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Pairwise Intersections of Słupecki Type Maximal Partial Clones

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Abstract. Let $k \geq 2$ and $\mathbf{k}$ be a k-element set. We study the pairwise intersections of all maximal partial clones of Słupecki type on $\mathbf{k}$. More precisely, we show that with one exception, if M and M' are two strong maximal partial clones of Słupecki type on $\mathbf{k}$, then $M \cap M'$ is covered by both M and M'. We also show that the situation is quite different if the non-strong maximal partial clone on $\mathbf{k}$ is involved in the intersection.

1. Introduction

Let $k \geq 2$ and $\mathbf{k} := \{0, \ldots, k-1\}$. Denote by $\mathrm{Par}(\mathbf{k})$ the set of all partial functions and $\mathrm{Op}(\mathbf{k})$ the set of all (total) functions on $\mathbf{k}$, that is $\mathrm{Op}(\mathbf{k})$ consists of all everywhere defined functions on $\mathbf{k}$. A *partial clone* on $\mathbf{k}$ is a subset of $\mathrm{Par}(\mathbf{k})$ closed under composition and containing all the projections on $\mathbf{k}$. If a partial clone is contained in $\mathrm{Op}(\mathbf{k})$, then it is called a *clone* on $\mathbf{k}$. For example if ρ is any relation on $\mathbf{k}$, then the set $\mathrm{pPol}\,\rho$ ($\mathrm{Pol}\,\rho$) of all partial functions (of all functions) that preserve ρ is a partial clone (a clone) on $\mathbf{k}$ (this and other concepts will be precisely defined in Section 2). The partial clones on $\mathbf{k}$ (the clones on $\mathbf{k}$), ordered by inclusion, form a dually atomic lattice, that is a bounded lattice where every proper partial clone (proper clone) can be extended to a maximal one and contains a minimal one (see e.g., [2], [7] and [15]). The foundation of a partial clone is the set of its unary functions. A

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famous result on clones whose foundation is $\mathrm{Op}^{(1)}(\mathbf{k})$, the set of all unary functions on $\mathbf{k}$, was established by Słupecki in 1939 ([20]). This result is known as the Słupecki criterion and may be formulated as follows. For $3 \leq h \leq k$, let

$$\tau_h := \{(x_1, \ldots, x_h) \in \mathbf{k}^h \mid x_i = x_j \text{ for some } 1 \leq i < j \leq h\}.$$

Then $\mathrm{Pol}\,\tau_k$ is the unique maximal clone that contains the set $\mathrm{Op}^{(1)}(\mathbf{k})$ of all unary functions on $\mathbf{k}$. Motivated by the remarkable Słupecki criterion, we call a partial clone on $\mathbf{k}$ *of Słupecki type* if it contains the set $\mathrm{Op}^{(1)}(\mathbf{k})$ of all unary functions on $\mathbf{k}$.

In [14], A.I. Mal'tsev gives an excellent proof of the Słupecki criterion and moreover improves it by showing that

$$\mathrm{Pol}\,\tau_3 \subset \ldots \subset \mathrm{Pol}\,\tau_k \subset \mathrm{Op}(\mathbf{k})$$

is the unique unrefinable finite chain of clones containing $\mathrm{Pol}\,\tau_3$. Later on Burle in [4] uses Mal'tsev results to show that the interval of all clones of Słupecki type on $\mathbf{k}$ is simply the chain

$$\langle \mathrm{Op}^{(1)}(\mathbf{k})\rangle \subset \mathrm{Pol}\,R_1 \subset \mathrm{Pol}\,\tau_3 \subset \ldots \subset \mathrm{Pol}\,\tau_k \subset \mathrm{Op}(\mathbf{k}),$$

where $\langle \mathrm{Op}^{(1)}(\mathbf{k})\rangle$ is the clone generated by the set $\mathrm{Op}^{(1)}(\mathbf{k})$ and

$$R_1 = \{(x, y, z, t) \in \mathbf{k}^4 \mid [x = y \text{ and } z = t] \text{ or } [x = z \text{ and } y = t] \text{ or } [x = t \text{ and } y = z]\}.$$

As $\mathrm{Op}^{(1)}(\mathbf{k})$ is a finite set, one can easily obtain (e.g., see [3]) that every clone of Słupecki type on $\mathbf{k}$ is finitely generated.

The situation is quite different for the partial case. Denote by M_k the set of all partial functions that are either everywhere or nowhere defined on $\mathbf{k}$. It is shown in [7] that this is the unique non-strong maximal partial clone on $\mathbf{k}$ and it is clearly of Słupecki type. A partial clone is strong if it is closed under formation of suboperations. Now from the description of all maximal partial clones on $\mathbf{k}$ given in [6] and [8], one can easily see that there are k strong maximal partial clones of Słupecki type on $\mathbf{k}$, they consist of $\mathrm{pPol}\,R_1$, $\mathrm{pPol}\,R_2$, $\mathrm{pPol}\,\tau_3, \ldots, \mathrm{pPol}\,\tau_k$, where

$$R_2 = \{(x, y, z, t) \in \mathbf{k}^4 \mid [x = y \text{ and } z = t] \text{ or } [x = t \text{ and } y = z]\}.$$

We point out here that these results were found independently by Romov in [17] and [19]. Moreover it is shown in citeinfinite that M_k is the only finitely generated maximal partial clone of Słupecki type on k. Finally, it is shown in [10] that for $k \geq 2$, the interval of all partial clones of Słupecki type on $\mathbf{k}$ has the cardinality of the continuum.

In this paper, we focus our attention on maximal partial clones of Słupecki type over $\mathbf{k}$. We study the pairwise intersection of such clones and show that with one exception, the intersection of any two strong maximal partial clones of Słupecki type on $\mathbf{k}$ is covered by both these two maximal partial clones. The exception is the case of the two maximal partial clones $\mathrm{pPol}\,R_1$ and $\mathrm{pPol}\,R_2$ since $\mathrm{pPol}\,R_1 \cap \mathrm{pPol}\,R_2$ is not covered by $\mathrm{pPol}\,R_2$. Furthermore, we show that such results do not hold if the non-strong maximal partial clone M_k on $\mathbf{k}$ is considered. Here we show that if M is a strong maximal partial clone of Słupecki type, then the interval of partial clones $[M_k \cap M, M]$ is of continuum cardinality on $\mathbf{k}$. A part of the proof here is based on results established in [1] and [10].

2. Preliminaries

Let $k \geq 2$ be an integer and $\mathbf{k} := \{0, 1, \ldots, k-1\}$. For a positive integer n, an n-ary partial function on $\mathbf{k}$ is a map $f : \text{dom}\,(f) \to \mathbf{k}$ where $\text{dom}\,(f) \subseteq \mathbf{k}^n$ is called the *domain* of f. Let $\text{Par}^{(n)}(\mathbf{k})$ denote the set of all n-ary partial operations on $\mathbf{k}$ and let $\text{Par}(\mathbf{k}) := \bigcup\limits_{n\geq 1} \text{Par}^{(n)}(\mathbf{k})$. Moreover set $\text{Op}^{(n)}(\mathbf{k}) := \{f \in \text{Par}^{(n)}(\mathbf{k}) \mid \text{dom}\,(f) = \mathbf{k}^n\}$ and let $\text{Op}(\mathbf{k}) := \bigcup\limits_{n\geq 1} \text{Op}^{(n)}(\mathbf{k})$, i.e., $\text{Op}(\mathbf{k})$ is the set of all total functions on $\mathbf{k}$. In the sequel we will say "function" for "total function".

For every positive integer n, and every $1 \leq i \leq n$, we denote by e_i^n the n-ary function *i-th projection* defined by $e_i^n(x_1, \ldots, x_n) := x_i$ for all $(x_1, \ldots, x_n) \in \mathbf{k}^n$. Furthermore let

$$J_k := \{e_i^n \mid 1 \leq i \leq n < \infty\}$$

be the set of all projections on $\mathbf{k}$.

Definitions. 1. A *partial clone* on $\mathbf{k}$ is a composition closed subset of $\text{Par}(\mathbf{k})$ containing the set of all projections J_k. For a formal definition, we refer the reader to one of the papers [2], [6] or [8].

2. A partial function $g \in \text{Par}^{(n)}(\mathbf{k})$ is a *subfunction* of $f \in \text{Par}^{(n)}(\mathbf{k})$ (in symbols $g \leq f$ or $g = f|_{\text{dom}\,(g)}$) if $\text{dom}\,(g) \subseteq \text{dom}\,(f)$ and $g(\vec{a}) = f(\vec{a})$ for all $\vec{a} \in \text{dom}\,(g)$. A partial clone C is *strong* if it contains all subfunctions of its partial functions, i.e., if for every $f \in C$ and $g \in \text{Par}(\mathbf{k})$, $g \leq f \Longrightarrow g \in C$. It is well known (e.g., see [3]) that a partial clone C is a strong partial clone if and only if it contains the set of all partial projections on $\mathbf{k}$.

3. Let $h \geq 1$ and ρ be an h-ary relation on $\mathbf{k}$, (i.e., $\rho \subseteq \mathbf{k}^h$), and let f be an n-ary partial function on $\mathbf{k}$. Denote by $\mathcal{M}(\rho, \text{dom}(f))$ $(\rho \neq \emptyset)$ the set of all $h \times n$ matrices M whose columns $M_{*j} \in \rho$, for $j = 1, \ldots, n$ and whose rows $M_{i*} \in \text{dom}\,(f)$ for $i = 1, \ldots, h$. We say that f *preserves* ρ if for every $M \in \mathcal{M}(\rho, \text{dom}(f))$, the h-tuple $f(M) := (f(M_{1*}), \ldots, f(M_{h*})) \in \rho$. Set $\text{pPol}\,\rho := \{f \in \text{Par}(\mathbf{k}) \mid f \text{ preserves } \rho\}$ and $\text{Pol}\,\rho = \text{pPol}\,\rho \cap \text{Op}(\mathbf{k})$ (i.e., $\text{Pol}\,\rho$ is the set of all (total) functions that preserve the relation ρ). It is well known that for every relation ρ, $\text{pPol}\,\rho$ ($\text{Pol}\,\rho$) is a strong partial clone (a clone) on $\mathbf{k}$. Now an h-ary relation ρ is said to be *repetition-free* if for all $0 \leq i < j \leq h-1$, there exists $(a_0, \ldots, a_{h-1}) \in \rho$ with $a_i \neq a_j$. Moreover ρ is said to be *irredundant* if it is repetition-free and has no fictitious components, i.e., there is no $i \in \{0, \ldots, h-1\}$ such that $(a_0, \ldots, a_{h-1}) \in \rho \Longrightarrow (a_0, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_{h-1}) \in \rho$ for all $x \in \mathbf{k}$. It is easy to see that if μ is any relation, then one can find an irredundant relation ρ such that $\text{pPol}\,\mu = \text{pPol}\,\rho$ (see [8] for details).

4. The partial clones on $\mathbf{k}$, ordered by inclusion, form an algebraic lattice ([17]) in which every meet is the set-theoretical intersection. A partial clone C *covers* a partial clone D if $D \subset C$ and the strict inclusions $D \subset C' \subset C$ hold for no partial clone C' on $\mathbf{k}$. Furthermore a partial clone M is a *maximal partial clone* if M is covered by $\text{Par}(\mathbf{k})$. For $F \subseteq \text{Par}(\mathbf{k})$, the partial clone $\langle F \rangle$ generated by F, is the intersection of all partial clones containing the set F.

