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Immersions with a Parallel Normal Field

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Abstract. In this paper we investigate the restrictions which are placed on the focal set of a submanifold of Euclidean space under the assumption that the submanifold has a parallel normal field.

1. Introduction

Throughout this paper $f : M^m \to \mathbb{R}^{m+k}$ is a C^∞ immersion of a connected C^∞ m-dimensional manifold M without boundary into Euclidean $(m+k)$-space.

The tangent space of M at a point p will be denoted by T_pM. So $df_p : T_pM \to T_{f(p)}(\mathbb{R}^{m+k}) = \{f(p)\} \times \mathbb{R}^{m+k} \equiv \mathbb{R}^{m+k}$ is an injection.

There is a standard inner product $<,>$ on $\mathbb{R}^{m+k}$. The total space of the normal bundle of f is defined by

$$N(f) = \{(p,v) \in M \times \mathbb{R}^{m+k} : \ < v, df_p(X) >= 0 \quad \forall X \in T_pM\} \quad ,$$

and the affine normal k-plane to $f(M)$ at p is $\nu_p(f) = \{f(p) + v : (p,v) \in N(f)\}$. Note that $N(f)$ is an $(m+k)$-dimensional manifold. The endpoint map $\eta(p,v) = f(p) + v$. A point $x \in \mathbb{R}^{m+k}$ is a *focal point* of M with base p if η is singular at $\big(p, x - f(p)\big)$. The focal point has *multiplicity* $\mu > 0$ if $\mathrm{rank}(\mathrm{Jac}\ \eta) = m + k - \mu$ at that point. The set of focal points of f with base p will be denoted by $F_p(f)$. This is an algebraic variety in $\nu_p(f)$. We remark that by [9], $x \in F_p(f)$ if and only if $x \in \nu_p(f)$, $x = f(p) + \frac{1}{\lambda}\xi$ where $\xi = \frac{x - f(p)}{\|x - f(p)\|}$, and λ is an eigenvalue of the shape operator $A_\xi : T_pM \to T_pM$, i.e. λ is a principal curvature of f at p in the normal direction ξ.

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If f has flat normal bundle it is known that for each $p \in M$, $F_p(f)$ is a union of at most m hyperplanes in the normal plane $\nu_p(f)$ [17], [10]. The origin of this fact can be traced back to [18]. By Corollary 1.5 in [14], if there is a unit parallel normal field ξ for f such that the shape operator A_ξ has m distinct eigenvalues, then the normal bundle of f is flat. Also for any $k \geq 2$, if there exists locally an orthonormal set of $(k-1)$ parallel normal fields $\xi_1, \dots \xi_{k-1}$ for f at each point of M, then f has flat normal bundle [17].

In this paper we consider the case where f has a parallel normal field and show that for $k \geq 2$ and for all $p \in M$, each 2-plane in $\nu_p(f)$ which contains the normal line at $f(p)$ in the direction of this parallel normal field intersects $F_p(f)$ in the union of at most m straight lines. We use this to compare critical points of height and distance functions for f with those for parallel immersions to f.

2. Parallel immersions and focal sets

Let $f : M^m \to \mathbb{R}^{m+k}$ be an immersion and assume that there exists a parallel normal field $\xi : M^m \to \mathbb{R}^{m+k}$ for f. The map

$$f_\xi : M^m \to \mathbb{R}^{m+k} \quad \text{is defined by} \quad f_\xi(p) = f(p) + \xi(p).$$

If f_ξ is an immersion, it is called a *parallel immersion* to f and ξ is said to be *immersive*. We state the following important relations between f and f_ξ. Formal proofs can be found in [11], [13].

Theorem 2.1. *Let $f : M^m \to \mathbb{R}^{m+k}$ be an immersion and assume that there exists a parallel normal field for f, $\xi : M^m \to \mathbb{R}^{m+k}$, then*

(i) *f_ξ is an immersion if and only if, for all $p \in M, f_\xi(p) \notin F_p(f)$,*

(ii) *$\nu_p(f_\xi) = \nu_p(f)$ for all $p \in M$,*

(iii) *$x \in \mathbb{R}^{m+k}$ is a focal point of f_ξ with base p if and only if x is a focal point of f with base p, i.e. $F_p(f_\xi) = F_p(f)$ for all $p \in M$.*

When f_ξ is an immersion, the *index* of f_ξ, ind f_ξ is defined to be the total multiplicity of the focal points of f with base p on the line segment between $f(p)$ and $f_\xi(p)$. As observed in [16], [2], it follows from the above theorem that this integer is a constant for all $p \in M$. We also call this integer the index of ξ.

