Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 41 (2000), No. 2, 401-409.

The Densest Packing of 12 Congruent Circles in a Circle

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Abstract. The densest packings of n congruent circles in a circle are known for $n \leq 11$ and $n = 19$. In this paper we exhibit the densest packing of 12 congruent circles in a circle. In fact, we show that the optimal configuration is the same as the one Kravitz [10] conjectured. We use a technique developed from a method of Bateman and Erdős [1].

1. Preliminaries and results

We shall denote the points of the Euclidean plane $\mathbb{E}^2$ by capitals, sets of points by script capitals, and the distance of two points by $d(P, Q)$. We use PQ for the line through P, Q, and $\overline{PQ}$ for the segment with endpoints P, Q. $\angle POQ$ denotes the convex angle determined by the three points P,O,Q in this order. $C(r)$ means the closed disc of radius r with center O. By an annulus $r < \rho \leq s$ we mean all points P such that $r < d(P, O) \leq s$. A sector POQ is a closed convex subset of $\mathbb{E}^2$ which is bounded by the rays OP and OQ. We also make use of the linear structure of $\mathbb{E}^2$ by identifying each point P with the vector $\overrightarrow{OP}$, where O is the origin. For a point P and a vector $\vec{a}$ by $P + \vec{a}$ we always mean the vector $\overrightarrow{OP} + \vec{a}$.

The problem of finding the densest packing of congruent circles in a circle arose in the 1960s. The question was phrased the following way. What is the smallest circle in which we can pack n congruent unit circles, or equivalently, what is the smallest circle in which we can place n points with mutual distances at least 1? In 1967 dense circle packings were given by Kravitz [10] for $n = 2, \dots, 16$. In 1969 Pirl [13] proved that in fact the arrangements given by Kravitz are optimal for $n \leq 9$ and he also found the optimal configuration for $n = 10$. Pirl also conjectured dense configurations for $11 \leq n \leq 19$. For

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$n \leq 6$ proofs were given independently by Graham in response to a problem posed by Coxeter in the American Mathematical Monthly [3]. A proof for $n = 6$ and 7 was also given by Crilly and Suen [4]. Subsequent improvements were presented by Goldberg [7] for $n = 14, 16$ and 17. He also found a new packing with 20 circles. In 1975 Reis [14] used an ingenious mechanical contraption to generate remarkably good packings up to 25 circles. Recently, Graham et al. [8], [9] using modern computers established packings with more than 100 circles and improved the packing of 25 circles. In 1994 Melissen [11] proved Pirl's conjecture for $n = 11$ and the author [6] proved it for $n = 19$. We shall find the optimal configuration for $n = 12$. Our goal is to prove the following theorem.

Theorem 1. *The smallest circle $\mathcal{C}$ which can accommodate* 12 *points with mutual distances at least* 1 *has radius $R = 1.5148\ldots$. Moreover, the* 12 *points form the unique configuration shown in Figure 1.*

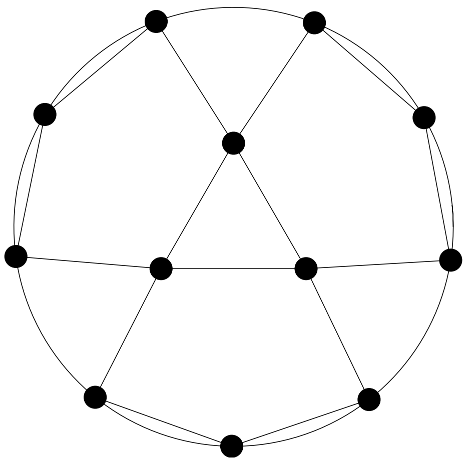

Figure 1. *The optimal configuration for $n = 12$*

Note that two points are connected with a straight line segment on Figure 1 if and only if their distance is exactly 1. In Theorem 1, $R = \frac{1}{\sqrt{3x_0}}$, where x_0 is the smallest positive root of $9x^5 - 15x^4 + 7x^3 - 3x + 1 = 0$.

The proof of Theorem 1 consists of three parts. We shall divide $C(R)$ into a smaller circle $C(S)$ and an annulus $S < \rho \leq R$, choosing S such that there can be at most 5 points in $C(S)$ and at most 9 points $S < \rho \leq R$. This leaves us three different cases. Namely when there are $3, 4, 5$ points in $C(S)$. We deal with the latter two cases first showing that they cannot occur. Then we prove that the three points in $C(S)$ form a regular triangle of unit side length centered at O.

In the course of our proof we shall utilize the following two statements. Lemma 1, slightly modified here, originates from a paper of Bateman and Erdős [1] where they used it to prove an improved version of a theorem of Besicovitch. Lemma 2 is used by the author [6] to show Pirl's conjecture for $n = 19$.