5. Let $h \geq 2$ and let E_h denote the set of all equivalence relations on the set $\{1, \ldots, h\}$. For $\varepsilon \in E_h$, put

$$\Delta_\varepsilon = \{(x_1, \ldots, x_h) \in \mathbf{k}^h \mid (i,j) \in \varepsilon \Rightarrow x_i = x_j\},$$

thus Δ_ε consists of all h-vectors over $\mathbf{k}$ constant on every equivalence class of ε. An h-ary relation ρ is *diagonal* if there exists $\varepsilon \in E_h$ such that $\rho = \Delta_\varepsilon$. It is well known (see e.g. [8]) that a non-empty relation λ on $\mathbf{k}$ is a diagonal if and only if pPol $(\lambda) = \mathrm{Par}(\mathbf{k})$.

6. A partial clone is called of *Słupecki type* if it contains all unary (total) functions on $\mathbf{k}$ (see [3], [10] and [18]). It is shown in [10], that if λ is an irredundant relation on $\mathbf{k}$, then pPol λ is a Słupecki type partial clone if and only if λ is a union of diagonal relations. Such relations are called *primitive* relations in [8] and [10].

For $k \geq 2$, let $M_k := \mathrm{Op}(\mathbf{k}) \cup \{p_n \mid n \geq 1\}$, where p_n is the n-ary partial function on $\mathbf{k}$ with empty domain. Notice that M_k is a partial clone of Słupecki type of $\mathbf{k}$, however it is not a strong partial clone and cannot be written as pPol ρ, where ρ is some relation over $\mathbf{k}$ (see [7]). We need the following relations. For $3 \leq h \leq k$, let

$$\tau_h := \{(x_1, \ldots, x_h) \in \mathbf{k}^h \mid |\{x_1, \ldots, x_h\}| \leq h-1\},$$

$$R_1 = \{(x,y,z,t) \in \mathbf{k}^4 \mid [x=y \text{ and } z=t] \text{ or } [x=z \text{ and } y=t] \text{ or } [x=t \text{ and } y=z]\},$$

and

$$R_2 = \{(x,y,z,t) \in \mathbf{k}^4 \mid [x=y \text{ and } z=t] \text{ or } [x=t \text{ and } y=z]\}.$$

Recall that an h-ary relation ρ on $\mathbf{k}$ is *totally symmetric* if for every $(x_1, \ldots, x_h) \in \mathbf{k}^h$ and every permutation π on the set $\{1, \ldots, h\}$, we have

$$(x_1, \ldots, x_h) \in \rho \Longleftrightarrow (x_{\pi(1)}, \ldots, x_{\pi(h)}) \in \rho.$$

Notice that R_1, $\tau_3, \ldots, \tau_k$ are totally symmetric relations on $\mathbf{k}$. This fact will be used later on in this paper.

The following result comes from [8] (see also [17]). It gives a description of all maximal partial clones of Słupecki type on $\mathbf{k}$.

Theorem 1. *Let $k \geq 3$. Then there are exactly $k+1$ maximal partial clones of Słupecki type on $\mathbf{k}$, namely M_k,* pPol R_1, pPol R_2, pPol $\tau_3, \ldots,$ pPol τ_k.

We want to study the pairwise intersections of maximal partial clones of Słupecki type over $\mathbf{k}$. We will employ the Definability Lemma shown in [17] (see Lemma 1.7 in [8] for a weak version of it). We need the following terminology to state it:

Let $h_1, \ldots, h_n \geq 1$ be integers and $\varrho_1, \ldots, \varrho_n$ be n relations on a set B, each of arity h_i, $i = 1, \ldots, n$. We say that the family $\mathcal{F} := \{\varrho_1, \ldots, \varrho_n\}$ *covers* the set B if for every $x \in B$, there is an $i \in \{1, \ldots, n\}$ such that x appears in at least one h_i-tuple of the relation ϱ_i.

We have

Lemma 2. [17] (Definability Lemma) *Let $h_1, \ldots, h_n \geq 1$, $t \geq 1$ be integers, ρ_i be an h_i-ary relation on $\mathbf{k}$, $i = 1, \ldots, n$ and λ be a t-ary irredundant relation on $\mathbf{k}$. Then $\bigcap_{i=1}^{n}$* pPol $\rho_i \subseteq$ pPol λ *if and only if there are n auxiliary relations $\varrho_1, \ldots, \varrho_n$ on the set $\{1, \ldots, t\}$, each of arity h_i, such that the family $\mathcal{F} := \{\varrho_1, \ldots, \varrho_n\}$ covers the set $\{1, \ldots, t\}$ and*

$$\lambda = \{(x_1, \ldots, x_t) \in \mathbf{k}^t \mid (x_{i_1^j}, \ldots, x_{i_{h_j}^j}) \in \rho_j \text{ for all } (i_1^j, \ldots, i_{h_j}^j) \in \varrho_j,\ j = 1, \ldots, n\}. \qquad \square$$

Example. Let ρ_1 be a binary and ρ_2 be a ternary relation on $\mathbf{k}$. Suppose that the 5-ary irredundant relation λ is defined by

$$\lambda := \{(x_1, \dots, x_5) \in \mathbf{k}^5 \mid (x_1, x_2) \in \rho_1, (x_3, x_3) \in \rho_1, (x_2, x_5, x_4) \in \rho_2\},$$

then $\mathrm{pPol}\,\rho_1 \cap \mathrm{pPol}\,\rho_2 \subseteq \mathrm{pPol}\,\lambda$. Here $n = 2, j_1 = 2, j_2 = 3$, $\varrho_1 = \{(1,2),(3,3)\}$ and $\varrho_2 = \{(2,5,4)\}$.

We will also employ the following result established by Romov.

Lemma 3. [17]. *Let C be a strong partial clone on $\mathbf{k}$. There exists a non-empty family of irredundant relations $\{\rho_i \mid i \in I\}$ such that $C = \bigcap_{i \in I} \mathrm{pPol}\,\rho_i$.*

Our study is divided into two main parts. In the first part we consider only strong maximal partial clones of Słupecki type, while in the second part the intersection of these partial clones with M_k is considered.

3. Intersections of strong partial clones

Consider two (distinct) maximal partial clones $\mathrm{pPol}\,\rho$ and $\mathrm{pPol}\,\theta$ and suppose that

$$[\mathrm{pPol}\,\rho \cap \mathrm{pPol}\,\theta \subseteq \mathrm{pPol}\,\lambda] \Longrightarrow [\mathrm{pPol}\,\lambda \subseteq \mathrm{pPol}\,\rho \text{ or } \mathrm{pPol}\,\lambda = \mathrm{pPol}\,\theta]$$

holds for every irredundant relation λ. Then the partial clone $\mathrm{pPol}\,\rho \cap \mathrm{pPol}\,\theta$ is covered by the maximal partial clone $\mathrm{pPol}\,\theta$. Indeed let C be a partial clone on $\mathbf{k}$ such that $\mathrm{pPol}\,\rho \cap \mathrm{pPol}\,\theta \subseteq C \subseteq \mathrm{pPol}\,\theta$. As both $\mathrm{pPol}\,\rho$ and $\mathrm{pPol}\,\theta$ contain all partial projections, the same holds for C and so C is a strong partial clone. By Lemma 3, there exists a family of irredundant relations $\{\rho_i \mid i \in I\}$ such that $C = \bigcap_{i \in I} \mathrm{pPol}\,\rho_i$. Then $\mathrm{pPol}\,\rho \cap \mathrm{pPol}\,\theta \subseteq \mathrm{pPol}\,\rho_i$ and by the above property we have $\mathrm{pPol}\,\rho_i \subseteq \mathrm{pPol}\,\rho$ or $\mathrm{pPol}\,\rho_i = \mathrm{pPol}\,\theta$ for each $i \in I$. If $\mathrm{pPol}\,\rho_i = \mathrm{pPol}\,\theta$ holds for all $i \in I$, then $C = \mathrm{pPol}\,\theta$. On the other hand, if $\mathrm{pPol}\,\rho_i \subseteq \mathrm{pPol}\,\rho$ for some $i \in I$, then as $C \subseteq \mathrm{pPol}\,\theta$ we get $C = \bigcap_{i \in I} \mathrm{pPol}\,\rho_i \subseteq \mathrm{pPol}\,\rho \cap \mathrm{pPol}\,\theta$ and consequently $C = \mathrm{pPol}\,\rho \cap \mathrm{pPol}\,\theta$. This shows that

$$[\mathrm{pPol}\,\rho \cap \mathrm{pPol}\,\theta \subseteq C \subseteq \mathrm{pPol}\,\theta] \Longrightarrow [C = \mathrm{pPol}\,\rho \cap \mathrm{pPol}\,\theta \text{ or } C = \mathrm{pPol}\,\theta]$$

holds for every partial clone C on $\mathbf{k}$, i.e., the partial clone $\mathrm{pPol}\,\rho \cap \mathrm{pPol}\,\theta$ is covered by the maximal partial clone $\mathrm{pPol}\,\theta$. This fact will be used later on in this paper.

In the sequel we will assume $k \geq 3$. We start with the following

Theorem 4. *Let $\varrho \in \{R_1, R_2\}$, $\theta \in \{R_1, \tau_3, \dots, \tau_k\}$ with $\rho \neq \theta$. Then the partial clone $\mathrm{pPol}\,\rho \cap \mathrm{pPol}\,\theta$ is covered by the maximal partial clone $\mathrm{pPol}\,\theta$.*

Proof. Let $t \geq 1$ and λ be a t-ary irredundant relation such that

$$\mathrm{pPol}\,\rho \cap \mathrm{pPol}\,\theta \subseteq \mathrm{pPol}\,\lambda.$$

By Lemma 2, there is a 4-ary relation ϱ and an ℓ-ary relation ϑ, (where $\ell = 4$ if $\theta = R_1$ and $\ell = h$ if $\theta = \tau_h$ for some $h = 3, \dots, k$), with $\{\varrho, \vartheta\}$ covering the set $\{1, \dots, t\}$ and such that

$$\lambda = \{(x_1, \dots, x_t) \in \mathbf{k}^t \mid (x_{i_1}, \dots, x_{i_4}) \in \rho \text{ for all } (i_1, \dots, i_4) \in \varrho \text{ and } (x_{j_1}, \dots, x_{j_\ell}) \in \theta \text{ for all } (j_1, \dots, j_\ell) \in \vartheta\}.$$

Note that if $\varrho = \emptyset$, then λ can be defined from θ and by Lemma 2, $\mathrm{pPol}\,\theta \subseteq \mathrm{pPol}\,\lambda$, thus $\mathrm{pPol}\,\theta = \mathrm{pPol}\,\lambda$ by the maximality of $\mathrm{pPol}\,\theta$. So assume $\varrho \neq \emptyset$. We have

Claim 1. *If* $|\{i_1,\dots,i_4\}| \le 3$ *for all* $(i_1,\dots,i_4) \in \varrho$*, then* $\mathrm{pPol}\,\lambda = \mathrm{pPol}\,\theta$.