For a given immersion $f : M \to \mathbb{R}^{m+k}$ we may not have a parallel normal field on M, even locally. But we can compose f with an inclusion of $\mathbb{R}^{m+k}$ in $\mathbb{R}^{m+k+1}$ or with an inverse stereographic projection from $\mathbb{R}^{m+k}$ to $\mathbb{S}^{m+k} \subset \mathbb{R}^{m+k+1}$ to get a parallel normal field, namely define

$$\begin{aligned} g &: M \to \mathbb{R}^{m+k+1} && \text{by} \quad g(p) = (f(p), 0) \quad \text{and} \\ \tilde{g} &: M \to \mathbb{S}^{m+k} && \text{by} \quad \tilde{g}(p) = \Big(\frac{2f(p)}{\|f(p)\|^2+1}, \frac{\|f(p)\|^2-1}{\|f(p)\|^2+1}\Big)\,. \end{aligned}$$

Then $\xi = (0, 0, \dots, 0, 1)$ and $\tilde{\xi} = \tilde{g}$ are parallel normal fields for $g, \tilde{g}$ respectively. Now, for $p \in M$, let $Q \subset \nu_p(g)$ and $\tilde{Q} \subset \nu_p(\tilde{g})$ be 2-planes containing $g(p), g_\xi(p)$ and $\tilde{g}(p), \tilde{g}_{\tilde{\xi}}(p)$

respectively. Then one can calculate that $Q \cap F_p(g)$ and $\tilde{Q} \cap F_p(\tilde{g})$ are either empty or a union of at most m lines; in fact $Q \cap F_p(g)$ is either empty or a union of at most m parallel lines and $\tilde{Q} \cap F_p(\tilde{g})$ can only be the union of m lines (counting multiplicities) intersecting at the centre of the unit sphere $\mathbb{S}^{m+k}$.

The aim of this section is to prove the following theorem which is motivated by the above two examples. When there can be no confusion we abbreviate $\nu_p(f), F_p(f)$ to ν_p, F_p.

Theorem 2.2. *Let $f : M^m \to \mathbb{R}^{m+k}$ be an immersion with $k \geq 2$ and assume that there exists a parallel normal field for f, $\xi : M^m \to \mathbb{R}^{m+k}$. For $p \in M$, let $Q \subset \nu_p$ be a 2-subplane containing $f(p)$ and $f_\xi(p)$. Then either $Q \cap F_p = \emptyset$ or $Q \cap F_p$ is the union of at most m straight lines.*

Note that this theorem could be proved from the fact that the shape operator A_ξ commutes with the shape operator for any other normal direction. A more sophisticated method is to use a partial tube as defined in [3] with type fibre a $(k-2)$-sphere with a line as axis. The axis is defined by the given parallel normal field and the methods of [4] show that this partial tube has flat normal bundle. The fibres of this normal bundle can be identified with the 2-planes referred to in the theorem and the lines in these planes together with the axes of the spheres are the focal sets of the partial tube. For more details on this method and a generalisation of Theorem 2.2 see [1]. However here we prefer to use a more direct geometrical approach which throws light on the structure of focal sets, and for this we need the following two propositions.

Proposition 2.3. *Let $f : M^m \to \mathbb{R}^{m+k}$ be an immersion, let $p \in M$ and let Q be an n-subplane of ν_p through $f(p)$. Then $Q \cap F_p$ is isometric to the zero set in $\mathbb{R}^n$ of a polynomial $h \in \mathbb{R}[t_1, \dots, t_n]$ of degree at most m.*

Proof. Let $p \in M$, then without loss of generality we can assume that $f(p)$ is at the origin in $\mathbb{R}^{m+k}$. Let $\{e_1, e_2, \dots, e_n\}$ be an orthonormal basis of Q. Then $y \in Q$ is a focal point of f with base p if and only if

$$y = \sum_{i=1}^{n} t_i e_i \quad \text{and} \quad det\Big(I - \sum_{i=1}^{n} t_i A_i\Big) = 0$$

where $A_i : T_pM \to T_pM$ is the shape operator of f at p in the normal direction e_i . So $Q \cap F_p$ is isometric to the zero set of the polynomial $h(t_1, \dots, t_n) = det\big(I - \sum_{i=1}^{n} t_i A_i\big)$. □

P. J. Ryan proved that given a codimension one immersion of a manifold M with unit parallel normal field ξ, there exist continuous functions $\{\lambda_i : M \to \mathbb{R}\}_{1 \leq i \leq m}$ such that each $\lambda_i(p)$ is an eigenvalue of the shape operator $A_{\xi(p)}$ [12]. By a similar technique we prove that in the higher codimension case, for each $p \in M$, the focal points of f with base p on normal lines through $f(p)$ vary continuously with respect to the normal direction.

Proposition 2.4. *Let $f : M^m \to \mathbb{R}^{m+k}$ be an immersion, let $p \in M$ and let Q be a 2-subplane of ν_p through $f(p)$ with orthonormal basis $\{e_1, e_2\}$. Then there exist continuous functions $\{\lambda_i : \mathbb{S}^1 \to \mathbb{R}\}_{1 \leq i \leq m}$ such that, for all $\theta \in \mathbb{S}^1$, $\lambda_1(\theta), \dots, \lambda_m(\theta)$ are the eigenvalues of the shape operator $A_{\eta(\theta)}$ at p in the direction $\eta(\theta) = \cos\theta e_1 + \sin\theta e_2$, and $\lambda_1(\theta) \geq \lambda_2(\theta) \geq \dots \geq \lambda_m(\theta)$.*

Proof. Let $g(t,\theta) = t^m + a_1(\theta)t^{m-1} + \cdots + a_m(\theta)$ be the characteristic polynomial of $A_{\eta(\theta)}$. Consider $g(z,\theta)$ as a complex polynomial in z.

