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Rational Summation of Rational Functions

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Abstract. In this article we characterize rational functions for which their indefinite sum is again a rational function.

1. Introduction

Let k be a field of characteristic 0 and let r be a rational function in one variable over k. Pick an integer j_0 such that $r(j)$ is defined for $j \geq j_0$, and consider

$$y_x = \sum_{j=j_0}^{x} r(j), \quad x > j_0.$$

If there exists a rational function y such that $y(x) = y_x$ for all $x > j_0$, we say that r is *rationally summable*. Notice that r is rationally summable if and only if the difference equation

$$\Delta y = y(x+1) - y(x) = r(x) \tag{1}$$

has a solution in the field $k(x)$ of rational functions in one variable. Of course, this is equivalent to requiring the equation

$$\nabla t = t(x) - t(x-1) = r(x)$$

to have a rational solution.

We consider the question, suggested by Alberto Grünbaum, of finding necessary and sufficient conditions for rational summability. This is accomplished in Theorems 10 and 11.

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The rational summation problem has been studied by Abramov ([1]), Paule ([2] and [3]), Pirastu and Strehl ([4] and [5]). These authors give algorithms, based on either the Gosper-Petkovšek representation or the shift saturated representation of a rational function, to decide whether a rational function is rationally summable or not. In the affirmative case, a rational solution to (1) is found. Otherwise, the output is a rational function, closest to a solution of (1) in some suitable sense.

Example 1. Let r be a polynomial of degree d in $k[x]$. We can write $r(n) = \sum_{i=0}^{d} a_i \binom{n}{i}$, where $a_i \in k$. Define $y(n) = \sum_{i=0}^{d} a_i \binom{n}{i+1}$. Then $\Delta y = r$.

In view of the division algorithm and Example 1, we assume from now on that $r = p/q$, with p, q polynomials, and $\deg(p) < \deg(q)$.

2. Characterizing rational summability

We define the *forward shift operator*, denoted by σ. That is,

$$(\sigma y)(n) = y(n+1).$$

Definition 2. *Let f be a polynomial. The dispersion of f is*

$$\begin{aligned} \operatorname{disp} f &= \max\{|l| : \ l \in \mathbf{Z} \ , \ \gcd(f, \sigma^l f) \neq 1\} \\ &= \max\{|a-b| \ : \ a, b \in \overline{k} \text{ are roots of } f \text{ and } a-b \in \mathbf{Z}\} \end{aligned}$$

where $\overline{k}$ is an algebraic closure of k. If $\operatorname{disp} f = 0$*, then we say that f is shift-free.*

The concept of dispersion was first introduced by Abramov ([1]).

Lemma 3. *Let $y \in k(x)$ and $0 \neq \Delta y = \frac{P}{Q}$, with P and Q not necessarily coprime polynomials. Then* $\operatorname{disp} Q > 0$.

Proof. Say $y = f/g$ in lowest terms (that is, f and g are relatively prime polynomials). Let a be a root of g (in an algebraic closure of k), and t a nonnegative integer such that $g(a-t) = 0$, but $g(a+1)$ and $g(a-t-1)$ are nonzero. We claim that $Q(a) = Q(a-t-1) = 0$. Thus $\operatorname{disp} Q \geq t+1 > 0$. To see this, look at

$$\frac{P}{Q} = \Delta \frac{f}{g} = \frac{g(x)f(x+1) - g(x+1)f(x)}{g(x)g(x+1)}$$

The numbers a and $a-t-1$ are roots of $g(x)g(x+1)$, but not of $g(x)f(x+1) - g(x+1)f(x)$, so the factors $(x-a)$, $(x-a+t+1)$ of $g(x)g(x+1)$ could not have been cancelled. □

Now we investigate the rational summability of functions of the form

$$r = \sum_{i=0}^{l-1} \frac{p_i}{\sigma^{-i} q} \ ; \ \ p_i, \, q \in k[x]$$

where the numerators and denominators are not necessarily coprime.

Lemma 4. *Let $r = \sum_{i=0}^{l-1} p_i/\sigma^{-i}q$. Consider the rational function*

$$r^* = \sum_{i=0}^{l-1} \frac{\sigma^i p_i}{q} = \frac{\sum_{i=0}^{l-1} \sigma^i p_i}{q}.$$

Then r is rationally summable if and only if r^* is rationally summable.