Proof (of the claim). Let $(i_1,\dots,i_4) \in \varrho$ be such that $|\{i_1,\dots,i_4\}| \le 3$. Since λ is repetition-free, the inequality $|\{i_1,\dots,i_4\}| \le 3$ implies that the condition $(x_{i_1},x_{i_2},x_{i_3},x_{i_4}) \in \rho$ is superfluous. This fact is shown in detail in [8]: Proposition 3.8 for the case $\rho = R_2$ and Proposition 3.10 for $\rho = R_1$. We give a sketch of the proof for the case $\rho = R_1$. Suppose $|\{i_1,\dots,i_4\}| \le 3$. Since R_1 is totally symmetric, we may assume $i_1 = i_2$. If $i_3 \ne i_4$, then the condition $(x_{i_1},x_{i_1},x_{i_3},x_{i_4}) \in R_1$ gives $x_{i_3} = x_{i_4}$ for all $(x_1,\dots,x_t) \in \lambda$, contradicting the fact that λ is an irredundant relation. On the other hand if $i_3 = i_4$, then $(x_{i_1},x_{i_2},x_{i_3},x_{i_4}) \in R_1$ reduces to $(x_{i_1},x_{i_1},x_{i_3},x_{i_3}) \in R_1$, which holds for every $x_{i_1},x_{i_3} \in \mathbf{k}$, i.e., the condition $(x_{i_1},x_{i_2},x_{i_3},x_{i_4}) \in \rho$ is superfluous. Now if $|\{i_1,\dots,i_4\}| \le 3$ holds for all $(i_1,\dots,i_4) \in \varrho$, then λ may be defined from θ and so $\mathrm{pPol}\,\theta \subseteq \mathrm{pPol}\,\lambda$. Again as $\mathrm{pPol}\,\theta$ is a maximal partial clone, we deduce that $\mathrm{pPol}\,\theta = \mathrm{pPol}\,\lambda$. □

We now suppose that the t-ary relation λ satisfies $\mathrm{pPol}\,\rho \cap \mathrm{pPol}\,\theta \subseteq \mathrm{pPol}\,\lambda$ and $\mathrm{pPol}\,\lambda \ne \mathrm{pPol}\,\theta$. According to Claim 1, the 4-ary relation ϱ contains at least one 4-tuple $(i_1,\dots,i_4)$ with $|\{i_1,\dots,i_4\}| = 4$. Thus $t \ge 4$. For notational ease we assume that $(1,2,3,4) \in \varrho$. We have

Claim 2. *Let* $\rho = R_1$*. Then there are two 0-1 t-vectors* $\underset{\sim}{a} = (a_1,\dots,a_t)$ *and* $\underset{\sim}{b} = (b_1,\dots,b_t) \in \lambda$ *such that* $(a_1,\dots,a_4) \ne (b_1,\dots,b_4)$ *and* $(a_1,\dots,a_4),(b_1,\dots,b_4) \in \{(0,0,1,1),$ $(0,1,1,0),(0,1,0,1)\}$.

Proof (of the claim). Since λ is repetition-free, there is a $\underset{\sim}{u} = (u_1,\dots,u_t) \in \lambda$ with $u_1 \ne u_2$. As $(u_1,\dots,u_4) \in R_1$, we have either $[u_1 = u_3$ and $u_2 = u_4]$ or $[u_1 = u_4$ and $u_2 = u_3]$. Define the unary partial function φ by setting $\mathrm{dom}\,(\varphi) = \{u_1,\dots,u_t\}$, $\varphi(u) = 0$ if $u \ne u_2$ and $\varphi(u_2) = 1$. As λ is a primitive relation, $\varphi \in \mathrm{pPol}\,\lambda$ and so $(\varphi(u_1),\dots,\varphi(u_t)) \in \lambda$ and satisfies $(\varphi(u_1),\dots,\varphi(u_4)) \in \{(0,1,0,1),(0,1,1,0)\}$. In a similar fashion, one can use the fact that λ admits two t-vectors $\underset{\sim}{v} = (v_1,\dots,v_t), \underset{\sim}{w} = (w_1,\dots,w_t)$ with $v_1 \ne v_3$ and $w_1 \ne w_4$, to show that there are two 0-1 t-vectors $(x_1,\dots,x_t),(y_1,\dots,y_t) \in \lambda$ such that $(x_1,\dots,x_4) \in \{(0,1,1,0),(0,0,1,1)\}$ and $(y_1,\dots,y_4) \in \{(0,1,0,1),(0,1,0,1)\}$. As the three sets $\{(0,1,0,1),(0,1,1,0)\}$,$\{(0,1,1,0),(0,0,1,1)\}$ and $\{(0,0,1,1),(0,1,0,1)\}$ have no common element, the result follows. □

Using the definition of R_2, we can show in a similar fashion the following

Claim 2'. *Let* $\rho = R_2$*. Then there are two 0-1 t-tuples* $\underset{\sim}{a} = (a_1,\dots,a_t)$ *and* $\underset{\sim}{b} = (b_1,\dots,b_t) \in \lambda$ *such that* $a_1 = a_2 = 0;\ a_3 = a_4 = 1$ *and* $b_1 = b_4 = 0;\ b_2 = b_3 = 1$. □

We turn to the proof of our theorem. We have

Claim 3. $\mathrm{pPol}\,\lambda \subseteq \mathrm{pPol}\,\rho$.

Proof (of the claim). We argue the contrapositive. Suppose $\mathrm{pPol}\,\lambda \nsubseteq \mathrm{pPol}\,\rho$ and let $\varphi \in \mathrm{pPol}\,\lambda \setminus \mathrm{pPol}\,\rho$ be n-ary. Then there is a $4 \times n$ matrix $M = [M_{ij}] \in \mathcal{M}(\rho, \mathrm{dom}\,(\varphi))$ such that $(\varphi(M_{1*}),\varphi(M_{2*}),\varphi(M_{3*}),\varphi(M_{4*})) \notin \rho$. Put $D := \{M_{1*},M_{2*},M_{3*},M_{4*}\}$ and let $\varphi' := \varphi|_D$. As $\mathrm{pPol}\,\lambda$ is a strong partial clone, $\varphi' \in \mathrm{pPol}\,\lambda$. Define n binary partial functions $g_1,\dots,g_n$ by

$$\mathrm{dom}\,(g_1) = \ldots = \mathrm{dom}\,(g_n) = \{(0,0),(0,1),(1,1),(1,0)\},$$

and

$$\begin{pmatrix} g_i(0,\ 0) \\ g_i(0,\ 1) \\ g_i(1,\ 1) \\ g_i(1,\ 0) \end{pmatrix} = \begin{pmatrix} M_{1i} \\ M_{2i} \\ M_{3i} \\ M_{4i} \end{pmatrix} \text{ for all } i = 1, \dots, n.$$

As $(M_{1i}, M_{2i}, M_{3i}, M_{4i}) \in \rho$, we have that $|\{M_{1i}, M_{2i}, M_{3i}, M_{4i})\}| \leq 2$ (recall that ρ is either the relation R_1 or R_2). Consequently $|\mathrm{Im}\ (g_i)| \leq 2$ and so $g_i \in \mathrm{pPol}\ \tau_h$ for all $i = 1, \dots, n$, and all $h = 3, \dots, k$. We treat the cases $\rho = R_1$ and $\rho = R_2$ separately.

Case 1. $\rho = R_1$. In this case $\theta \in \{\tau_3, \dots, \tau_k\}$. First note that R_1 is a *totally symmetric* relation, that is

$$(x_1, \dots, x_4) \in R_1 \Longleftrightarrow (x_{\pi(1)}, \dots, x_{\pi(4)}) \in R_1,$$

for every permutation π on the set $\{1, \dots, 4\}$. Since $(M_{1i}, M_{2i}, M_{3i}, M_{4i}) \in R_1$ and R_1 is totally symmetric, we see that $g_i \in \mathrm{pPol}\ R_1$ for all $i = 1, \dots, n$. Consequently $g_i \in \mathrm{pPol}\ \rho \cap \mathrm{pPol}\ \theta$, and thus $g_i \in \mathrm{pPol}\ \lambda$, for all $i = 1, \dots, n$. It follows that the binary partial function $\overline{\varphi} := \varphi'[g_1, \dots, g_n]$ satisfies $\overline{\varphi} \in \mathrm{pPol}\ \lambda$. Note that $\mathrm{dom}\ (\overline{\varphi}) = \{(0,0), (0,1), (1,1), (1,0)\}$ and $\overline{\varphi}(0,0) = \varphi(M_{1*})$, $\overline{\varphi}(0,1) = \varphi(M_{2*})$, $\overline{\varphi}(1,1) = \varphi(M_{3*})$ and $\overline{\varphi}(1,0) = \varphi(M_{4*})$. Denote by $N := [\underset{\sim}{a} \mid \underset{\sim}{b}]$ the 0-1 matrix of size $t \times 2$ with first column $N_{*1} = \underset{\sim}{a}$ and second column $N_{*2} = \underset{\sim}{b}$, where $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are constructed in Claim 2. Since $\mathrm{dom}\ (\overline{\varphi}) = \{(0,0), (0,1), (1,1), (1,0)\}$, all rows of the matrix N belong to the domain of $\overline{\varphi}$. Moreover as $\underset{\sim}{a}, \underset{\sim}{b} \in \lambda$, we have $N \in \mathcal{M}(\lambda, \mathrm{dom}\ (\overline{\varphi}))$. Now $(\overline{\varphi}(N_{1*}), \dots, \overline{\varphi}(N_{4*})) = (\varphi(M_{\pi(1)*}), \varphi(M_{\pi(2)*}), \varphi(M_{\pi(3)*}), \varphi(M_{\pi(4)*}))$ for some permutation π on the set $\{1, \dots, 4\}$. Since R_1 is a totally symmetric relation and since $(\varphi(M_{1*}), \varphi(M_{2*}), \varphi(M_{3*}), \varphi(M_{4*})) \notin R_1$, we get $(\overline{\varphi}(N_{1*}), \dots, \overline{\varphi}(N_{4*})) \notin R_1$. It follows that $(\overline{\varphi}(N_{1*}), \dots, \overline{\varphi}(N_{t*}) \notin \lambda$ contradicting $\overline{\varphi} \in \mathrm{pPol}\ \lambda$.

Case 2. $\rho = R_2$. In this case $\theta \in \{R_1, \tau_3, \dots, \tau_k\}$. As in Case 1, one may use the symmetries of R_2 and the fact that $(M_{1i}, M_{2i}, M_{3i}, M_{4i}) \in R_2$ to show that $g_i \in \mathrm{pPol}\ R_1 \cap \mathrm{pPol}\ R_2$, for all $i = 1, \dots, n$. Consequently, for $\rho = R_2$ and $\theta \in \{R_1, \tau_3, \dots, \tau_k\}$, we have $g_i \in \mathrm{pPol}\ \lambda$ for all $i = 1, \dots, n$. The rest of the proof is similar to the one given for the Case 1, with the difference that $(\overline{\varphi}(N_{1*}), \dots, \overline{\varphi}(N_{4*})) = (\varphi(M_{1*}), \varphi(M_{2*}), \varphi(M_{3*}), \varphi(M_{4*}))$ in this case. □

From Claim 3 we deduce that for every $\rho \in \{R_1, R_2\}$, $\theta \in \{R_1, \tau_3, \dots, \tau_k\}$ with $\rho \neq \theta$, the inclusions $\mathrm{pPol}\ \rho \cap \mathrm{pPol}\ \theta \subset C \subset \mathrm{pPol}\ \theta$ hold for no partial clone C. This concludes the proof of our theorem. □

Remark. From Theorem 4, we have that the partial clone $\mathrm{pPol}\ R_1 \cap \mathrm{pPol}\ R_2$ is covered by the maximal partial clone $\mathrm{pPol}\ R_1$. One can easily verify that this results holds for $\mathbf{k} = \{0, 1\}$. The proof given above fails for the case $\rho = R_1$ and $\theta = R_2$. Indeed there is no guarantee that the partial functions g_i defined in Claim 3 above preserve the relation R_2. In fact the dual result for R_2 does not hold, indeed there is at least one partial clone that strictly lies between $\mathrm{pPol}\ R_1 \cap \mathrm{pPol}\ R_2$ and $\mathrm{pPol}\ R_2$. To show this, we define the 7-ary relation

$$\lambda := \{(x_1, \dots, x_7) \in \mathbf{k}^7 \mid (x_1, x_2, x_3, x_4) \in R_1, (x_1, x_2, x_5, x_6) \in R_2 \text{ and } (x_2, x_4, x_6, x_7) \in R_2\}.$$