For $\theta_0 \in \mathbb{S}^1$, let $\{\mu_i(\theta_0)\}_{1\le i\le r}$ be the distinct eigenvalues of $A_{\eta(\theta_0)}$ and let $\{m_i\}_{1\le i\le r}$ be their respective multiplicities. So $\sum_{i=1}^r m_i = m$. Assume $\mu_1(\theta_0) > \mu_2(\theta_0) > \cdots > \mu_r(\theta_0)$.

Let $\epsilon > 0$ be arbitrary, let $\epsilon_0 = \min\big\{\epsilon, \frac{1}{2}, \frac{1}{2}\min\limits_{\substack{1\le i,j\le r\\ i\ne j}} |\mu_i(\theta_0) - \mu_j(\theta_0)|\big\}$ and let $C_i = \{z \in \mathbb{C} : |z - \mu_i(\theta_0)| = \epsilon_0\}$.

Clearly $g(z,\theta_0) \neq 0$ on C_i. Now choose $\delta_0 > 0$ so that if $|\theta - \theta_0| < \delta_0$ then $g(z,\theta) \neq 0$ on C_i. Now

$$m_i = \frac{1}{2\pi i}\int_{C_i} \frac{g'(z,\theta_0)}{g(z,\theta_0)} dz,$$

and

$$\Big|\frac{1}{2\pi i}\int_{C_i}\Big(\frac{g'(z,\theta_0)}{g(z,\theta_0)} - \frac{g'(z,\theta)}{g(z,\theta)}\Big)dz\Big| \le \epsilon_0 \sup_{z\in C_i}\Big|\frac{g'(z,\theta_0)}{g(z,\theta_0)} - \frac{g'(z,\theta)}{g(z,\theta)}\Big|.$$

When $\theta \to \theta_0$, the right hand side converges uniformly to 0 on C_i, thus there exists $\delta < \delta_0$ such that $|\theta - \theta_0| < \delta$ implies

$$\sup_{z\in C_i}\Big|\frac{g'(z,\theta_0)}{g(z,\theta_0)} - \frac{g'(z,\theta)}{g(z,\theta)}\Big| < 1.$$

Hence, as the integral $\frac{1}{2\pi i}\int_{C_i}\frac{g'(z,\theta)}{g(z,\theta)}dz$ is the number of zeros of $g(z,\theta)$ inside C_i, it is integer valued and is equal to m_i.

So $|\theta-\theta_0| < \delta$ implies that $g(z,\theta)$ has m_i roots inside C_i. For each $\theta \in \mathbb{S}^1$, let $\{\lambda_i(\theta)\}_{1\le i\le m}$ be the m eigenvalues of $A_{\eta(\theta)}$ with $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_m$. They are real as $A_{\eta(\theta)}$ is real symmetric, and $\{\lambda_1(\theta_0), \lambda_2(\theta_0), \cdots, \lambda_m(\theta_0)\} = \{\mu_1(\theta_0), \mu_2(\theta_0), \cdots, \mu_r(\theta_0)\}$.

So if $|\theta - \theta_0| < \delta$ then $|\lambda_i(\theta) - \lambda_i(\theta_0)| < \epsilon_0 \le \epsilon$. So each λ_i is continuous. □

Proof of Theorem 2.2. By Proposition 2.3 we can identify Q with $\mathbb{R}^2, f(p)$ with $(0,0)$ and $F_p^Q = F_p \cap Q$ with $\{(x,y) \in \mathbb{R}^2 : h(x,y) = 0\}$ where h is a polynomial in 2 variables of degree $\le m$. By the Unique Factorization Theorem [8] we can write h as

$$h(x,y) = c\prod_{i=1}^r h_i(x,y)^{m_i}$$

where c is a real number, each h_i is an irreducible polynomial in $\mathbb{R}[x,y]$, for $i \ne j$ h_i is not a factor of h_j and $m_1 + m_2 + \cdots + m_r \le m$. Then F_p^Q is the zero set of $\prod\limits_{i=1}^r h_i(x,y)$. We know that for $i \ne j$, h_i and h_j have finitely many intersection points and also each h_i has finitely many singular points [7]. Let W denote the complement in F_p^Q of this finite set of points. Assume that the factors of h are ordered so that $W \cap h_i^{-1}(0) \ne \emptyset$ if and only if $i = 1, \ldots, s$. Then the zero set of $\prod\limits_{i=s+1}^r h_i(x,y)$ is a finite set of points, P say, and for $i = 1, \ldots, s$ the zero set of each h_i is a piecewise smooth curve with finitely many components.

Fix $i \in \{1, \dots, s\}$ and let $(a, b) \in W$ be such that $h_i(a, b) = 0$ and (a, b) is a regular point of h_i. Then, without loss of generality, we can assume $\frac{\partial h_i(a,b)}{\partial y} \neq 0$, and therefore by the Implicit Function Theorem we get a smooth function β such that $h_i\big(x, \beta(x)\big) = 0$ and $\big(x, \beta(x)\big) \in W$ for all $x \in (a - \epsilon, a + \epsilon)$ for some $\epsilon > 0$. Define the smooth curve α by $\alpha(x) = \big(x, \beta(x)\big)$. We will show that the curvature of α, $\kappa(x)$, is zero for all $x \in (a - \epsilon, a + \epsilon)$.

