb{R}^n \text{ and positive on } \mathbb{R}^n \setminus \{0\}\Big\}.$$

*For $s \in E$, let $K(s) := B + \sum_{i=1}^p s_i K_i$ be the convex body whose support function is $h_B + \sum_{i=1}^p s_i h_i$. Then for $s \in E$, the Santaló point $y(s)$ of $K(s)$ satisfies*

$$y(s) = \sum_{i=1}^{p} s_i y_i + \sum_{1\le i,j\le p} s_i s_j y_{ij} + \big(\sum_{i=1}^{p} s_i^2\big)\varepsilon(s),$$

*where $\varepsilon(s) \to 0$ when $s \to 0$, $s \in E$, and for $1 \le i \le p$, $y_i$ is the Steiner point of $K_i$ and $y_{ij} \in \mathbb{R}^n$, $1 \le i, j \le p$. Moreover for $s \in E$,*

$$\left|K(s)^{*y(s)}\right| = |B| - (n+1)\sum_{i=1}^{p} s_i \int_B h_i(x)\,dx$$

$$+\frac{(n+1)(n+2)}{2}\sum_{i,j=1}^{p} s_i s_j \int_B (h_i(x) - \langle x, y_i\rangle)(h_j(x) - \langle x, y_j\rangle)\,dx + \Big(\sum_{i=1}^{p} s_i^2\Big)\varepsilon(s)$$

*where* $\varepsilon(s) \to 0$ *when* $s \to 0$, $s \in E$.

**Lemma 7.** *With the notations of Lemma 6, assume that each* $K_i$ *has a* $C^2$ *boundary with positive curvature ; then there exists* $a > 0$ *such that* $]-a, +\infty[^p \subset E$. *Moreover, if* $V_i = V(K_i, B, \ldots, B)$ *and* $V_{ij} = V(K_i, K_j, B, \ldots, B)$, *one has for all* $s \in ]-a, +\infty[^p$,

$$|B + s_1 K_1 + \ldots + s_p K_p| = |B| + n\sum_{i=1}^{p} V_i s_i + \frac{n(n-1)}{2}\sum_{i,j=1}^{p} V_{ij} s_i s_j + \Big(\sum_{i=1}^{p} s_i^2\Big)\,\varepsilon(s),$$

*where* $\varepsilon(s) \to 0$ *when* $s \to 0$.

*Proof.* For the first fact, see [8], Theorem 2.5.4. The second part follows from the extension of mixed volumes to $n$-tuples of differences of support functions (see [8], Section 5.2, or another proof in [3]). □

Now using Lemmas 6 and 7 and the same calculations as in Lemma 3, observing that $nv_n V_i = (n+1)v_n \int_B h_i(x)\,dx$, we get

**Lemma 8.** *With the notations of Lemma 7,*

$$\mathcal{P}(K(s)) = \mathcal{P}(B) + \sum_{i,j=1}^{p} c_{ij} s_i s_j + \Big(\sum_{i=1}^{p} s_i^2\Big)\varepsilon(s),$$

*where* $\varepsilon(s) \to 0$ *when* $s \to 0$ *and for* $1 \le i, j \le p$,

$$c_{ij} = \frac{n(n-1)}{2}\,v_n V_{ij} + \frac{(n+1)(n+2)}{2}\,v_n \int_B (h_i(x) - \langle x, y_i\rangle)(h_j(x) - \langle x, y_j\rangle)\,dx$$

$$- n(n+1)V_i \int_B h_j(x)\,dx.$$

**Theorem.** *Let* $K_1, \ldots, K_p$ *be convex bodies in* $\mathbb{R}^n$ *and let* $B$ *be the Euclidean ball. For* $1 \le i, j \le p$, *let* $V_i = V(K_i, B, \ldots, B)$, $V_{ij} = V(K_i, K_j, B, \ldots, B)$, $h_i = h_{K_i}$, *let* $B_i$ *be the Steiner ball of* $K_i$ *and*

$$a_{ij} = V_i V_j - v_n V_{ij} - \frac{n+1}{n(n-1)}\,v_n \int_{S_{n-1}} (h_i(x) - h_{B_i}(x))(h_j(x) - h_{B_j}(x))\,d\sigma(x).$$