Then from Lemma 2 we have $\mathrm{pPol}\ R_1 \cap \mathrm{pPol}\ R_2 \subseteq \mathrm{pPol}\ \lambda$. Define the ternary partial function φ by setting

$$\mathrm{dom}\ (\varphi) := \{(1,0,0), (0,1,0), (0,0,1), (1,1,1)\},$$

and

$$\begin{pmatrix} \varphi(1,0,0) \\ \varphi(0,1,0) \\ \varphi(0,0,1) \\ \varphi(1,1,1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then clearly $\varphi \notin \mathrm{pPol}\, R_1$. The mathematical software Maple V was used to check that there is no 7×4 matrix A with rows from the set $\{(1,0,0,0),(0,1,0,0),(0,0,1,0),(1,1,1,1)\}$ and such that the first three columns of A are in λ while the fourth is not, thus $\varphi \in \mathrm{pPol}\, \lambda$. Consequently $\mathrm{pPol}\, \lambda \neq \mathrm{pPol}\, R_1$ and in view of the Theorem 4 we have $\mathrm{pPol}\, \lambda \subseteq \mathrm{pPol}\, R_2$. Define now the 4-ary partial function ψ by

dom $(\psi) := \{(0,0,0,0),(1,0,1,0),(0,0,1,1),(0,1,1,0),(1,0,0,1),(1,1,1,1),(1,1,0,0)\}$

and

$$\begin{pmatrix} \psi(1,0,0,1) \\ \psi(0,0,0,0) \\ \psi(1,0,1,0) \\ \psi(0,0,1,1) \\ \psi(0,1,1,0) \\ \psi(1,1,1,1) \\ \psi(1,1,0,0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

From the above matrix we see that $\psi \notin \mathrm{pPol}\, \lambda$ (notice that $(1,0,0,0,0,1,1) \notin \lambda$ since $(1,0,0,0) \notin R_1$). We used Maple V to check that there is no 4×5 matrix with all rows from the set

$$\{(0,0,0,0,0),(1,0,1,0,0),(0,0,1,1,0),(0,1,1,0,0),(1,0,0,1,1),(1,1,1,1,1),(1,1,0,0,1)\}$$

and such that its first 4 columns belong to R_2 while the fifth row does not, thus $\psi \in \mathrm{pPol}\, R_2$. This shows that $\mathrm{pPol}\, R_1 \cap \mathrm{pPol}\, R_2 \subset \mathrm{pPol}\, \lambda \subset \mathrm{pPol}\, R_2$.

We now focus our attention on the family of maximal partial clones $\{\mathrm{pPol}\, \tau_h \mid 3 \leq h \leq k\}$. We have

Theorem 5. *Let $h \in \{4,\ldots,k\}$, $\theta \in \{R_1, R_2, \tau_3, \ldots, \tau_{h-1}\}$. Then the partial clone* $\mathrm{pPol}\, \tau_h \cap \mathrm{pPol}\, \theta$ *is covered by the maximal partial clone* $\mathrm{pPol}\, \theta$. *Moreover for $i = 1, 2$, the partial clone* $\mathrm{pPol}\, \tau_3 \cap \mathrm{pPol}\, R_i$ *is covered by the maximal partial clone* $\mathrm{pPol}\, R_i$.

Proof. Let $3 \leq h \leq k$, $\theta \in \{R_1, R_2, \tau_3, \ldots, \tau_{h-1}\}$ if $h \geq 4$ and $\theta \in \{R_1, R_2\}$ if $h = 3$. Let $\varphi \in \mathrm{pPol}\, \theta \setminus (\mathrm{pPol}\, \tau_h \cap \mathrm{pPol}\, \theta)$ be n-ary. As τ_h and θ are primitive relations, $n \geq 2$. Moreover since $\varphi \notin \mathrm{pPol}\, \tau_h$, there is an $h \times n$ matrix $M = M_{ij} \in \mathcal{M}(\tau_h, \mathrm{dom}(\varphi))$ such that the h-vector $(\varphi(M_{1*}),\ldots,\varphi(M_{h*})) \notin \tau_h$. Thus $|\{\varphi(M_{1*}),\ldots,\varphi(M_{h*})\}| = h$ and so the $\varphi(M_{i*})$ are pairwise distinct. We show that $\langle(\mathrm{pPol}\, \tau_h \cap \mathrm{pPol}\, \theta) \cup \{\varphi\}\rangle = \mathrm{pPol}\, \theta$. Let φ_1 be the n-ary partial function defined by dom $(\varphi_1) = \{M_{1*},\ldots,M_{h*}\}$ and $\varphi_1(M_{j*}) = j-1$, for $j = 1,\ldots,h$. We have

Claim 1. $\varphi_1 \in \langle(\mathrm{pPol}\, \tau_h \cap \mathrm{pPol}\, \theta) \cup \{\varphi\}\rangle$.

Proof (of the claim). Since $\langle(\mathrm{pPol}\,\tau_h \cap \mathrm{pPol}\,\theta) \cup \{\varphi\}\rangle$ is a strong partial clone, it contains the partial function $\varphi' := \varphi|_D$ where $D := \{M_{1*}, \ldots, M_{h*}\}$. Let α be the unary partial function defined by $\mathrm{dom}\,(\alpha) = \{\varphi(M_{1*}), \ldots, \varphi(M_{h*})\}$ and $\alpha(\varphi(M_{j*})) := j-1$ for all $j = 1, \ldots, h$. As $\mathrm{pPol}\,\tau_h \cap \mathrm{pPol}\,\theta$ contains all unary partial functions, we have $\alpha \in \langle(\mathrm{pPol}\,\tau_h \cap \mathrm{pPol}\,\theta) \cup \{\varphi\}\rangle$ and so $\varphi_1 := \alpha[\varphi'] \in \langle(\mathrm{pPol}\,\tau_h \cap \mathrm{pPol}\,\theta) \cup \{\varphi\}\rangle$. □

Claim 2. *The partial clone* $\langle(\mathrm{pPol}\,\tau_h \cap \mathrm{pPol}\,\theta) \cup \{\varphi\}\rangle$ *contains an n-ary partial function* $\overline{\varphi}$ *that satisfies* $\mathrm{dom}\,(\overline{\varphi}) \subseteq \{0, \ldots, h-2\}^n$, $|\mathrm{dom}\,(\overline{\varphi})| = h$ *and* $\mathrm{Im}\,(\overline{\varphi}) = \{0, \ldots, h-1\}$.

Proof (of the claim). Let $j \in \{1, \ldots, h\}$ and $M_{*j} = \{M_{1j}, \ldots, M_{hj}\}$ be the j-th column of the matrix M. Since $M_{*j} \in \tau_h$ we have $|\{M_{1j}, \ldots, M_{hj}\}| \leq h-1$, and so one can define a surjective unary function $\alpha_j : \{0, \ldots, h-2\} \to \{M_{1j}, \ldots, M_{hj}\}$. As $\mathrm{pPol}\,\tau_h$ and $\mathrm{pPol}\,\theta$ are Słupecki type partial clones, we have $\alpha_j \in \mathrm{pPol}\,\tau_h \cap \mathrm{pPol}\,\theta$. Let $B_{ij} \in \{0, \ldots, h-2\}$ be such that $\alpha_j(B_{ij}) = M_{ij}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, h$. Consider now the n-ary partial function $\overline{\varphi} := \varphi_1[a_1, \ldots, \alpha_n]$. Then $\overline{\varphi} \in \langle(\mathrm{pPol}\,\tau_h \cap \mathrm{pPol}\,\theta) \cup \{\varphi\}\rangle$, $\mathrm{dom}\,(\overline{\varphi}) = \{(B_{11}, \ldots, B_{1n}), \ldots, (B_{h1}, \ldots, B_{hn})\}$ and $\mathrm{Im}(\overline{\varphi}) = \mathrm{Im}(\varphi_1) = \{0, 1, \ldots, h-1\}$. □

We turn to the proof of the theorem. For $4 \leq \ell < k$, denote by γ_ℓ the ℓ-ary partial function whose domain consists of the cyclic permutations of the ℓ-vector $(0, 0, 1, \ldots, \ell-2)$, (thus $|\mathrm{dom}\,(\gamma_\ell)| = \ell$), and defined by

$$\gamma_\ell(x_1, \ldots, x_\ell) = \begin{cases} \ell-1, & \text{if } (x_1, \ldots, x_\ell) = (0, 0, 1, \ldots, \ell-2). \\ x_1, & \text{otherwise} \end{cases}$$

Note that $\mathrm{Im}\,(\gamma_\ell) = \{0, \ldots, \ell-2, \ell-1\}$. We have

Claim 3. $\gamma_\ell \in \mathrm{pPol}\,\tau_h \cap \mathrm{pPol}\,R_1 \cap \mathrm{pPol}\,R_2$ *for all* $3 \leq h < \ell \leq k$.

Proof (of the claim). First notice that if $(x_1, \ldots, x_\ell), (y_1, \ldots, y_\ell) \in \mathrm{dom}\,(\gamma_\ell)$ are such that $x_i = y_i \neq 0$ for some $i = 1, \ldots, \ell$, then $(x_1, \ldots, x_\ell) = (y_1, \ldots, y_\ell)$. Moreover as $\ell \geq 4$, any matrix with all rows in $\mathrm{dom}\,(\gamma_\ell)$ contains at least two non-zero entries in each of its rows. Using these facts, one can easily show that if $M \in \mathcal{M}(R_i, \mathrm{dom}(\gamma_\ell))$ is $4 \times \ell$ matrix, then $(\gamma_\ell(M_{1*}), \gamma_\ell(M_{2*})\gamma_\ell(M_{3*})\gamma_\ell(M_{4*})) \in R_i$, for $i = 1, 2$. Consequently $\gamma_\ell \in \mathrm{pPol}\,R_i$ for $i = 1, 2$. We now show that $\gamma_\ell \in \mathrm{pPol}\,\tau_h$ for all $3 \leq h < \ell < k$. Let $B \in \mathcal{M}(\tau_h, \mathrm{dom}(\gamma_\ell))$ be an $h \times \ell$ matrix. Then every column of B has at least two equal entries. If a column contains two equal entries that are not zeros, then by the observation above the two rows containing these entries are equal, and consequently $(\gamma_\ell(B_{1*}), \ldots, \gamma_\ell(B_{h*})) \in \tau_h$. So suppose that every column of B contains at least two zeros. It follows that at least 2ℓ entries of B are zeros. On the other hand, as every ℓ-vector in $\mathrm{dom}\,(\gamma_\ell)$ contains exactly 2 zeros, we have that B contains exactly $2h$ zeros with $h < \ell$, a contradiction. Therefore $\gamma_\ell \in \mathrm{pPol}\,\tau_h$ for every $3 \leq h < \ell \leq k$. □

Set $m := \dfrac{k!\,n}{h!}$. We have

Claim 4. *The partial clone* $\langle(\mathrm{pPol}\,\tau_h \cap \mathrm{pPol}\,\theta) \cup \{\varphi\}\rangle$ *contains an m-ary partial function* ψ *that satisfies* $\mathrm{dom}\,(\psi) \subseteq \{0, \ldots, h-2\}^m$, $|\mathrm{dom}\,(\psi)| = k$ *and* $(\mathrm{Im}\,\psi) = \mathbf{k}$.

Proof (of the claim). For $i \geq 0$ let $m_i := \dfrac{(h+i)!}{h!}n$, thus $m_i = (h+i)m_{i-1}$ for every $i \geq 1$. We construct recursively, for every $0 \leq i \leq k-h$, an m_i-ary partial function ψ_i as follows.