Let ℓ denote the line $\{f(p) + t\xi(p) : t \in \mathbb{R}\}$ and for $x \in (a - \epsilon, a + \epsilon)$ let ℓ_x denote the tangent line to $\alpha(a - \epsilon, a + \epsilon)$ at $\alpha(x)$.

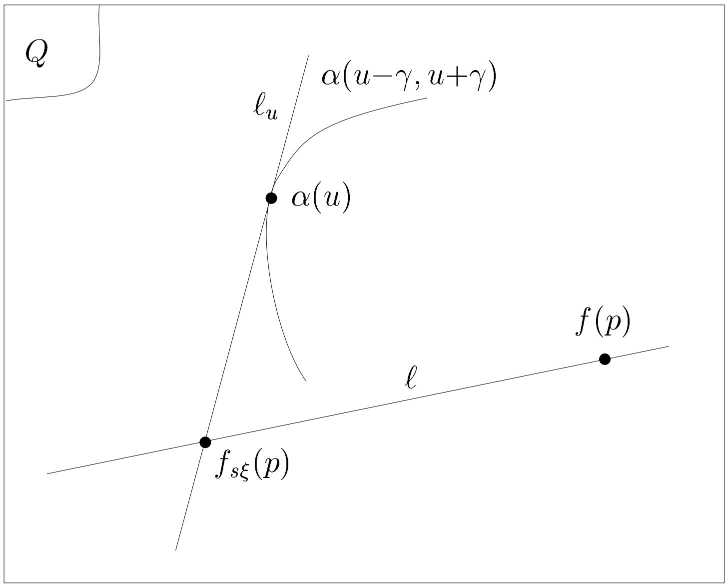

Figure 1.

Suppose $\kappa_\alpha(x_0) \neq 0$ for some $x_0 \in (a - \epsilon, a + \epsilon)$, then there exists $\delta > 0$ such that $\kappa(x) \neq 0$ for all $x \in (x_0 - \delta, x_0 + \delta) \subset (a - \epsilon, a + \epsilon)$. Then an elementary calculation shows that $\{\ell \cap \ell_x : x \in (x_0 - \delta, x_0 + \delta)\}$ contains an open interval. So, as there are only a finite number of focal points with base p on ℓ there exits $u \in (x_0 - \delta, x_0 + \delta)$ such that $\ell \cap \ell_u \neq \emptyset$ and $\ell \cap \ell_u \notin F_p^Q$.

Let $\ell \cap \ell_u = f(p) + s\xi(p) = f_{s\xi}(p)$. Since $f_{s\xi}(p) \notin F_p$ we can use Theorem 2.1 to show that there exists a neighbourhood U of p in M such that $f_{s\xi}|U$ is an immersion and has the same focal set as $f|U$. Thus F_p^Q is the focal set of $f_{s\xi}|U$ at p.

Now as $\kappa(u) \neq 0$ there exists $\gamma > 0$ such that $\alpha(u - \gamma, u + \gamma)$ lies on one side of the tangent line ℓ_u, see Figure 1. But, as $\ell_u \subset Q$, it is a normal line to $f_{s\xi}$ at p. So $\alpha(u) = f_{s\xi}(p) + \frac{1}{\lambda(\theta)}\eta(\theta)$ for some principal curvature function λ of $f_{s\xi}$ as defined in Proposition 2.4, where $\eta(\theta) = \frac{\alpha(u) - f_{s\xi}(p)}{\|\alpha(u) - f_{s\xi}(p)\|}$ determines the unit direction of ℓ_u. But λ is not continuous at θ since the focal points of $f_{s\xi}$ at p do not change continuously as the normal direction changes continuously (there is no focal point near $\alpha(u)$ on normal lines moving in one direction away from ℓ_u). This contradicts Proposition 2.4 for $f_{s\xi}$.

Hence $\kappa(x) = 0$ for all $x \in (a - \epsilon, a + \epsilon)$ and therefore $\alpha(a - \epsilon, a + \epsilon)$ is a straight line segment. Since this is in the zero set of h_i and h_i is irreducible it follows that the zero set of h_i is a straight line, Λ_i say, and degree of $h_i = 1$.

So the focal set F_p^Q consists of the focal lines $\Lambda_1, \dots, \Lambda_s$ with corresponding multiplicities $m_1, \dots, m_s$, together with the finite set of points P. Now we are going to show that $P = \emptyset$.