Proof. First notice that $r - \sigma^{-l}r^*$ is rationally summable. To see this write:

$$r - \sigma^{-l}r^* = \sum_{i=0}^{l-1} \left(\frac{p_i}{\sigma^{-i}q} - \sigma^{i-l} \left(\frac{p_i}{\sigma^{-i}q} \right) \right).$$

Call $s_i = p_i/\sigma^{-i}q$. The telescoping identity

$$s_i - \sigma^{i-l}s_i = \sum_{j=0}^{l-i-1} (\sigma^{-j}s_i - \sigma^{-j-1}s_i) = \nabla \left(\sum_{j=0}^{l-i-1} \sigma^{-j}s_i \right)$$

implies that each summand of $r - \sigma^{-l}r^*$ is rationally summable, so that $r - \sigma^{-l}r^*$ is rationally summable as well.

Pick a rational function s such that $r - \sigma^{-l}r^* = \Delta s$. If r is rationally summable, then there exists a rational function y with $\Delta y = r$. This means that $\Delta(y - s) = \sigma^{-l}r^*$, and hence $r^* = \Delta\sigma^l(y - s)$. Conversely, if r^* is rationally summable, then there exists a rational function u such that $\Delta u = r^*$. Hence $r = \Delta(s - \sigma^{-l}u)$. □

Proposition 5. *Let $r = \sum_{i=0}^{l-1} p_i/\sigma^{-i}q$, and suppose q is shift-free. Then r is rationally summable if and only if $p_0 + \sigma p_1 + \cdots + \sigma^{l-1}p_{l-1} = 0$. In that case, the rational function*

$$y := \sum_{i=0}^{l-2} \frac{\sum_{j=0}^{i} \sigma^{j-(i+1)} p_j}{\sigma^{-(i+1)}q}$$

satisfies $\Delta y = r$.

Proof. Assume that r is rationally summable. By Lemma 4, so is r^*, whose denominator is shift-free. From Lemma 3 we conclude that $r^* = 0$, so that $\sum_{i=0}^{l-1} \sigma^i p_i = 0$. The converse follows by direct verification of the formula for y. □

Notice that if we do not assume q to be shift-free, the previous condition is still sufficient, but no longer necessary, as the following example shows.

Example 6. Let

$$r(x) = \frac{-(x+1)(x-2)}{(x+1)x(x-1)(x-2)} + \frac{-3}{x(x-1)(x-2)(x-3)}.$$

The condition of Proposition 5 is not satisfied; however,

$$r = \Delta \left(\frac{1}{x(x-1)(x-2)} + \frac{1}{x(x-1)} \right).$$

The goal now is to break up any given $r \in k(x)$ into summands of the form of Proposition 5. If f is a polynomial, we denote:

$$[f]^{\underline{l}} = f\,(\sigma^{-1}f)\cdots(\sigma^{-(l-1)}f).$$

Definition 7. *Two polynomials f and g are called shift-coprime if*

$$\gcd(\sigma^i f, g) = 1 \;\; \forall i \in \mathbf{Z}.$$

Take $r = p/q$ written in lowest terms, with q monic. We want to have control over the factors of q that differ by an integer. To achieve that, we are going to replace q by a polynomial $Q = [Q_1]^{\underline{1}} \cdots [Q_l]^{\underline{l}}$ such that:

1. q divides Q.
2. The polynomials Q_i are all shift-free and pairwise shift-coprime.

Following Peter Paule, such a Q is called a *shift-saturated* multiple of q. For any polynomial, there is a unique shift-saturated multiple of minimal degree. An algorithm for calculating it was proposed in [2]. Unfortunately, the *RISC* report is not available, and the algorithm does not appear in the published version ([3]). Paule's algorithm has been implemented in Mathematica by Christian Mallinger, and it is from the comments in the code (kindly sent to me by Peter Paule) that the reference to [2] was obtained. Another algorithm can be found in [4].

Example 8. Let

$$q = x(x-10)^{15}(x-17)(x-25)^2(x-1/2)(x-3/2)(x-1/3)(x-4/3)^2(x-1/5).$$

The polynomials

$$Q = [x-1/5][(x-1/2)(x-1/3)^2]^{\underline{2}}[x^{15}]^{\underline{26}}$$

and

$$Q' = [x-1/5][((x-1/2)(x-1/3))^2]^{\underline{5}}[(x-1/7)x^{15}]^{\underline{26}}$$

are shift-saturated multiples of q. Among all such multiples, Q has minimal degree.

Now suppose $r = p/q$. Compute $Q = [Q_1]^{\underline{1}} \cdots [Q_l]^{\underline{l}}$ a shift-saturation of q. Then $r = H/Q$ for $H = pQ/q$. Since the Q_i are pairwise shift-coprime, we can write (using the extended gcd):

$$r = \sum_{i=1}^{l} \frac{h_i}{[Q_i]^{\underline{i}}}$$

for some polynomials h_i.

Proposition 9. *An r as above is rationally summable if and only if all $h_i/[Q_i]^{\underline{i}}$ are rationally summable.*

Proof. The following claim gives a slightly stronger statement than what we need for an induction on l.