Let $\overline{\varphi}$ be the n-ary partial function constructed in Claim 2 and set $\psi_0 := \overline{\varphi}$. Consider now $1 \leq i \leq k-h$. Define the m_i-ary partial function ψ_i from the (m_{i-1})-ary partial function ψ_{i-1} and the $(h+i)$-ary partial function γ_{h+i} by setting

$(x_1, \ldots, x_{m_i}) \in \mathrm{dom}\,(\psi_i) \Longleftrightarrow$

1) $(x_1, \ldots, x_{m_{i-1}}) \in \mathrm{dom}\,(\psi_{i-1}), \ldots, (x_{(h+i-1)m_{i-1}+1}, \ldots, x_{m_i}) \in \mathrm{dom}\,(\psi_{i-1})$ and

2) $(\psi_{i-1}(x_1, \ldots, x_{m_{i-1}}), \ldots, \psi_{i-1}(x_{(h+i-1)m_{i-1}+1}, \ldots, x_{m_i})) \in \mathrm{dom}\,(\gamma_{h+i})$ and

$\psi_i(x_1, \ldots, x_{m_i}) := \gamma_{h+i}[\psi_{i-1}(x_1, \ldots, x_{m_{i-1}}), \ldots, \psi_{i-1}(x_{(h+i-1)m_{i-1}+1}, \ldots, x_{m_i})]$.

It is easy to verify that $|\mathrm{dom}\,(\psi_i)| = |\mathrm{dom}\,(\psi_{i-1})| + 1$ holds for every $1 \leq i \leq k-h$. As $|\mathrm{dom}\,(\psi_0)| = h$ (recall that $\psi_0 = \overline{\varphi}$ constructed in Claim 2), we have $|\mathrm{dom}\,(\psi_i)| = h + i$ for all $0 \leq i \leq k-h$.

Now from Claim 3 we have $\gamma_\ell \in \mathrm{pPol}\,\tau_h \cap \mathrm{pPol}\,R_1 \cap \mathrm{pPol}\,R_2$ whenever $\ell > h$, and so $\gamma_{h+i} \in \mathrm{pPol}\,\tau_h \cap \mathrm{pPol}\,R_1 \cap \mathrm{pPol}\,R_2$ for every $i \geq 1$. Consequently

$[\psi_i \in \langle(\mathrm{pPol}\,\tau_h \cap \mathrm{pPol}\,\theta) \cup \{\varphi\}\rangle \Longrightarrow \psi_{i+1} \in \langle(\mathrm{pPol}\,\tau_h \cap \mathrm{pPol}\,\theta) \cup \{\varphi\}\rangle]$, for all $i \geq 0$.

As $\psi_0 = \overline{\varphi} \in \langle(\mathrm{pPol}\,\tau_h \cap \mathrm{pPol}\,\theta) \cup \{\varphi\}\rangle$, we have $\psi_{i+1} \in \langle(\mathrm{pPol}\,\tau_h \cap \mathrm{pPol}\,\theta) \cup \{\varphi\}\rangle$ for all $0 \leq i < k-h$.

Furthermore, one can check that for every $0 \leq i < k-h$, the equality $\mathrm{Im}\,(\psi_i) = \mathrm{Im}\,(\gamma_{h+i})$ implies $\mathrm{Im}\,(\psi_{i+1}) = \mathrm{Im}\,(\gamma_{h+i+1})$. This together with $\mathrm{Im}\,(\psi_0) = \mathrm{Im}\,(\gamma_h) = \{0, \ldots, h-1\}$ gives $\mathrm{Im}\,(\psi_{k-h}) = \mathrm{Im}\,(\gamma_k) = \{0, \ldots, k-1\} = \mathbf{k}$. Moreover it is straightforward to verify that $\mathrm{dom}\,(\psi_{k-h}) \subseteq \{0, \ldots, h-2\}^{m_{k-h}}$ and $|\mathrm{dom}\,(\psi_{k-h})| = k$. Put $\psi := \psi_{k-h}$ and note that $m_{k-h} = \dfrac{(h+(k-h))!\,n}{h!} = m$. This completes the proof of Claim 4. □

We now can complete the proof of our theorem. Denote by Ψ the $(k \times m)$ matrix whose rows consist of $\mathrm{dom}\,(\psi)$ and such that $\psi(\Psi_{1*}) = 0, \ldots, \psi(\Psi_{k*}) = k-1$. Note that by construction of the partial function ψ, we have that every entry of the matrix Ψ belongs to the set $\{0, \ldots, h-2\}$. Let $t \geq 2$ and $g \in \mathrm{pPol}\,\theta$ be t-ary. We show that $g \in \langle(\mathrm{pPol}\,\tau_h \cap \mathrm{pPol}\,\theta) \cup \{\varphi\}\rangle$. First define m unary total functions $\alpha_1, \ldots, \alpha_m$ by $a_j(r) := \Psi_{(r+1)j}$, for all $r \in \mathbf{k}$ and all $j = 1, \ldots, m$. Notice that $\mathrm{Im}\,(\alpha_j) = \Psi_{*j}$, the j-th column of the matrix Ψ. As $\mathrm{pPol}\,\theta$ is Słupecki type partial clone, we have $\alpha_j \in \mathrm{pPol}\,\theta$ for every $j = 1, \ldots, m$. Next define m t-ary partial functions $g_1, \ldots, g_m$ by setting $g_j(x_1, \ldots, x_t) := \alpha_j[g(x_1, \ldots, x_t)]$, for all $j = 1, \ldots, m$ and all $(x_1, \ldots, x_t) \in \mathrm{dom}\,(g)$. Since $\alpha_1, \ldots, \alpha_m$ are total functions we have

$$\mathrm{dom}\,(g_1) = \ldots, \mathrm{dom}\,(g_m) = \mathrm{dom}\,(g).$$

Moreover as $g \in \mathrm{pPol}\,\theta$, we have $g_j = \alpha_j[g] \in \mathrm{pPol}\,\theta$ for all $j = 1, \ldots, m$.

As $\mathrm{Im}\,(g_j) \subseteq \mathrm{Im}\,(\alpha_j)$, the partial function g_j takes on values from Ψ_{*j}, the j-th column of the matrix Ψ. Thus $|\mathrm{Im}\,(g_j)| \leq h-1$ and so $g_j \in \mathrm{pPol}\,\tau_h$ for all $j = 1, \ldots, m$. It follows that $g_j \in \mathrm{pPol}\,\tau_h \cap \mathrm{pPol}\,\theta$ for all $j = 1, \ldots, m$, and thus $\psi[g_1, \ldots, g_m] \in \langle(\mathrm{pPol}\,\tau_h \cap \mathrm{pPol}\,\theta) \cup \{\varphi\}\rangle$.

We show that $g = \psi[g_1, \ldots, g_m]$. Clearly $\mathrm{dom}\,(\psi[g_1, \ldots, g_m]) \subseteq \mathrm{dom}\,(g_1) = \mathrm{dom}\,(g)$. Let $\underset{\sim}{x} \in \mathrm{dom}\,(g)$, then $\underset{\sim}{x} \in \bigcap_{j=1}^{n} \mathrm{dom}\,(g_j)$ and $(g_1(\underset{\sim}{x}), \ldots, g_m(\underset{\sim}{x}) = (\alpha_1[g(\underset{\sim}{x})], \ldots, \alpha_m[g(\underset{\sim}{x})])$ which is a row of the matrix Ψ, i.e., $(g_1(\underset{\sim}{x}), \ldots, g_m(\underset{\sim}{x}) \in \mathrm{dom}\,(\psi)$. Now let $\underset{\sim}{x} \in \mathrm{dom}\,(g)$ and let $g(\underset{\sim}{x}) = r \in \mathbf{k}$. Then

$$\begin{aligned}
\psi[g_1, \ldots, g_m](\underset{\sim}{x}) &= \psi[\alpha_1(r)), \ldots, \alpha_m(r)] \\
&= \psi[\Psi_{(r+1)1}, \ldots, \Psi_{(r+1)m}] \text{ (by definition of the functions } \alpha_1, \ldots, \alpha_m) \\
&= r = g(\underset{\sim}{x}) \text{ (by definition of the matrix } \Psi).
\end{aligned}$$

This shows that $g \in \langle(\text{pPol}\,\tau_h \cap \text{pPol}\,\theta) \cup \{\varphi\}\rangle$ and concludes the proof of our theorem. □

Remark. The proof above cannot be used to show that $\text{pPol}\,\tau_h \cap \text{pPol}\,\tau_{h'}$ is covered by $\text{pPol}\,\tau'_h$ if $h < h'$. Indeed take $\theta = \tau_{h'}$ and let i in Claim 4 take the value $h' - h$. But then $\gamma_{h+(h'-h)} \notin \text{pPol}\,\tau_{h'}$ and so $\psi_{h'-h}$ will not necessarily belong to the partial clone $\langle(\text{pPol}\,\tau_h \cap \text{pPol}\,\theta) \cup \{\varphi\}\rangle$. However the result holds, we use Romov's definability Lemma to prove it. We have

Theorem 6. *Let $3 \le h < h' \le k$. Then the partial clone $\text{pPol}\,\tau_h \cap \text{pPol}\,\tau_{h'}$ is covered by the maximal partial clone $\text{pPol}\,\tau_{h'}$.*

Proof. The proof here is similar to that of Theorem 4. Let λ be a t-ary irredundant relation such that

$$\text{pPol}\,\tau_h \cap \text{pPol}\,\tau_{h'} \subseteq \text{pPol}\,\lambda.$$

We show that $[\text{pPol}\,\lambda \neq \text{pPol}\,\tau_{h'} \implies \text{pPol}\,\lambda \subseteq \text{pPol}\,\tau_h]$. By Lemma 2, there is an h-ary relation ϱ and an h'-ary relation ϱ' with $\{\varrho, \varrho'\}$ covering the set $\{1, \dots, t\}$ and such that

$$\lambda = \{(x_1, \dots, x_t) \in \mathbf{k}^t \mid (x_{i_1}, \dots, x_{i_h}) \in \tau_h \text{ for all } (i_1, \dots, i_h) \in \varrho \text{ and } (x_{j_1}, \dots, x_{j_{h'}}) \in \tau_{h'} \text{ for all } (j_1, \dots, j_{h'}) \in \varrho'\}.$$

As in Theorem 4, we may assume that $\varrho \neq \emptyset \neq \varrho'$ (otherwise $\text{pPol}\,\lambda = \text{pPol}\,\tau_h$ or $\text{pPol}\,\lambda = \text{pPol}\,\tau_{h'}$). Moreover we may assume that no condition $(x_{i_1}, \dots, x_{i_h}) \in \tau_h$ or $(x_{j_1}, \dots, x_{j_{h'}}) \in \tau_{h'}$ is superfluous. Thus $|\{i_1, \dots, i_h\}| = h$ for all $(i_1, \dots, i_h) \in \varrho$ and $|\{j_1, \dots, j_{h'}\}| = h'$ for all $(j_1, \dots, j_{h'}) \in \varrho'$. Consequently $t \ge h' > h$. For notational ease let $(1, \dots, h) \in \varrho$. We have

Claim. $(a_1, \dots, a_h, a_1, \dots, a_1) \in \lambda$ *for all* $(a_1, \dots, a_h) \in \tau_h$.

Proof (of the claim). Let $(a_1, \dots, a_h) \in \tau_h$. Then $(a_1, \dots, a_h, a_1, \dots, a_1) \in \lambda$ if and only if for every $(i_1, \dots, i_h) \in \varrho$ and every $(j_1, \dots, j_{h'}) \in \varrho'$, we have $(b_{i_1}, \dots, b_{i_h}) \in \tau_h$ and $(b_{j_1}, \dots, b_{j_{h'}}) \in \tau_{h'}$, where $b_s = a_s$ for $s \in \{1, \dots, h\}$ and $b_s = a_1$ for $s \in (\{i_1, \dots, i_h\} \cup \{j_1, \dots, j_{h'}\}) \setminus \{1, \dots, h\}$.