For $\theta \in \mathbb{R}$ let ℓ_θ denote the line $\{f(p) + t(\cos\theta e_1 + \sin\theta e_2) : t \in \mathbb{R}\}$, where e_1, e_2 is an orthonormal basis of Q and let

$$\Theta = \{\theta \in \mathbb{R} : \ell_\theta \cap P = \emptyset \quad \text{and} \quad \ell_\theta \cap \Lambda_i \neq \emptyset, \ \ \forall i = 1, \dots, s\}.$$

Then $\mathbb{R} \backslash \Theta$ is finite and $\forall \theta \in \Theta$ the total multiplicity of focal points with base p on ℓ_θ is $\sum_{i=1}^{s} m_i$. Hence there is a "focal point at infinity on ℓ_θ" with multiplicity $m - \sum_{i=1}^{s} m_i$, i.e. the shape operator of f at p in direction $\eta(\theta) = \cos\theta e_1 + \sin\theta e_2$ has an eigenvalue 0 with multiplicity $m - \sum_{i=1}^{s} m_i$. It follows from Proposition 2.4 that the line at infinity in Q (extending Q to a projective plane) is a focal line with multiplicity $m - \sum_{i=1}^{s} m_i$, and $\forall \theta \in \mathbb{R}$, the total multiplicity of the eigenvalues of $A_{\eta(\theta)}$ at p arising from the focal lines is m. (Note if $\ell_\theta \cap \Lambda_i = \emptyset$ then the multiplicity m_i transfers to the eigenvalue 0). Hence $P = \emptyset$. Thus if $F_p^Q \neq \emptyset$ it is a union of straight lines. $\square$

Note that we need only the existence of a parallel normal field in a neighbourhood of p to obtain this result.

3. Parallel immersions, distance functions and height functions

In this section we obtain relations between the critical points of distance or height fuctions for f and those for f_ξ. As an application we prove the following theorem, which generalises Theorem 3.6(i) from [2].

Theorem 3.1. *Let $f : M^m \to \mathbb{R}^{m+k}$ be an immersion of a compact manifold, and suppose that there exists an immersive parallel normal field for f with odd index. Then the Euler characteristic $\chi(M) = 0$.*

Definition 3.2. *Let $f : M^m \to \mathbb{R}^{m+k}$ be an immersion, let $\xi : M^m \to \mathbb{R}^{m+k}$ be an immersive parallel normal field for f and let $x \in \mathbb{R}^{m+k}$. The distance functions $L_x : M^m \to \mathbb{R}$, $\hat{L}_x : M^m \to \mathbb{R}$ for f, f_ξ respectively are defined by $L_x(p) = \|x - f(p)\|^2$, $\hat{L}_x(p) = \|x - f_\xi(p)\|^2$ for all $p \in M$.*

Theorem 3.3. *Let $f : M^m \to \mathbb{R}^{m+k}$ be an immersion, let $\xi : M^m \to \mathbb{R}^{m+k}$ be an immersive parallel normal field for f with index λ and let $x \in \mathbb{R}^{m+k}$. Then p is a (nondegenerate)critical point of L_x if and only if p is a (nondegenerate)critical point of $\hat{L}_x$. Further, if p is a nondegenerate critical point of L_x with index μ then $\exists\ \ell \in \mathbb{N}$, $\max\{0, \lambda + \mu - m\} \leq \ell \leq \min\{\lambda, \mu\}$ such that p is a nondegenerate critical point of $\hat{L}_x$ with index $\lambda + \mu - 2\ell$.*

Proof. We know that p is a critical point of L_x if and only if $x \in \nu_p(f)$, and further p is nondegenerate if and only if $x \notin F_p(f)$ [9]. The proposition then follows immediately from Theorem 2.1 which states that $\nu_p(f) = \nu_p(f_\xi)$ and $F_p(f) = F_p(f_\xi)$.

So now assume that p is a nondegenerate critical point of L_x with index μ. We will write $\widehat{ac}$ for the open line segment between points a and c in $\nu_p(f)$ and $\#\widehat{ac}$ for the total multiplicity of focal points with base p on $\widehat{ac}$.

To simplify the notation we will write $a = f(p)$ and $c = f_\xi(p)$. Now $x \in \nu_p(f)$ and $\#\widehat{xa} = \mu$ [9]. Also, since ind f_ξ is λ we have $\#\widehat{ac} = \lambda$.

If x is on the line through $f(p)$ and $f_\xi(p)$, then the index of p as a nondegenerate critical point of $\hat{L}_x$ is

$$\#\widehat{xc} = \begin{cases} \lambda + \mu & \text{if } x \in \{f(p) + t\xi(p) : t < 0\} \\ \lambda - \mu & \text{if } x \in \{f(p) + t\xi(p) : 0 \leq t \leq 1\} \\ \mu - \lambda & \text{if } x \in \{f(p) + t\xi(p) : t > 1\} \end{cases}$$

which corresponds to $l = 0, \mu$ or λ respectively.

Otherwise, if x is not on this line, consider the triangle, $\triangle$, with vertices x, $f(p)$,$f_\xi(p)$ in ν_p as in Figure 2, and let Q be the 2-subplane of ν_p which contains $\triangle$.

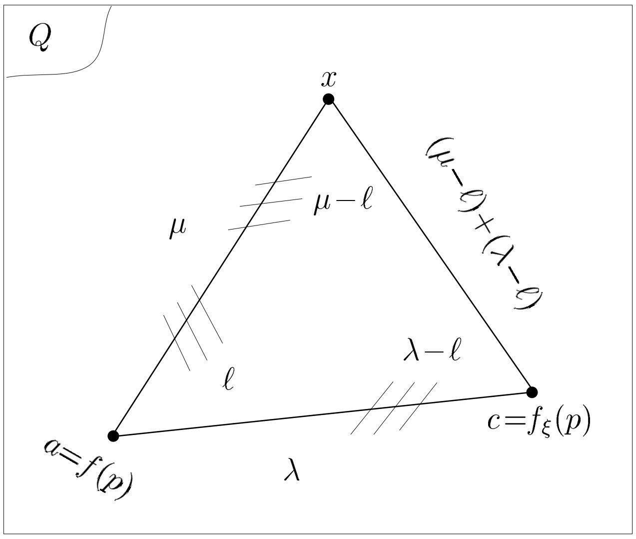

Figure 2.

