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Self-similar Simplices

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Abstract. A Euclidean d-simplex $\mathcal{S}$ is called k-self-similar if $\mathcal{S}$ can be dissected into $k \geq 2$ simplices, each similar to $\mathcal{S}$. Each triangle ($d = 2$) is k-self-similar for $k = 4$ and $k \geq 6$ whereas for $d > 2$ most d-simplices are not self-similar. A first class of 3-simplices which are m^3-self-similar for all positive integers m is characterized.

1. Introduction

The concept of self-similarity comes from fractal geometry, cf. [2]. Let φ_i be similarities of the Euclidean d-space $\mathbb{R}^d$, i.e.

$$\bigwedge_{x,y\in\mathbb{R}^d} \Big(|\varphi_i(x) - \varphi_i(y)| = \lambda_i|x - y|\Big)$$

where $0 < \lambda_i < 1$ $(i = 1, \dots, k)$. Then a subset $\mathcal{M}$ of $\mathbb{R}^d$ is called *k-self-similar* $(k \geq 2)$ if $\mathcal{M}$ is invariant under $\varphi_1, \dots, \varphi_k$, i.e. if

$$\mathcal{M} = \bigcup_{i=1}^k \varphi_i(\mathcal{M}).$$

We look for k-self-similar d-simplices. A *d-simplex* $\mathcal{S}$ is the convex hull of $d + 1$ affinely independent points $p_0, \dots, p_d \in \mathbb{R}^d$:

$$\mathcal{S} = \operatorname{conv}\{p_0, \dots, p_d\} := \{x \in \mathbb{R}^d : x = \textstyle\sum_{i=0}^d \lambda_i p_i \wedge \sum_{i=0}^d \lambda_i = 1 \wedge \lambda_i \geq 0\,(i = 0, \dots, d)\}$$

(we don't distinguish points and vectors in notation). Thus, the set $\operatorname{vert}(\mathcal{S}) := \{p_0, \dots, p_d\}$ is the set of vertices of $\mathcal{S}$. Another way of specifying a d-simplex will be useful, namely

$$\mathcal{S} = \langle p_0; a_1, \dots, a_d\rangle = \{x : x = p_0 + \textstyle\sum_1^d \lambda_i a_i \;\wedge 1 \geq \lambda_1 \geq \dots \geq \lambda_d \geq 0\},$$

where $a_i := p_i - p_{i-1}$ $(i = 1, \ldots, d)$ denote the edges (edge-vectors) of a maximal simple edge path beginning in the vertex p_0 (cf. Figure 1).

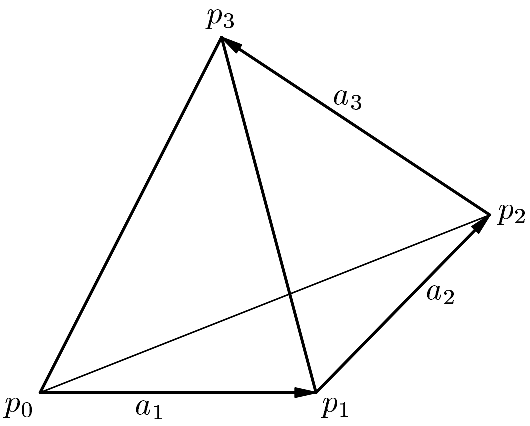


Figure 1

Furthermore, we say a set $\mathcal{S}$ is dissected or $\mathcal{S}$ admits a *dissection* into sets $\mathcal{S}_1, \ldots, \mathcal{S}_k$

$$\mathcal{S} = \sum_{i=1}^{k} \mathcal{S}_i \quad :\Longleftrightarrow \quad \mathcal{S} = \bigcup_{i=1}^{k} \mathcal{S}_i \quad \wedge \quad \mathrm{int}(\mathcal{S}_i \cap \mathcal{S}_l) = \emptyset \ (i \neq l). \tag{1}$$

2. General k-self-similarity

Now, we define the (general) self-similarity of simplices in a slightly more special manner than above:

Definition 1. *A d-simplex $\mathcal{S}$ is called k-self-similar if $\mathcal{S}$ admits a dissection into $k \geq 2$ simplices, each similar to $\mathcal{S}$.*

For $d = 2$ one has a complete classification of self-similar simplices (triangles), cf. [3, 6]:

Proposition 1. a) *Each triangle is k-self-similar with $k = 4$ and $k \geq 6$.*
b) *A triangle $\mathcal{S}$ is 2-self-similar if and only if $\mathcal{S}$ is a right triangle.*
c) *A triangle $\mathcal{S}$ is 3-self-similar if and only if $\mathcal{S}$ is a right triangle.*
d) *A triangle $\mathcal{S}$ is 5-self-similar if and only if $\mathcal{S}$ is a right triangle or $\mathcal{S}$ has angles of size $\frac{2\pi}{3}$ and $\frac{\pi}{6}$.*

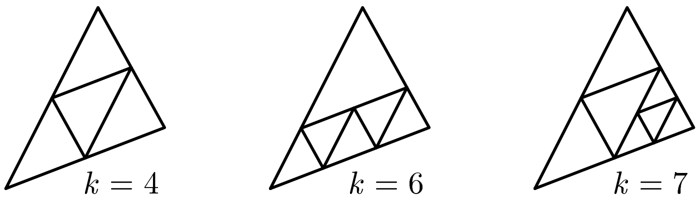


Figure 2

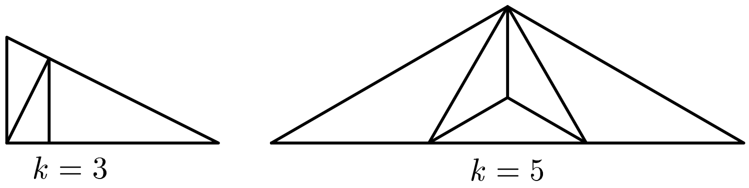


Figure 3

Figures 2 and 3 show the sufficency of the conditions in Proposition 1 (the principles of construction).
Also, in the two-dimensional case each simplex is (general-) self-similar. It's remarkable that the situation for $d > 2$ is quite different:

Lemma 1. *Most d-simplices are not k-self-similar for $d > 2$.*

Indeed, each k-self-similar d-simplex $\mathcal{S}$ admits a dissection (tiling) of the whole space $\mathbb{R}^d$. By a theorem of Debrunner [1], such a simplex $\mathcal{S}$ must be equidissectable to a d-cube. Hence, $\mathcal{S}$ has vanishing Dehn-functionals (Lemma 1 holds for any d-polyhedra with $d \geq 3$).

3. Perfect k-self-similarity

Now we restrict our consideration to a first special case of k-self-similar simplices.

Definition 2. *A simplex $\mathcal{S}$ is said to admit a perfect k-self-similar dissection, or, in short, $\mathcal{S}$ is called k-perfect, if $\mathcal{S}$ admits a dissection* (1) *into $k \geq 2$ simplices $\mathcal{S}_i$ that are mutually incongruent (but similar to $\mathcal{S}$).*

For $d = 2$ one has the following results:

Proposition 2. a) *Each non-equilateral triangle is $2m$-perfect for all $m \geq 4$.*
b) *The equilateral triangle is non-k-perfect for any $k \geq 2$.*
c) *A triangle $\mathcal{S}$ is k-perfect for all $k \geq 2$ if and only if $\mathcal{S}$ is a non-isoceles right triangle.*

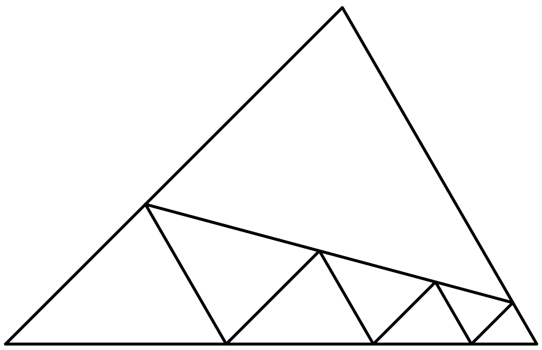


Figure 4

The principle of construction in case a) is shown in Figure 4, cf. [7]. Statement b) is a consequence of the fact that there is no dissection of $\mathbb{R}^2$ into mutually incongruent equilateral

triangles, one of them being minimal [8], cf. also [10]. Concerning statement c) see Figure 3 ($k = 3$).
What happens in the situation of $d > 2$? We have only the following

Conjecture 1. *For $d \geq 3$ there isn't any perfect d-simplex.*

4. Reptiles

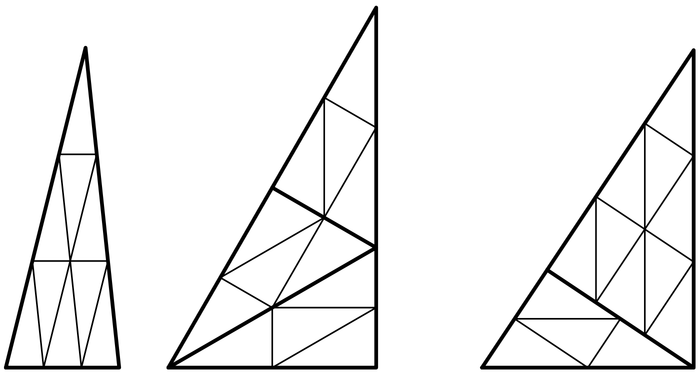
Finally, we consider the following special case of self-similarity:

Definition 3. *A d-simplex $\mathcal{S}$ is called a replicating tile, or, in short, $\mathcal{S}$ is called a k-reptile if $\mathcal{S}$ admits a dissection* (1) *into $k \geq 2$ simplices $\mathcal{S}_i$ that are mutually congruent (and similar to $\mathcal{S}$).*

For $d = 2$ the k-reptiles (triangles) are well known, cf. [9]:

Proposition 3. *A triangle $\mathcal{S}$ is a k-reptile if and only if*
a) $k = m^2$ ($m \geq 2$; *any triangle), or*
b) $k = 3m^2$ ($m \geq 1$; *$\mathcal{S}$ is a right triangle with acute angles $\frac{\pi}{3}$ and $\frac{\pi}{6}$), or*
c) $k = m^2 + l^2$ ($m, l \geq 1$; *$\mathcal{S}$ is a right triangle with cathetuses in the length ratio $m : l$.*

Examples for the three cases are shown in Figure 5.



$k = m^2$ $k = 3 \cdot m^2$ $k = m^2 + l^2$

Figure 5

Thus, each triangle $\mathcal{S}$ is a k-reptile (with $k = m^2$). The corresponding dissection should be called standard: Divide each side (edge) of $\mathcal{S}$ by $m - 1$ points into m parts of equal length. Then dissect $\mathcal{S}$ by straight lines through these points parallel to the sides of $\mathcal{S}$.

The situation for dimensions $d \geq 3$ is rather more difficult (see Lemma 1). But we can apply the above standard dissection to a 3-simplex: Divide each edge $p_i p_k$ of the tetrahedron

$$\mathcal{S} = \operatorname{conv}\{p_0, p_1, p_2, p_3\}$$

into congruent parts and dissect $\mathcal{S}$ by planes through these points parallel to the facets of $\mathcal{S}$. If we assume that $\mathcal{S}$ can be dissected in this way (on the analogy of proposition 3 a) into

m^3 congruent tetrahedra, each similar to $\mathcal{S}$, then $\mathcal{S}$ also admits such a dissection into $8 = 2^3$ tetrahedra. Let m_i be the midpoints of the edges of $\mathcal{S}$ $(i = 1, \ldots, 6)$, cf. Figure 6.

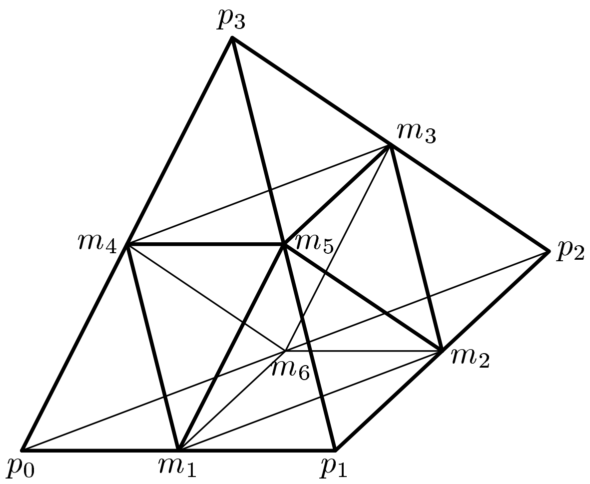


Figure 6

Thus, $\mathcal{S}$ is dissected into the four tetrahedra

$$\mathcal{S}_0 := \operatorname{conv}\{p_0, m_1, m_4, m_6\}, \quad \mathcal{S}_1 := \operatorname{conv}\{m_1, p_1, m_2, m_5\},$$

$$\mathcal{S}_2 := \operatorname{conv}\{m_2, p_2, m_3, m_6\}, \quad \mathcal{S}_3 := \operatorname{conv}\{m_3, p_3, m_4, m_5\}$$

and the "middle octahedron" $\mathcal{O} := \operatorname{conv}\{m_1, \ldots, m_6\}$. Obviously, the middle octahedron of any tetrahedron is centrally symmetric. We need the following

Lemma 2. *If a centrally symmetric octahedron $\mathcal{O}$ is divided into four tetrahedra*

$$\mathcal{O} = \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4$$

then two of them form a quadrangular pyramid, and hence the others do as well.