First we show that $(b_{i_1}, \dots, b_{i_h}) \in \tau_h$ for all $(i_1, \dots, i_h) \in \varrho$. If $\{i_1, \dots, i_h\} = \{1, \dots, h\}$ then $(b_{i_1}, \dots, b_{i_h}) \in \tau_h$ holds trivially. Suppose therefore that $\{i_1, \dots, i_h\} \neq \{1, \dots, h\}$. If $|\{i_1, \dots, i_h\} \cap \{1, \dots, h\}| \le h - 2$, then a_1 appears at least twice in the sequence $b_{i_1}, \dots, b_{i_h}$ and consequently $(b_{i_1}, \dots, b_{i_h}) \in \tau_h$. Suppose now that $|\{i_1, \dots, i_h\} \cap \{1, \dots, h\}| = h - 1$. If $1 \in \{i_1, \dots, i_h\}$, then again a_1 appears at least twice in the sequence $b_{i_1}, \dots, b_{i_h}$ and we are done. So let $1 \notin \{i_1, \dots, i_h\}$. As $|\{i_1, \dots, i_h\} \cap \{1, \dots, h\}| = h - 1$, we may assume $\{i_1, \dots, i_h\} = \{i_1, 2, \dots, h\}$ with $i_1 \notin \{1, \dots, h\}$. Then $b_{i_1} = a_1$ and thus $(b_{i_1}, \dots, b_{i_h}) = (a_1, \dots, a_h) \in \tau_h$.

We now show that $(b_{j_1}, \dots, b_{j_{h'}}) \in \tau_{h'}$ for all $(j_1, \dots, j_{h'}) \in \varrho'$. If $\{1, \dots, h\} \subseteq \{j_1, \dots, j_{h'}\}$, then $|\{b_{j_1}, \dots, b_{j_{h'}}\}| \le h' - 1$, proving $(b_{j_1}, \dots, b_{j_{h'}}) \in \tau_{h'}$. Otherwise $|\{1, \dots, h\} \cap \{j_1, \dots, j_{h'}\}| \le h - 1$, and so a_1 appears at least $h' - (h + 1)$ times (thus at least twice as $h' \ge h + 1$) in the sequence $b_{j_1}, \dots, b_{j_{h'}}$. Again this shows that $(b_{j_1}, \dots, b_{j_{h'}}) \in \tau_{h'}$ and completes the proof of our claim. □

We return to the proof of our theorem. Set

$$\mu := \{(x_1, \dots, x_h) \in \mathbf{k}^h \mid (x_1, \dots, x_h, x_1, \dots, x_1) \in \lambda\},$$

then in view of Lemma 2 we have $\text{pPol}\,\lambda \subseteq \text{pPol}\,\mu$. We show that $\mu = \tau_h$. Let $(x_1, \dots, x_h) \in \tau_h$. Then from the claim above $(x_1, \dots, x_h, x_1, \dots, x_1) \in \lambda$ and so $(x_1, \dots, x_h) \in \mu$. Conversely if $(x_1, \ldots, x_h) \in \mu$, then $(x_1, \ldots, x_h, x_1, \ldots, x_1) \in \lambda$ and in view of $(1, \ldots, h) \in \varrho$ we have $(x_1, \ldots, x_h) \in \tau_h$, proving $\mu = \tau_h$.

We have shown that if λ is an irredundant relation such that $\mathrm{pPol}\,\tau_h \cap \mathrm{pPol}\,\tau_{h'} \subseteq \mathrm{pPol}\,\lambda$, then $[\mathrm{pPol}\,\lambda \neq \mathrm{pPol}\,\tau_{h'} \Longrightarrow \mathrm{pPol}\,\lambda \subseteq \mathrm{pPol}\,\tau_h]$. Consequently

$$\mathrm{pPol}\,\tau_h \cap \mathrm{pPol}\,\tau_{h'} \subset C \subset \mathrm{pPol}\,\tau_{h'}$$

for no partial clone C, hence the partial clone $\mathrm{pPol}\,\tau_h \cap \mathrm{pPol}\,\tau_{h'}$ is covered by the maximal partial clone $\mathrm{pPol}\,\tau_{h'}$. □

Combining Theorems 4, 5 and 6 we get

Corollary 7. *Let $k \geq 2$ and $M \neq M'$ be two strong maximal partial clones of Słupecki type on $\mathbf{k}$. If $(M, M') \neq (\mathrm{pPol}\,R_1, \mathrm{pPol}\,R_2)$, then the partial clone $M \cap M'$ is covered by the maximal partial clone M'.* □

It is shown in [3] that no strong maximal partial clone of Słupecki type on $\mathbf{k}$ is finitely generated (this result was shown earlier in [12] for $k = 2$). As the intersection of two strong maximal partial clones of Słupecki type is covered by (at least) one of them, we get

Corollary 8. *Let $\mathbf{k} \geq 2$ and $M \neq M'$ be two strong maximal partial clones of Słupecki type on $\mathbf{k}$. Then the partial clone $M \cap M'$ is not finitely generated.* □

4. Intersections of Słupecki type partial clones with M_k

In this section, we consider the intervals of partial clones $[M_k \cap \mathrm{pPol}\,\theta, \mathrm{pPol}\,\theta]$ and $[M_k \cap \mathrm{pPol}\,\theta, M_k]$ on $\mathbf{k}$, where $\theta \in \{R_1, R_2, \tau_3, \ldots, \tau_k\}$. First notice that intervals of the type $[M_k \cap \mathrm{pPol}\,\theta, M_k]$ are quite easy to describe. Indeed the equality $\mathrm{Pol}\,R_2 = \langle \mathrm{Op}^{(1)}(\mathbf{k}) \rangle$ follows from Burle's Theorem ([4]) and is shown in [10]. Again from Burle's Theorem we have that the unrefinable chain

$$\begin{aligned}\mathrm{Pol}\,R_2 \cup \{p_n \mid n \geq 1\} \subset \mathrm{Pol}\,R_1 \cup \{p_n \mid n \geq 1\} \subset \mathrm{Pol}\,\tau_3 \cup \{p_n \mid n \geq 1\} \subset \ldots \\ \subset \mathrm{Pol}\,\tau_k \cup \{p_n \mid n \geq 1\} \subset \mathrm{Op}(\mathbf{k}) \cup \{p_n \mid n \geq 1\},\end{aligned}$$

is the interval of partial clones $[\mathrm{Pol}\,R_2 \cup \{p_n \mid n \geq 1\}, \mathrm{Op}(\mathbf{k}) \cup \{p_n \mid n \geq 1\}]$. Thus for $\theta \in \{R_1, R_2, \tau_3, \ldots, \tau_k\}$, the interval $[\mathrm{pPol}\,\theta \cap M_k, M_k]$ is simply a subchain of the unrefinable chain

$$\mathrm{pPol}\,R_2 \cap M_k \subset \mathrm{pPol}\,R_1 \cap M_k \subset \mathrm{pPol}\,\tau_3 \cap M_k \subset \ldots \subset \mathrm{pPol}\,\tau_k \cap M_k \subset M_k.$$

We now focus on intervals of the type $[M_k \cap \mathrm{pPol}\,\theta, \mathrm{pPol}\,\theta]$. We first need to recall the descriptions of the clones $\mathrm{Pol}\,\theta$ for $\theta \in \{R_1, \tau_3, \ldots, \tau_k\}$.

Recall that an n-ary function f is said to be *essentially unary* if there are $1 \leq i \leq n$ and $g \in \mathrm{Op}^{(1)}(\mathbf{k})$ such that $f(x_1, \ldots, x_n) = g(x_i)$ holds for all $(x_1, \ldots, x_n) \in \mathbf{k}^n$. Moreover f is *quasilinear* if there are $\phi_0 : \{0, 1\} \to \mathbf{k}$ and $\phi_i : \mathbf{k} \to \{0, 1\}$ $(i = 1, \ldots, n)$ such that

$$f(x_1, \ldots, x_n) = \phi_0(\phi_1(x_1) + \ldots + \phi_n(x_n))$$

for all $(x_1, \ldots, x_n) \in \mathbf{k}^n$, where the sum is mod 2 on $\{0, 1\}$.

It is well known that $\langle \mathrm{Op}^{(1)}(\mathbf{k}) \rangle$ is the set of all essentially unary functions on $\mathbf{k}$ and $\mathrm{Pol}\,R_1$ is the set of all functions that are essentially unary or quasilinear. Furthermore for $3 \leq h \leq k$, $\mathrm{Pol}\,\tau_h$ is the set of all functions f that are essentially unary or satisfy $|\mathrm{Im}\,f| \leq h-1$ (see e.g., [16] and [21] for details).

We start with the maximal partial clone pPol R_1. We will need the following notation here. If C is a total clone on $\mathbf{k}$, then we denote the *strong closure of* C by Str (C), that is Str $(C) := \{g \in \mathrm{Par}(\mathbf{k}) \mid g \leq f$ for some $f \in C\}$. Notice that for every clone C, Str (C) is a strong partial clone on $\mathbf{k}$. Moreover, if ρ is an irredundant relation on $\mathbf{k}$, then Str (Pol ρ) $\subseteq$ pPol ρ, with the equality holding if and only if every partial function that preserve ρ can be extended to a total function that preserve ρ (see [3] for details). We have

Theorem 9. *Let* $k \geq 2$. *Then the interval of partial clones* [Str (Pol R_1), pPol R_1] *is of continuum cardinality on* $\mathbf{k}$.

Proof. This theorem is shown in [1] for $k = 2$. We use the proof given in [1] to establish the result for $k \geq 3$. In the proof below, we will use the definition of a partial clone involving the three Mal'tsev operations ζ, τ, Δ as well as the composition $\star$ of partial functions (see [8],[9] for partial clones and [15] for the general theory). Moreover, we will denote by Op($\mathbf{2}$) (Par($\mathbf{2}$)) the set of all functions (partial functions) on $\{0,1\}$. Furthermore let $\mathrm{Pol}_2\ R_1 \subset \mathrm{Op}(\mathbf{2})$ ($\mathrm{pPol}_2\ R_1 \subset \mathrm{Par}(\mathbf{2})$) be the clone (the partial clone) of all functions (partial functions) that preserve the relation R_1 on $\{0,1\}$. Let

$$\mathcal{Q} := \{C \subseteq \mathrm{Par}(\mathbf{2}) \mid C \text{ is a partial clone and } \mathrm{Str}\ (\mathrm{Pol}_2\ R_1) \subseteq C \subseteq \mathrm{pPol}_2\ R_1\}.$$

The equality $|\mathcal{Q}| = 2^{\aleph_0}$ is shown in [1]. Our main idea is to construct a one-to-one map from the subset $\mathcal{Q}$ of Par($\mathbf{2}$) to the interval of partial clones [Str (Pol R_1), pPol R_1] on $\mathbf{k}$. For every $C \in \mathcal{Q}$ we define a set $\widehat{C} \subseteq \mathrm{Par}(\mathbf{k})$ as follows:

$$\widehat{C} := \langle \mathrm{Par}^{(1)}(\mathbf{k}) \rangle \quad \cup$$

$$\bigcup_{n\geq 1} \{f \in \mathrm{Par}^{(n)}(\mathbf{k}) \mid \text{ for some } t \geq 1,\ \exists\ F \in C \cap \mathrm{Par}^{\ (t)}(\mathbf{2}),\ \exists \varphi_0, \varphi_1, \ldots, \varphi_t \in \mathrm{Op}^{(1)}(\mathbf{k})\ ,$$
$$\exists \{i_1, i_2, \ldots, i_t\} \subseteq \{1,2,\ldots,n\} \text{ such that the following hold :}$$

$|\mathrm{Im}\ (\varphi_0)| \leq 2,$ and
$\mathrm{Im}\ (\varphi_i) \subseteq \{0,1\}\ \forall i \in \{1,\ldots,t\}$ and,
$f(x_1,\ldots,x_n) = \varphi_0(F(\varphi_1(x_{i_1}), \varphi_2(x_{i_2}), \ldots, \varphi_n(x_{i_t})))$ for all $(x_1,\ldots,x_n) \in \mathrm{dom}\ (f)\}$.