Let $R = R_1 + R_2 \in k(x)$; $R_i = p_i/q_i$, with q_1 and q_2 shift coprime. Then R is rationally summable if and only if R_1 and R_2 are rationally summable.

The "if" implication is trivial. For the "only if", we assume that R_i are in lowest terms (cancellation does not effect shift-coprimeness). Let $y = f/g$, with f, g coprime, such that $\Delta y = R$. Then $g = ABC$, where $A = \prod(x-\alpha)^{e_\alpha}$, over the α that are the roots of g (in $\overline{k}$) and of some integer shift of q_1; analogously for B and q_2, and C is the leftover. Notice that A, B, C are polynomials over k, because $A = \gcd(g, ([q_1]^{\underline{t+1}})^N)$, for $t = \operatorname{disp} q_1$, and N a sufficiently large integer. Likewise for B.

As q_1 and q_2 are shift coprime, A, B and C are pairwise relatively prime. Then we can write (using the extended gcd) $y = \frac{A_1}{A} + \frac{B_1}{B} + \frac{C_1}{C}$, for some polynomials A_1, B_1, C_1, and then

$$0 = \Delta y - R = \left(\Delta\frac{A_1}{A} - R_1\right) + \left(\Delta\frac{B_1}{B} - R_2\right) + \Delta\frac{C_1}{C}.$$

So we have three rational functions that add up to zero, and whose denominators are pairwise coprime. Taking into account that the denominators of the first two summands are nonconstant (since the q_i are nonconstant) we see that this cannot happen unless those rational functions are zero. In particular $R_1 = \Delta\frac{A_1}{A}$ and $R_2 = \Delta\frac{B_1}{B}$. □

Theorem 10. *Let $r = \sum_{i=1}^{l} h_i/[Q_i]^{\underline{i}}$ in the form of Proposition 9. In particular, the Q_i are shift free, so that the $\sigma^{-j}Q_i$ are relatively prime. Using the extended* gcd, *write*

$$r = \sum_{i=1}^{l}\sum_{j=0}^{i-1}\frac{p_{i,j}}{\sigma^{-j}Q_i}$$

Then r is rational summable if and only if $p_{i,0} + \sigma p_{i,1} + \cdots + \sigma^{i-1}p_{i,i-1} = 0$ for all i. In that case, the rational function:

$$y := \sum_{i=1}^{l}\sum_{t=0}^{i-2}\frac{\sum_{j=0}^{t}\sigma^{j-(t+1)}p_{i,j}}{\sigma^{-(t+1)}Q_i}$$

satisfies $\Delta y = r$.

Proof. Propositions 5 and 9. □

3. Remarks

• Given $r = p/q$, it is possible (even symbolically) to find a shift free multiple of q, so that multiplying and dividing by an adequate polynomial, Proposition 5 applies. However, the denominator in such a representation of r will have much larger degree than the one we get by using Theorem 10.

• If we know the complete partial fraction decomposition of our rational function, the previous results can be restated as follows:

Theorem 11. *Suppose we are given:*

$$r(x) = \sum_{n=1}^{l} \sum_{i=0}^{s} \sum_{j=1}^{e} \frac{A_{n,i,j}}{(x - \alpha_n - i)^j}$$

where $\alpha_n - \alpha_m$ *is not an integer for* $n \neq m$ *and the* $A_{n,i,j}$ *are (not necessarily nonzero) constants. Then* r *is rational summable if and only if* $\sum_{i=0}^{s} A_{n,i,j} = 0$ *for all* j, n. *In that case, the rational function*

$$y(x) = \sum_{n=1}^{l} \sum_{j=1}^{e} \sum_{t=1}^{s} \frac{\sum_{i=0}^{t-1} A_{n,i,j}}{(x - \alpha_n - t)^j}$$

satisfies $\Delta y = r$.

Proof. The difficult implication is a straightforward consequence of Propositions 5 and 9. □

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References

[1] Abramov, S. A.: *The Rational Component of the Solution of a First-Order Linear Recurrence Relation with a Rational Right Side.* Zh. vychisl. Mat. mat. fis. no. 4, **15** (1975), 40–42.

[2] Paule, Peter: *Greatest Factorial Factorization and Symbolic Summation 1.* RISC-Linz Report Series No. 93-02 (1993).

[3] Paule, Peter: *Greatest Factorial Factorization and Symbolic Summation.* J. Symbolic Computation **20** (1995), 235–268.

[4] Pirastu, Roberto: *Algorithmen zur Summation rationaler Funktionen* (in German). Diploma thesis, University of Erlangen-Nürnberg 1992.

[5] Pirastu, Roberto; Strehl, Volker: *Rational Summation and the Gosper-Petkovšek Representation.* J. Symbolic Computation **20** (1995), 617–635.

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