Lemma 1. [1] *Let $r, s, (s \geq \frac{1}{2})$ be two positive real numbers and suppose that we have two points P and Q which lie in the annulus $r \leq \rho \leq s$ and which have mutual distance at* *least* 1. *Then the minimum* $\phi(r,s)$ *of the angle* $\angle POQ$ *has the following values:*

$$\phi(r,s) = \arccos \frac{s^2+r^2-1}{2rs}, \quad \text{if } \ 0 < s-1 \le r \le s-1/s;$$

$$\phi(r,s) = 2 \arcsin \frac{1}{2s}, \quad \text{if } \ 0 < s-1/s \le r \le s \ \text{ or } \ s \le 1.$$

Lemma 2. [6] *Let* $\mathcal{S}$ *be a set of* n $(n \ge 2)$ *points in the plane and* $\mathcal{C}$ *the smallest circle containing* $\mathcal{S}$. *Let* $\vec{a}$ *be a vector. There exist two points* P_1, P_2 *on the boundary of* $\mathcal{S}$, *such that* $d(P_1+\vec{a}, O) + d(P_2+\vec{a}, O) \ge 2r$, *where* r *is the radius and* O *is the center of* $\mathcal{C}$.

The problem of finding the densest packing of equal circles in a circle is also stated in the book of Croft, Falconer and Guy [5] on pp. 110–111. Packings of congruent circles in hyperbolic plane were treated by K. Bezdek [2]. Analogous results of packing n equal circles in an equilateral triangle and square can be traced down in the doctoral dissertation of Melissen [12].

2. Proofs

We prove Theorem 1 with a sequence of lemmas. Let $S = R - 1/R = 0.854\ldots$.

Lemma 3. *There can be at most* 9 *points in the annulus* $S < \rho \le R$.

Proof. This follows from the fact that $\phi(S,R) = \phi(R,R) = 38^\circ.5468\ldots$.
Thus, $10\phi(R,R) > 360^\circ$. □

We know from [10] that the radius of the circumcircle of 6 points with mutual distances at least 1 is at least 1, which is larger than S. Therefore there cannot be fewer than 7 points in the annulus $S < \rho \le R$, or there would be at least 6 points in $C(S)$. Thus, we have three cases to consider, namely, when there are $5, 4, 3$ points in $C(S)$. We shall deal with the three cases one-by-one.

Lemma 4. *There cannot be exactly* 7 *points in the annulus* $S < \rho \le R$.

Proof. If we have 7 points in the annulus $S < \rho \le R$ there must 5 points in $C(S)$. Consider the following assertion. There cannot be any of these 5 points in $C(0.835)$. Because $2\phi(0.835, S) = 145^\circ.343\ldots$ and $\phi(S,S) = 71^\circ.61\ldots$, we have that $3\phi(S,S) + 2\phi(0.835,S) = 360^\circ.175\ldots > 360^\circ$. Therefore we may assume that all 5 points are in the annulus $0.835 < \rho \le S$.

Connecting the 5 points in $C(S)$ to O we obtain 5 angular sectors. The minimal central angle of a sector is $\phi(S,S) = 71^\circ.61\ldots$. In one of these sectors, say P_iOP_{i+1}, there is a point P of the 7 points from the annulus $S < \rho \le R$. Then $\angle P_iOP, \angle P_{i+1}OP \ge \phi(0.835, R) = 76^\circ.1173\ldots$. Therefore $4\phi(S,S) + 2\phi(0.835,R) = 362^\circ.561\ldots > 360^\circ$. So the 5 points cannot all be in $C(S)$. □

Lemma 5. *There cannot be exactly* 8 *points in the annulus* $S < \rho \leq R$.

Proof. If we have 8 points in the annulus $S < \rho \leq R$, then there must be 4 points in $C(S)$. To accommodate 4 points in the plane with mutual distances at least 1 we need a circle of radius at least $r_4 = 1/\sqrt{2}$. Suppose that these 4 points are in the minimum radius circle $\mathcal{C}'$ with center O'. The radius of $\mathcal{C}'$ is at least r_4. We know from Lemma 2 that there are two of the 4 points, P and Q on the boundary of $\mathcal{C}'$, such that $d(P,O)+d(Q,O) \geq 2r_4 = \sqrt{2}$. The 8 points in the annulus $S < \rho \leq R$ connected to O divide $\mathcal{C}$ into 8 angular sectors. We can assume that P and Q are in the interior of two different sectors, say P is in P_iOP_{i+1} and Q in P_jOP_{j+1}, and the angle $\angle P_iOP_{i+1} \geq 2\phi(d(P,O),R)$, and $\angle P_jOP_{j+1} \geq 2\phi(d(Q,O),R)$.