Recall that $\langle \mathrm{Par}^{(1)}(\mathbf{k}) \rangle$ is the partial clone generated by all partial unary functions on $\mathbf{k}$. We show that $\widehat{C}$ is a partial clone on $\mathbf{k}$.

First $\{\zeta(f), \tau(f), \Delta(f)\} \subset \widehat{C}$ for every $f \in \widehat{C}$ follows from $\{\zeta(F), \tau(F), \Delta(F)\} \subset C$ for every $F \in C$. Now let $f \in \widehat{C}$, $g \in \widehat{C} \cap \mathrm{Par}^{(1)}(\mathbf{k})$. Then $g \star f \in \widehat{C}$ is immediate and $f \star g \in \widehat{C}$ follows from $\mathrm{Im}\ (\varphi_1) \subseteq \{0,1\} \Longrightarrow \mathrm{Im}\ (\varphi_1 \star g) \subseteq \{0,1\}$. Let now $f, g \in \widehat{C} \backslash \langle \mathrm{Par}^{(1)}(\mathbf{k}) \rangle$, where f is n-ary and g is m-ary. Then (w.l.o.g.) there are $F, G \in C$, $f_0, \ldots, f_n, g_0, \ldots, g_m \in \mathrm{Op}^{(1)}(\mathbf{k})$ with $|\mathrm{Im}\ (f_0)| \leq 2$, $|\mathrm{Im}\ (g_0)| \leq 2$, $\mathrm{Im}\ (h) \subseteq \{0,1\}$ for all $h \in \{f_1,\ldots,f_n,g_1,\ldots,g_m\}$ and

$$f(x_1,\ldots,x_n) = f_0(F(f_1(x_1), f_2(x_2), \ldots, f_n(x_n)))$$

and

$$g(x_1,\ldots,x_m) = g_0(G(g_1(x_1), g_2(x_2), \ldots, g_m(x_m))).$$

hold for every $(x_1,\ldots,x_n) \in \mathrm{dom}\ (f)$ and every $(x_1,\ldots,x_m) \in \mathrm{dom}\ (g)$. Thus

$$(f \star g)(x_1,\ldots,x_{m+n-1}) = f(g(x_1,\ldots,x_m), x_{m+1}, \ldots, x_{m+n-1}) =$$
$$f_0(F(f_1(g_0(G(g_1(x_1), g_2(x_2), \ldots, g_m(x_m)))), f_2(x_{m+1}), \ldots, f_n(x_{m+n-1}))).$$

Since $\mathrm{Par}^{(1)}(\mathbf{2}) \subset C$ and $G \in C$, there is a function $H \in C$ with

$$H(g_1(x_1), \ldots, g_m(x_m)) = f_1(g_0(G(g_1(x_1), g_2(x_2), \ldots, g_m(x_m)))).$$

Put $T := F \star H$. Then $T \in C$ and from

$$\begin{aligned}&(f \star g)(x_1, \ldots, x_{m+n-1}) = \\ &f_0(F(H(g_1(x_1), \ldots, g_m(x_m)), f_2(x_{m+1}), \ldots, f_n(x_{m+n-1}))) = \\ &f_0(T(g_1(x_1), g_2(x_2), \ldots, g_m(x_m), f_2(x_{m+1}), \ldots, f_n(x_{m+n-1}))),\end{aligned}$$

we get $f \star g \in \widehat{C}$. Hence $\widehat{C}$ is a partial clone on $\mathbf{k}$.

Furthermore, from the characterization of Pol R_1 given above, we see that $\widehat{C} \cap \mathrm{Op}(\mathbf{k}) =$ Pol R_1. Now from

$$\bigcup_{n\geq 1} \{f \in \mathrm{Par}(\mathbf{2}) \mid \exists f_1 \in \widehat{C} \cap \mathrm{Par}^{(n)}(\mathbf{k}) \text{ such that } \forall \vec{a} \in \{0,1\}^n,\ f_1(\vec{a}) = f(\vec{a})\} = C,$$

for all $C \in \mathcal{Q}$, we deduce

$$\forall C, C' \in \mathcal{Q};\ C \neq C' \Longrightarrow \widehat{C} \neq \widehat{C'}.$$

This concludes the proof of our theorem. □

Now as pPol $R_1 \cap \mathrm{Op}(\mathbf{k}) =$ Pol $R_1 \subset$ Str (Pol R_1) and $\{p_n \mid n \geq 1\} \subset$ Str (Pol R_1), we have pPol $R_1 \cap M_k \subset$ Str (Pol R_1) , and so from Theorem 9 we deduce

Corollary 10. *Let $k \geq 2$. The interval of partial clones* [pPol $R_1 \cap M_k$, pPol R_1] *has the cardinality of continuum on* $\mathbf{k}$. □

We now prove a result similar to Theorem 10 for the each member of the family $\{\mathrm{pPol}\ \tau_3, \ldots, \mathrm{pPol}\ \tau_k\}$. We have

Theorem 11. *Let $3 \leq h \leq k$. Then the interval of partial clones* [Str (Pol τ_h), pPol τ_h] *is of continuum cardinality on* $\mathbf{k}$.

Proof. For $n \geq 2$, denote by $\mathcal{F}_{2n+h}$ the family of all (binary) equivalence relations ε on the set $\{1, \ldots, 2n+h\}$ such that either 1) ε has at most $h-1$ equivalence classes or 2) ε has exactly h equivalence classes and these h classes satisfy the following: $h-2$ of them are singleton (thus consist of one element each), one has size 2 and finally one has size $2n$. For example, if $h = 5$ and $n = 3$, then the equivalence ε on the set $\{1, \ldots, 11\}$ with classes $\{1\}$; $\{2\}$; $\{3\}$; $\{4, 5\}$; and $\{6, 7, 8, 9, 10, 11\}$ belongs to $\mathcal{F}_{11}$.

Furthermore, for every $n \geq 2$, let σ_{2n+h} be the relation of arity $(2n+h)$ on the set $\{1, \ldots, 2n+h\}$ defined by

$$\sigma_{2n+h} := \bigcup_{\varepsilon \in \mathcal{F}_{2n+h}} \Delta_\varepsilon \quad .$$

Thus for every $(x_1, \ldots, x_{2n+h}) \in \mathbf{k}^{2n+h}$, $(x_1, \ldots, x_{2n+h}) \in \sigma_{2n+h} \Longleftrightarrow |\{x_1, \ldots, x_{2n+h}\}| \leq h-1$ or $|\{x_1, \ldots, x_{2n+h}\}| = h$ with 1) $h-2$ symbols occuring each once and 2) one symbol occuring exactly twice and 3) one symbol occuring $2n$ times in $x_1, \ldots, x_{2n+h}$.

Notice that σ_{2n+h} is a totally symmetric relation. Moreover since σ_{2n+h} is a primitive relation, pPol σ_{2n+h} contains all partial unary functions on $\mathbf{k}$, and consequently it contains the set $\langle \mathrm{Par}^{(1)}(\mathbf{k})\rangle$ of all essentially unary functions on $\mathbf{k}$. Moreover since any at most $(h-1)$-valued $(2n+h)$-vector of $\mathbf{k}$ belongs to σ_{2n+h}, pPol σ_{2n+h} contains all at most $(h-1)$-valued functions on $\mathbf{k}$, thus Pol $\tau_h \subseteq$ pPol σ_{2n+h}. As pPol τ_h is a strong partial clone, we have Str (Pol τ_h) $\subseteq$ pPol σ_{2n+h}. On the other hand

$$\tau_h = \{(x_1,\dots,x_h) \in \mathbf{k}^h | (x_1,\dots,x_1,x_2,\dots,x_h) \in \sigma_{2n+h}\} \ .$$

Indeed, if $(x_1,\dots,x_h) \notin \tau_h$, then the x_i's are all pairwise distinct, and so the vector $\vec{v} := (x_1,\dots,x_1,x_2,\dots,x_h)$ is h-valued. As the equivalence relation $\ker(\vec{v}) := \{(i,j) \in \{1,\dots,h\}^2 | x_i = x_j\}$ has $h-1$ equivalence classes of size 1, we have $\vec{v} \notin \sigma_{2n+h}$. On the other hand, if $(x_1,\dots,x_h) \in \tau_h$, then $|\{x_1,\dots,x_h\}| \leq h-1$ and so $\vec{v} := (x_1,\dots,x_1,x_2,\dots,x_h) \in \sigma_{2n+h}$ by definition of σ_{2n+h}. This shows that pPol $\sigma_{2n+h} \subseteq$ pPol τ_h, and consequently, Str (Pol τ_h) $\subseteq$ pPol $\sigma_{2n+h} \subseteq$ pPol τ_h for all $n \geq 1$.

Our goal is to show that the partial clones pPol σ_{2n+h} give a family of partial clones of continuum cardinality in the interval [Str (Pol τ_h), pPol τ_h].

We define a family of partial functions for this task. For every $m \geq 1$, denote by $\vec{v}_0$ the $(2m+h)$-vector over $\mathbf{k}$

$$\vec{v}_0 := (0,1,\dots,h-3,h-2,h-2,h-1,\dots,h-1).$$

Notice that the (2m+h)-vector $\vec{v}_0$ contains one symbol from each of $0,1,\dots,h-3$, two symbols $h-2$ and $2m$ symbols $h-1$. Furthermore, let α be the cyclic permutation $(0\ 1\ \dots\ 2m+h-1)$ and for $j = 0,\dots,2m+h-1$, let

$$\vec{v}_j := (x_{\alpha^j(0)}, x_{\alpha^j(1)},\dots,x_{\alpha^j(2m+h-1)}),$$

where

$$(x_0,\dots,x_{2m+h-1}) := \vec{v}_0.$$

Thus

$$\begin{aligned} \vec{v}_1 &= (1,\dots,h-3,h-2,h-2,h-1,\dots,h-1,0),\dots \\ \vec{v}_{h-2} &= (h-2,h-2,h-1,\dots,h-1,0,\dots,h-3), \\ \vec{v}_{h-1} &= (h-2,h-1,\dots,h-1,0,\dots,h-3,h-2), \end{aligned}$$

while

$$\vec{v}_h = (h-1,\dots,h-1,0,\dots,h-3,h-2,h-2),\dots$$

and

$$\vec{v}_{2m+h-1} = (h-1,0,\dots,h-3,h-2,h-2,h-1,\dots,h-1).$$

We now define the $(2m+h)$-ary partial function φ_{2m+h} by

$$\mathrm{dom}\ (\varphi_{2m+h}) := \{\vec{v}_0,\vec{v}_1,\dots,\vec{v}_{2m+h-1}\},$$

and

$$\varphi_{2m+h}(x_1,\dots,x_{2m+h}) = \begin{cases} h-1 & \text{if } (x_1,\dots,x_{2m+h}) = \vec{v}_{h-1} \\ x_1 & \text{otherwise} \end{cases}$$

Notice that $\varphi_{2m+h}(\vec{v}_j) = j$ for $j = 0,\dots,h-1$. Also $\varphi_{2m+h}(\vec{v}_\ell) = e_1^{2m+h}(\vec{v}_\ell)$ for all $\ell = 0,\dots,2m+h-1$, $\ell \neq h-1$.