*Then the quadratic form* $q : \mathbb{R}^p \mapsto \mathbb{R}$ *defined by* $q(s_1, \dots, s_p) = \sum_{i,j=1}^p a_{ij} s_i s_j$ *is non-negative.*

*Proof.* We approximate $K_i$ by bodies having $C^2$ boundaries with positive curvature. Using homogeneity, we may assume, due to Lemma 7, that $h_B + \sum_{i=1}^p s_i h_i$ is the support function of a convex body $K(s)$ for all $s = (s_1, \dots, s_p) \in [-1, 1]^p$. Then by Santaló's inequality, $\mathcal{P}(K(s)) \le \mathcal{P}(B)$. The result follows now from Lemmas 4 and 8, with the same proof as in Proposition 5 (we needed here the fact that $q(s) \ge 0$ for $s$ in a neighborhood of 0, this is what necessitated the introduction of Lemma 7, that is the extension of Steiner's formula to negative values of $s_i$). □

**Remark.** Note that Proposition 5 is a corollary of the theorem, since the latter implies $a_{ii} \ge 0$.

**Corollary 9.** ([7], [1]) *If the Steiner points and the mean widths of* $K_1$ *and* $K_2$ *are the same, then*

$$2V_{12} - V_{11} - V_{22} \ge \frac{n+1}{n(n-1)} \int_{S_{n-1}} (h_1(x) - h_2(x))^2 d\sigma(x).$$

*Hence, for any* $K_1$ *and* $K_2$ *one has*

$$V_{12}^2 - V_{11}V_{22} \ge \frac{n+1}{n(n-1)} w(K_1)^2 V_{22} \int_{S_{n-1}} (h_{\tilde{K}_1}(x) - h_{\tilde{K}_2}(x))^2 d\sigma(x)$$

*where* $\tilde{K}_1$, $\tilde{K}_2$ *are homothetic copies of* $K_1$, $K_2$, *with* $w(\tilde{K}_1) = w(\tilde{K}_2) = 1$ *and* $z_{\tilde{K}_1} = z_{\tilde{K}_2}$.

*Proof.* The second assertion is an easy consequence of the first one (see [7] and [1]). To prove the first assertion, one may assume that $B$ is the common Steiner ball of $K_1$ and $K_2$. Then, with $p = 2$ in the theorem,

$$a_{12} = v_n^2 - v_n V_{12} - \frac{n+1}{n(n-1)} v_n \int_{S_{n-1}} (h_1(x) - 1)(h_2(x) - 1)\, d\sigma(x).$$

From the non-negativity of the quadratic form we get $a_{11} + a_{22} \ge 2\sqrt{a_{11}a_{22}} \ge 2a_{12}$, which is the first assertion. □

## References

[1] Goodey, P. R.; Groemer, H.: *Stability results for first order projection bodies.* Proc. Amer. Math. Soc. **109** (1990), 1103-1114.

[2] Groemer, H.: *Geometric applications of Fourier series and spherical harmonics.* Encyclopedia Math. Appl. **61**, Cambridge Univ. Press, Cambridge 1996.

[3] Matheron, G.: *La formule de Steiner pour les érosions.* J. Appl. Probab. **15** (1978), 126–135.

[4] Meyer, M.; Pajor, A.: *On the Blaschke-Santaló inequality.* Arch. Math. **55** (1990), 82–93.

[5] Meyer, M.; Werner, E.: *The Santaló-regions of a convex body.* Trans. Amer. Math. Soc. **350** (1998), 4569–4591.

[6] Santaló, L. A.: *Un invariante afin para los cuerpos convexos del espacio de n dimensiones.* Portugal. Math. **8** (1949), 155–161.

[7] Schneider, R.: *Stability in the Aleksandrov-Fenchel-Jessen theorem.* Mathematika **36** (1989), 50–59.

[8] Schneider, R.: *Convex bodies: The Brunn-Minkowski Theory.* Encyclopedia Math. Appl. **44**, Cambridge Univ. Press, Cambridge 1993.

Received April 13, 1999; revised version August 16, 1999