We know by Theorem 2.2, that $Q \cap F_p$ is a union of at most m lines counting multiplicities. Since none of the vertices of $\triangle$ are in F_p it follows that none of the edges of $\triangle$ can be part of a focal line, and therefore any focal line which intersects one edge of the triangle must intersect another edge.

Let ℓ be the total multiplicity of focal lines which intersect both $\widehat{xa}$ and $\widehat{ac}$. So $0 \leq \ell \leq \min\{\lambda, \mu\}$. Then $\mu - \ell$ lines intersect both $\widehat{xa}$ and $\widehat{xc}$, and $\lambda - \ell$ lines intersect both $\widehat{ac}$ and $\widehat{xc}$. Therefore $\#\widehat{xc} = (\mu - \ell) + (\lambda - \ell) = \lambda + \mu - 2\ell \leq m$. Hence the index of p as a critical point of $\hat{L}_x$ is $\lambda + \mu - 2\ell$.
Also the total multiplicity of focal lines which intersect $\triangle$ is $\ell + (\lambda - \ell) + (\mu - \ell) = (\lambda + \mu - \ell) \leq m$. Hence $(\lambda + \mu - m) \leq \ell$. $\square$

We next prove corresponding results for height functions.

Definition 3.4. *Let $f : M^m \to \mathbb{R}^{m+k}$ be an immersion, let $\xi : M^m \to \mathbb{R}^{m+k}$ be an immersive parallel normal field for f and let $z \in \mathbb{S}^{m+k-1}$. The height functions $H_z : M^m \to \mathbb{R}$, $\hat{H}_z$:*

$M^m \to \mathbb{R}$ for f, f_ξ respectively are defined by $H_z(p) =< z, f(p) >$, $\hat{H}_z(p) =< z, f_\xi(p) >$ for all $p \in M$.

Theorem 3.5. *Let $f : M^m \to \mathbb{R}^{m+k}$ be an immersion, let $\xi : M^m \to \mathbb{R}^{m+k}$ be an immersive parallel normal field for f with index λ and let $z \in \mathbb{S}^{m+k-1}$. Then p is a (nondegenerate) critical point of H_z if and only if p is a (nondegenerate) critical point of $\hat{H}_z$. Further, if p is a nondegenerate critical point of H_z with index μ then $\exists\ \ell \in \mathbb{N}$ with $\max\{0, \lambda + \mu - m\} \leq \ell \leq \min\{\lambda, \mu\}$ such that p is a nondegenerate critical point of $\hat{H}_z$ with index $\lambda + \mu - 2\ell$.*

Proof. It is known that p is a critical point of H_z if and only if $f(p) + z \in \nu_p(f)$ [6]. This holds if and only if $f(p) + \xi(p) + z = f_\xi(p) + z \in \nu_p(f) = \nu_p(f_\xi)$, that is, if and only if p is a critical point of $\hat{H}_z$.

Now assume that p is a nondegenerate critical point of H_z with index μ. Let $f^-(p) = \{f(p) + tz : t < 0\}$ and $f^+(p) = \{f(p) + tz : t > 0\}$ with corresponding definitions for $f_\xi^-(p)$ and $f_\xi^+(p)$. Then, with the notation used in Theorem 3.3, $f(p) + z \in \nu_p(f)$, $\#f^-(p) = \mu$ and $\#f^+(p) = m - \mu$. Also we have $\#\widehat{ac} = \lambda$ where $a = f(p)$ and $c = f_\xi(p)$.

If $\xi(p) = tz$ for some $t \in \mathbb{R}$, then $\#f_\xi^-(p) + \#f_\xi^+(p) = \#f^-(p) + \#f^+(p) = m$, since $F_p(f_\xi) = F_p(f)$ and the line through $f_\xi(p)$ and $f_\xi(p) + z$ coincides with the line through $f(p)$ and $f(p) + z$. Hence p is a nondegenerate critical point of $\hat{H}_z$ and its index is

$$\#f_\xi^-(p) = \begin{cases} \lambda + \mu & \text{if } t \geq 0 \\ \mu - \lambda & \text{if } t < 0 \end{cases},$$

which correponds to $\ell = 0, \lambda$ in the statement of the theorem.

Otherwise, if $\xi(p) \neq tz$ for all $t \in \mathbb{R}$, let Q be the 2-subplane of ν_p which contains $f(p), f(p) + z$ and $f_\xi(p)$ as in Figure 3.