We have

Claim *Let $n,\ m \geq 1$. Then $\varphi_{2m+h} \in \mathrm{pPol}\,\sigma_{2n+h} \Longleftrightarrow n \neq m$.*

Proof. ($\Longrightarrow$) Let $n = m$ and consider the square matrix A of size $2n+h$ whose $2n+h$ rows are the vectors of the domain of φ_{2n+h}. Then in every column of A, the symbols $0, 1, \ldots, h-3$ appear exactly once, the symbol $h-2$ appears exactly twice while the symbol $h-1$ appears $2n$ times. Thus every column of A belongs to σ_{2n+h}. Applying φ_{2n+h} on the rows of A we get a $(2n+h)$-vector where each of $0, 1, \ldots, h-3, h-2$ appears exactly once while the symbol $h-1$ appears $2n+1$ times. So if $A_{[i*]}$ denotes the i-th row of A, we have

$$(\varphi_{2n+h}(A_{[1*]}), \ldots, \varphi_{2n+h}(A_{[2n+h*]})) \notin \sigma_{2n+h},$$

proving $\varphi_{2n+h} \notin \mathrm{pPol}\,\sigma_{2n+h}$.

($\Longleftarrow$) Let A be a matrix of size $(2n+h) \times (2m+h)$ with all rows $A_{[1*]}, \ldots, A_{[2n+h*]} \in \mathrm{dom}\,(\varphi_{2m+h})$, all columns $A_{[*1]}, \ldots, A_{[*2m+h]} \in \sigma_{2n+h}$ and such that

$$(\varphi_{2m+h}(A_{[1*]}), \ldots, \varphi_{2m+h}(A_{[2n+h*]})) \notin \sigma_{2n+h}.$$

We show that $m = n$. For $1 \leq j \leq 2m+h$, denote the $(2m+h)$-ary operation j-th projection by e_j. Since σ_{2n+h} is a totally symmetric relation, the rows of the matrix A may be listed in any convenient order. As

$$(\varphi_{2m+h}(A_{[1*]}), \ldots, \varphi_{2m+h}(A_{[2n+h*]})) \notin \sigma_{2n+h},$$

we have that $(\varphi_{2n+h}(A_{[1*]}), \ldots, \varphi_{2m+h}(A_{[2n+h*]}))$ is an h-valued vector over $\mathbf{k}$. Thus each of $\vec{v}_0, \vec{v}_1, \ldots, \vec{v}_{h-2}$ is a row of the matrix A. We will assume that the first $h-1$ rows of A are $\vec{v}_0, \vec{v}_1, \ldots, \vec{v}_{h-2}$ respectively. Now if $\vec{v}_{h-1}$ is not a row of A, then $\varphi_{2m+h}(\vec{v}) = e_1(\vec{v})$ for every row $\vec{v}$ of the matrix A, which is a contradiction with the choice of A. So let $\vec{v}_{h-1}$ be the h-th row of A. It follows that the last column $A_{[*\ 2m+h]}$ of A is an h-valued vector.

On the other hand, as $\varphi_{2m+h}(\vec{v}_i) = e_{2m+h}(\vec{v}_i) + 1 \pmod{h}$, for all $i = 0, \ldots, h$, at least one row of A is different from $\vec{v}_0, \vec{v}_1, \ldots, \vec{v}_h$. Now as every $\vec{v} \in \mathrm{dom}\,(\varphi_{2m+h})$ different from $\vec{v}_0, \vec{v}_1, \ldots, \vec{v}_h$ has both its first and last entries equal to $h-1$, we deduce on one hand that the first column $A_{[*1]}$ of A is an h-valued vector and, on the other hand, that the last column $A_{[*\ 2m+h]}$ of A contains at least two entries equal to $h-1$. Moreover as $\vec{v}_{h-2}$ and $\vec{v}_{h-1}$ are two rows of A, we deduce that at least two entries of $A_{[*1]}$ are $h-2$. We have three cases:

Case 1. There is a $j \in \{0, 1, \ldots, h-3\}$ such that j is repeated $2n$ times in $A_{[*1]}$. Since σ_{2n+h} is a totally symmetric relation, we may assume that

$$A_{[*1]} = (0, 1, \ldots, h-3, h-2, h-2, h-1, j \ldots, j),$$

and so

$$(\varphi_{2m+h}(A_{[1*]}), \ldots, \varphi_{2m+h}(A_{[2n+h*]})) = \\ (0, 1, \ldots, h-3, h-2, h-1, h-1, j \ldots, j) \in \sigma_{2n+h},$$

a contradiction with the choice of A.

Case 2. $h-2$ is repeated $2n$ times in $A_{[*1]}$. Thus $2n$ rows of A belong to the set $\{\vec{v}_{h-2}, \vec{v}_{h-1}\}$ and so $2n$ entries of the last column $A_{[*\ 2m+h]}$ belong to $\{h-3, h-2\}$. However since each of $h-3$ and $h-2$ appears at least once and $h-1$ appears twice in $A_{[*\ 2m+h]}$, we obtain the contradiction $A_{[*\ 2m+h]} \notin \sigma_{2n+h}$.

Case 3. $h-1$ appears $2n$ times in $A_{[*1]}$. Notice that in this case each of $\vec{v}_0, \vec{v}_1, \ldots, \vec{v}_{h-1}$ is a row exactly once in A. Now since every element of $\mathrm{dom}\,(\varphi_{2m+h})$ whose first entry is $h-1$ has its last entry from the set $\{h-2, h-1\}$, and since $h-1$ appears at least once in $A_{[*\ 2m+h]}$ (as $\vec{v}_0$ is a row of A), we deduce that at least one row of A has the form $(h-1, \ldots, h-1)$.

Suppose that $h-2$ appears $2n$ times in the last column $A_{[*\ 2m+h]}$. Then as $\vec{v}_{h-1} = (h-2, h-1, \ldots, h-1, 0, 1, \ldots, h-2)$ is a row of A, we have that $2n-1$ rows of A are equal to $\vec{v}_h$. Now as $\vec{v}_0, \vec{v}_1$ are rows of A, the symbol $h-1$ appears twice in the next to last column $A_{[*\ 2m+h-1]}$. If $\vec{v}_{h+1} = (h-1, \ldots, h-1, 0, \ldots, h-2, h-2, h-1)$ is not a row of A, then the symbol $h-2$ will occur an odd number of times (≥ 3) in the next to last column $A_{[*\ 2m+h-1]}$, a contradiction. It follows that $\vec{v}_{h+1}$ is a row of A and consequently the $2m+h-2$-th column $A_{[*\ 2m+h-2]}$ is an h-valued vector and contains exactly three symbols $h-1$, a contradiction.

It follows that the symbol $h-2$ appears exactly twice in the last column $A_{[*\ 2m+h]}$ and so $h-1$ appears $2n$ times in that last column $A_{[*\ 2m+h]}$. Moreover this shows that exactly one row of A is equal to $\vec{v}_h$.

Suppose now that $m=1$ and $n \neq m$, thus $n \geq 2$. As each of $\vec{v}_0$, $\vec{v}_1, \ldots, \vec{v}_{h-1}$ and $\vec{v}_h$ is a row exactly once of A, we have that $\vec{v}_{h+1}$ (in this case equal to $(h-1, 0, 1, \ldots, h-2, h-2, h-1)$) is a row $2n-1$ times of A. Then the symbol 0 appears $2n-1$ times in the second column $A_{[*2]}$, a contradiction.

We assume now that $m \geq 2$. So far we have shown each of the vectors $\vec{v}_0, \ldots, \vec{v}_h$ is a row exactly once in A. Now the symbol $h-1$ appears at least twice in the next to last column $A_{[*\ 2m+h-1]}$, and by a similar argument as above we can show that the vector $\vec{v}_{h+1}$ is a row exactly once of A. Now let $1 \leq i \leq 2n-1$ and suppose that $\vec{v}_0, \ldots, \vec{v}_h, \ldots, \vec{v}_{h+i}$ are the first $h+i+1$ rows of A and that each of these vectors is a row exactly once of A.

Then $A_{[*2n+h-(i)]}$ (i.e., the $(2n+h-(i))$-th column of A) is an h-valued vector and contains two symbols $h-2$ and at least 3 symbols $h-1$. Thus all other entries of that column must be $h-1$. On the other hand, $A_{[*2n+h-(i+1)]}$ is an h-valued vector and contains symbol $h-2$ and at least 4 symbols $h-1$. Thus this column must contain one more symbol $h-2$. As every row of A different from $\vec{v}_0, \ldots, \vec{v}_{h-1}, \vec{v}_h$ has its first entry as well as its last entry equal to $h-1$, we see that one row of A must be $(h-1, \ldots, h-2, h-2, h-1, \ldots, h-1)$ where the number of the consecutive $h-1$ on the right is $i+1$, i.e., $\vec{v}_{h+i+1}$ is also a row of A. If $\vec{v}_{h+i+1}$ appears more than once as a row of A, then each of the symbols $h-2$ and $h-1$ would be at least 3 times in the column $A_{[*2n+h-(i+1)]}$, a contradiction.

We have shown that each of the vectors in dom (φ_{2m+h}) is a row exactly once of A. Thus each row and each column of A contains exactly 2 symbols $h-2$. Counting in two different ways the number of symbols $h-2$ in A leads to $n=m$. □

We turn to the proof of our theorem. Let $P(\mathbf{N})$ be the power set of $\mathbf{N} = \{1, 2, \ldots\}$. From the above claim, the correspondence $\chi : P(\mathbf{N}) \to [\text{Str } (\text{Pol } \tau_h), \text{pPol } \tau_h]$ defined by $\chi(X) := \bigcap_{n \notin X} \text{pPol } \sigma_{2n+h}$ is a one-to-one mapping, which completes the proof of our Theorem. □

As for the case of the relation R_1, we deduce

Corollary 12. *Let $3 \leq h \leq k$. Then the interval of partial clones $[\text{pPol } \tau_h \cap M_k, \text{pPol } \tau_h]$ has the cardinality of continuum on $\mathbf{k}$.* □

On the other hand, a result similar to Corollary 10 is established for the relation R_2. The relations used for the proof can be found in [9], Theorem 11. We have

Corollary 13. ([10], Corollary 7) *Let $k \geq 2$. The interval of partial clones* $[\mathrm{pPol}\, R_2 \cap M_k, \mathrm{pPol}\, R_2]$ *has cardinality of the continuum on* $\mathbf{k}$. □

Combining Corollaries 10, 12 and 13, we get

Corollary 14. *Let $k \geq 2$,* $\mathrm{pPol}\, \theta$ *be any maximal partial clone of Słupecki type on* $\mathbf{k}$. *Then the interval of partial clones* $[\mathrm{pPol}\, \theta \cap M_k, \mathrm{pPol}\, \theta]$ *has cardinality of the continuum on* $\mathbf{k}$. □

Remark. This study yields several open problems, we mention some of them here.

1. Theorem 4 states that the interval of partial clones $[\mathrm{pPol}\, R_1 \cap \mathrm{pPol}\, R_2, \mathrm{pPol}\, R_1]$ consists of the two bounds only. As shown after Theorem 4, this does not hold for the interval $[\mathrm{pPol}\, R_1 \cap \mathrm{pPol}\, R_2, \mathrm{pPol}\, R_2]$. So one may ask for a description of this interval. In particular, is this interval finite?

2. The results established in Section 3 give families of partial clones covered by the maximal partial clones $\mathrm{pPol}\, R_1, \mathrm{pPol}\, R_2, \mathrm{pPol}\, \tau_3, \ldots, \mathrm{pPol}\, \tau_h$. So one may ask here for a description of all partial clones covered by $\mathrm{pPol}\, R_1, \mathrm{pPol}\, R_2, \mathrm{pPol}\, \tau_3, \ldots, \mathrm{pPol}\, \tau_h$.

3. Given a maximal partial clone of the form $\mathrm{pPol}\, \rho$, describe the interval of partial clones $[\mathrm{Pol}\, \rho, \mathrm{pPol}\, \rho]$. Notice that this problem was addressed in Section 4 for Słupecki type maximal partial clones. The description of all maximal partial clones on $\mathbf{k}$ is given in [6], [8] and [19].

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