Proof. Each edge of $\mathcal{S}_i$ is either an edge of $\mathcal{O}$ or its relative interior is in the interior of $\mathcal{O}$. Hence, each triangular facet of $\mathcal{O}$ is an "outer" facet of exactly one of the simplices $\mathcal{S}_i$. We consider any vertex p of $\mathcal{O} = \operatorname{conv}\{a, b, c, d, p, q\}$. At p there meet outer facets of a) four, b) three, or c) two tetrahedra.

In case a) the four simplices must be

$$\mathcal{S}_1 = \operatorname{conv}\{a, b, p, q\}, \quad \mathcal{S}_2 = \operatorname{conv}\{b, c, p, q\},$$

$$\mathcal{S}_3 = \operatorname{conv}\{c, d, p, q\} \quad \text{and} \quad \mathcal{S}_4 = \operatorname{conv}\{d, a, p, q\}.$$

Hence $\mathcal{S}_1 + \mathcal{S}_2$ is the pyramid $\mathcal{P} = \operatorname{conv}\{a, p, c, q, b\}$ with the parallelogram $apcq$ as base.

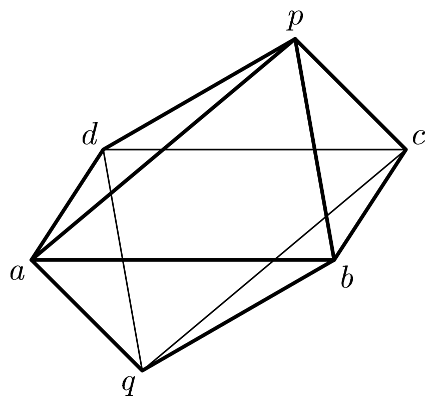


Figure 7

In b), for example, $\mathcal{S}_1$ has facets abp and bcp while cdp is a facet of $\mathcal{S}_2$ and dap is a facet of $\mathcal{S}_3$. Then $\mathcal{S}_2 + \mathcal{S}_3$ must form the pyramid $apcqd$ with the base $apcq$.
In c), let abp and bcp be facets of $\mathcal{S}_1$ and cdp and dap facets of $\mathcal{S}_2$. Then $\mathcal{S}_1 + \mathcal{S}_2$ is the pyramid $abcdp$ with the base $abcd$ which completes the proof.

Now we go back to the standard dissection of the tetrahedron $\mathcal{S}$ into four tetrahedra $\mathcal{S}_i$ and the middle octahedron $\mathcal{O}$. If $\mathcal{S}$ is an 8-reptile then $\mathcal{O}$ must admit a dissection into four mutually congruent tetrahedra

$$\mathcal{O} = \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4,$$

each of them similar to $\mathcal{S}$. Without loss of generality we will assume that $\mathcal{P} = \mathrm{conv}\{m_1, ..., m_5\}$ is the pyramid in accordance with Lemma 2, cf. Figure 6. This pyramid is dissected into two tetrahedra $\mathcal{T}_1$ and $\mathcal{T}_2$, each congruent to the tetrahedron

$$\mathcal{S}_1 = \mathrm{conv}\{m_1, p_1, m_2, m_5\}.$$

Hence, $\mathcal{P} + \mathcal{S}_1 = \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{S}_1$ is a triangular prism that can be dissected into three mutually congruent tetrahedra. Then $\mathcal{S}_1$, and also $\mathcal{S}$, must be a Hill-tetrahedron, cf. [5, 4]. A d-simplex

$$\mathcal{S} = \langle p_0; a_1, a_2, \ldots , a_d\rangle$$

is called a *Hill-simplex* (of the first type) if there exist real numbers $c > 0$ and α $(0 < \alpha < \frac{2\pi}{3})$ with

$$a_i \cdot a_k = \begin{cases} c^2 & \text{for } i = k, \\ c^2 \cos\alpha & \text{for } i \neq k. \end{cases}$$

Therefore, in contrast to the twodimensional case, we have only a very special class of 3-simplices which are m^3-reptiles by the standard construction:

Theorem. *Any 3-simplex $\mathcal{S}$ is an m^3-reptile using the standard dissection if and only if $\mathcal{S}$ is a Hill-simplex.*

Finally, we postulate the following

Conjecture 2. A tetrahedron $\mathcal{S}$ is a k-reptile if and only if $\mathcal{S}$ is a Hill-simplex.

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