Since $\#\big(f^-(p) \cup f^+(p)\big) = m$ it follows from Theorem 2.2 that $Q \cap F_p$ is a union of m lines counting multiplicity and each line intersects the line through $f(p)$ and $f(p) + z$ transversally and hence intersects the line through $f_\xi(p)$ and $f_\xi(p)+z$ transversally. Therefore $\#\big(f_\xi^-(p) \cup f_\xi^+(p)\big) = m$ and p is a nondegenerate critical point of $\hat{H}_z$.

Let ℓ be the total multiplicity of focal lines which intersect both $f^-(p)$ and $\widehat{ac}$. So $0 \leq \ell \leq \min\{\lambda, \mu\}$. Then these lines intersect $f_\xi^+(p)$. The remaining $\mu - \lambda$ lines which intersect $f^-(p)$ also intersect $f_\xi^-(p)$. Since λ lines must intersect $\widehat{ac}$ it follows that $\lambda - \ell$ lines intersect $f^+(p)$ and $f_\xi^-(p)$. So $0 \leq \lambda - \ell \leq m - \mu$, which gives $\ell \geq \lambda + \mu - m$.

We then get that $\#f_\xi^-(p) = (\mu - \ell) + (\lambda - \ell) = \lambda + \mu - 2\ell$. Hence the index of p as a critical point of $\hat{H}_z$ is $\lambda + \mu - 2\ell$.

If we start with a nondegenerate critical point p of $\hat{H}_z$, then applying the above argument to f_ξ and $(f_\xi)_{-\xi} = f$ shows that p is a nondegenerate critical point of H_z. □

Proof of Theorem 3.1. We are now assuming that M is compact. There exists $z \in \mathbb{S}^{m+k-1}$ such that H_z is a nondegenerate height function for f. Then by Theorem 3.5, $\hat{H}_z$ is a nondegenerate height function for f_ξ. Let C_μ (resp. D_μ) denote the number of critical points of H_z (resp. $\hat{H}_z$) with index μ. By Theorem 3.5, since λ is odd, p is an even (resp. odd) indexed critical point of H_z if and only if p is an odd (resp. even) indexed critical point of $\hat{H}_z$.

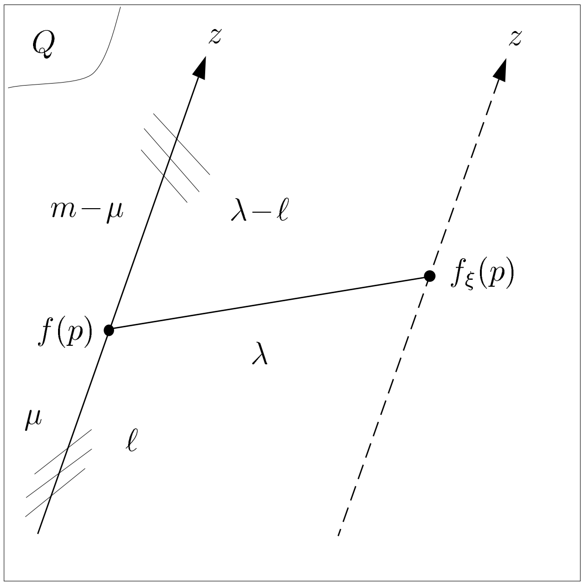

Figure 3.

Therefore

$$\sum_{\mu\ even} C_\mu = \sum_{\mu\ odd} D_\mu \quad \text{and} \quad \sum_{\mu\ odd} C_\mu = \sum_{\mu\ even} D_\mu \ .$$

Now, by the Weak Morse inequalities [9],

$$\begin{aligned} \chi(M) = \sum_{\mu=0}^{m}(-1)^\mu C_\mu &= \sum_{\mu\ even} C_\mu - \sum_{\mu\ odd} C_\mu = \sum_{\mu\ even} D_\mu - \sum_{\mu\ odd} D_\mu \\ &= \sum_{\mu\ odd} C_\mu - \sum_{\mu\ even} C_\mu \quad \text{(from the above relations)} \\ &= -\chi(M)\ . \end{aligned}$$

Hence $\chi(M) = 0$. □

4. Parallel immersions, tightness and tautness

Finally, in this section we draw some conclusions from the hypothesis that a taut or tight immersion has an immersive parallel normal field.

Proposition 4.1. *Let $f : M^m \longrightarrow \mathbb{R}^{m+k}$ be an immersion of a compact manifold and suppose f has an immersive parallel normal field with index λ. Then, $\forall \mu \in \{0, \lambda, m-\lambda, m\}$, any nondegenerate distance or height function for f has a critical point with index μ.*

Proof. Let H_z be a nondegenerate height function for f, for some $z \in \mathbb{S}^{m+k-1}$. Since M is compact, H_z and $\hat{H}_z$ have critical points with indices 0 and m. By Theorem 3.5 applied to f_ξ and $(f_\xi)_{-\xi} = f$, if p is a critical point of $\hat{H}_z$ with index 0 (resp. m) then p is a critical point of H_z with index λ corresponding to $\ell = 0$ (resp. $m - \lambda$ corresponding to $\ell = \lambda$).

Likewise using Theorem 3.3 a similar argument applies to distance functions. □

Corollary 4.2. *Let $f : M^m \longrightarrow \mathbb{R}^{m+k}$ be an immersion of a compact manifold and let $0 < \lambda < m$. If there exists a nondegenerate distance (resp. height) function for f which has no critical point with index λ then there does not exist an immersive parallel normal field for f with index λ nor with index $m - \lambda$.*

Definition 4.3. *The Morse number $\gamma(M)$ of a manifold M is defined by*

$$\gamma(M) = \inf \Big\{ \sum_{\lambda=0}^{m} C_\lambda(\alpha) : \ \alpha \textit{ is a nondegenerate function on } M \Big\}.$$

Then an immersion $f : M^m \longrightarrow \mathbb{R}^{m+k}$ is taut (resp. tight) if every nondegenerate distance (resp. height) function has $\gamma(M)$ critical points.

In particular if the number of critical points of every distance (resp. height) function is equal to the sum of Betti numbers of M with respect to some field then f is taut (resp. tight) by using the Weak Morse Inequalities [9].

Proposition 4.4. *Let $f : M^m \longrightarrow \mathbb{R}^{m+k}$ be an immersion. Suppose that there exists an immersive parallel normal field for f, $\xi : M^m \longrightarrow \mathbb{R}^{m+k}$. Then f is taut (resp. tight) if and only if f_ξ is taut (resp. tight).*

Proof. This is immediate from the Theorems 3.3 and 3.5 since L_x and $\hat{L}_x$ (resp. H_z and $\hat{H}_z$) have the same number of critical points. □

Proposition 4.5. *Let $f : M^m \longrightarrow \mathbb{R}^{m+k}$ be a tight or taut immersion of a compact manifold such that f has an immersive parallel normal field with index λ for some $0 < \lambda < m$. Then $\gamma(M) \geq 4$.*

Proof. This is a straightforward application of Proposition 4.1. Note that if $\lambda = m - \lambda$ then H_z must have 2 critical points with index λ corresponding to an index 0 and an index m critical point of $\hat{H}_z$. Similarly for L_x. □

Finally we consider a $2m$-dimensional, compact, $(m-1)$-connected manifold M. The diffeomorphism types of such taut submanifolds are determined in [15] . Here we use the methods of [5].

Theorem 4.6. *Let $f : M^{2m} \longrightarrow \mathbb{R}^{2m+k}$ be a substantial taut embedding of an $(m-1)$-connected, compact, $2m$-dimensional manifold such that f has a parallel normal field ξ then either $k = 1$ or $f(M) \subset \mathbb{S}^{2m+k-1}$.*

Proof. Suppose that $k > 1$ and let $p \in M$. Since f is substantial every normal line through $f(p)$ meets the focal set F_p [5]. Further as each distance function for f can only have nondegenerate critical points with index $0, m$ or $2m$ it follows that on each normal line through $f(p)$ there can be either one focal point with multiplicity m or $2m$, or two focal points each with multiplicity m. Now let $Q \subset \nu_p$ be a 2-subplane which contains $f(p)$ and $f_\xi(p)$. Then by Theorem 2.2, $Q \cap F_p \subset \nu_p$ is a union of lines, and by the above comments it must be two intersecting lines each with multiplicity m. Hence the point of intersection of the two lines is an umbilic in F_p. Thus f is spherical [5]. □

Theorem 4.7. *Let $f : M^{2m} \longrightarrow \mathbb{R}^{2m+k}$ be a substantial taut embedding of an $(m-1)$-connected (not m-connected) compact manifold such that f has flat normal bundle then $k \leq 2$ and if $k = 1$, f has an immersive parallel normal field with index m.*

Proof. As f has flat normal bundle it follows that for all p in M, F_p is the union of at most $2m$ hyperplanes in ν_p [10].

For $k \geq 2$, F_p must consist of 2 intersecting hyperplanes in ν_p each with multiplicity m otherwise, as in the proof of Theorem 4.6, f would not be substantial or there would exist a distance function for f which has a nondegenerate critical point with index not equal to $0, m$ or $2m$. Further if $k > 2$ there is a line l in the intersection of the two hyperplanes and then the line l' in ν_p through $f(p)$ and parallel to the line l has no focal point on it. This contradicts f is substantial. Hence $k \leq 2$.

Now assume $k = 1$ and let $n : M^{2m} \longrightarrow \mathbb{R}^{2m+1}$ be the inward pointing unit normal. For each $p \in M$, consider the inward pointing normal ray $\nu_p^+ = \{f(p) + tn(p) : t > 0\}$. As in the proof of Theorem 3.8 of [5], $F_p \cap \nu_p^+ \neq \emptyset$ and consists of at most two points each with multiplicity m. Let $f(p) + t_1(p)n(p)$ be the first point of $F_p \cap \nu_p^+$, and, if it exists, let $f(p) + t_2(p)n(p)$ be the second such point, so $t_1(p) < t_2(p)$. Put

$$\begin{aligned} d &= \sup\{t_1(p) : p \in M\} \\ e &= \inf\{t_2(p) : p \in M\} \quad . \end{aligned}$$

It is shown in [5] that $0 < d < e < \infty$. Now choose $c \in \mathbb{R}$ such that $d < c < e$, then for all $p \in M$, $f(p) + cn(p) \notin F_p$. Hence $\xi = cn$ is an immersive parallel normal field with index m. □

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