Suppose that $d(Q,O) \geq d(P,O)$. We may assert then that one of the points in $C(S)$ other than Q is at least 0.66 from O. To see this we show that the three points in $C(S)$ other than Q do not fit in $C(0.66)$. On the contrary let us suppose that they do. Connecting the three points to O we obtain three angular sectors with angles at least $\phi(0.66, 0.66) = 98°.5019\ldots$. Q lies in one of these sectors, say in P_1OP_2. Then $\angle P_1OP_2 \geq 2\phi(0.66, S) = 163°.074\ldots$, so $2\phi(0.66,0.66)+2\phi(0.66,S) = 360°.079\ldots > 360°$. Therefore the three points cannot all be in $C(0.66)$.

Now the sum of the angles of the 8 sectors, determined by the points in the annulus $S < \rho \leq R$, is at least $6\phi(R,R) + 2\phi(d(P,O),R) + 2\phi(d(Q,O),R)$, where we assume that $d(P,O) + d(Q,O) = \sqrt{2}$ and, by the above argument, that $d(P,O) \geq 0.66$. It is easy to check that the minimum of the sum occurs when $d(P,O) = 0.66$ and it is $362°.16\ldots > 360°$.

Therefore there cannot be exactly 8 points in the annulus $S < \rho \leq R$. □

In the subsequent paragraphs we shall deal with the remaining case when we have only 3 points, P_1, P_2, P_3, in $C(S)$. Let d_i, $i = 1,2,3$ denote the distance $d(P_i,O)$, $i = 1,2,3$. In the following lemma we summarize several properties of these three points.

Lemma 6.

a) $d_i \geq \sqrt{3}/3$ *for some* i *and* $0.43 \leq d_i \leq d = 0.6122$ *for all* $i = 1,2,3$.
b) $C(0.56)$ *cannot contain two of* P_i, $i = 1,2,3$.
c) *A sector, determined by two points of the annulus* $S < \rho \leq R$, *contains at most one of* P_i, $i = 1,2,3$.
d) *A sector, determined by* P_i *and* P_j, *contains exactly three points from the annulus* $S < \rho \leq R$.
e) *If* $d_i \leq 0.5635$, *then* P_i *does not contribute to the minimal angle of the sector, determined by two points of the annulus* $S < \rho \leq R$, *in which it lies.*

Proof. a) The radius of the circumcircle of three points with mutual distances 1 is $\sqrt{3}/3$, therefore at least one of d_i is not less than $\sqrt{3}/3$. Suppose that $d_i \geq 0.6122$ for some i. Then adding up the angles of the nine sectors determined by the points in the annulus $S < \rho \leq R$ we obtain $8\phi(R,R) + 2\phi(d,R) = 308°.37\ldots + 51°.66 = 360°.83$.

As $d_i \leq d$ for all i, no side of the triangle $\triangle P_1P_2P_3$ can be longer than $2d = 1.2244\ldots$, so all side lengths are in between 1 and $2d$. It is easy to see that such a triangle is always strictly acute and all three heights are at least $\sin\arccos(0.6122) = 0.79\ldots$. This shows that O must be inside $\triangle P_1P_2P_3$ otherwise it would be farther than d from one of the vertices. Now, suppose that $d_i \le 0.43$ for some i. Then one of the other two d_j is at least $\sqrt{(0.79-0.43)^2+1/4} = 0.616\ldots > d$.

b) If $d_i, d_j \le 0.56$, then adding up the angles of the three sectors determined by P_i, $I = 1,2,3$ we obtain $2\phi(0.56, d) + \phi(0.56, 0.56) = 234°.06 + 126°.468 = 360°.52$.

c) Let us assume, on the contrary, that in one sector there are two of P_i, $i = 1,2,3$. Then the central angle of this sector is at least $\phi(d,d) = 109°.516\ldots$. Thus, if we sum up the angles of the 9 sectors determined by the points in the annulus $S < \rho \le R$ we get $8\phi(R,R) + \phi(d,d) = 417°.8\ldots > 360°$.

d) Now, we will show that each of the 3 sectors determined P_i, $i = 1,2,3$ contains exactly three of the points from the annulus $S < \rho \le R$. To see this suppose, on the contrary, that in the sector determined by P_1 and P_2 there are four points. We may suppose that $d_1 \ge 0.56$. If $d_2 \ge 0.525$, then the angle of the sector is at least $3\phi(R,R) + \phi(0.56, R) + \phi(0.525, R) = 115°.64\ldots + 18°.57\ldots + 9°.16\ldots = 143°.37\ldots$. Therefore the total angle of the three sectors is at least $2\phi(d,d) + 140°.87\ldots = 362°.4\ldots$. If $d_2 < 0.525$, then the total angle of the three sectors is $3\phi(R,R) + \phi(0.56,R) + \phi(d,d) + \phi(0.525,d) = 115°.64\ldots + 18°.57\ldots + 109°.516\ldots + 122°.9\ldots = 366°.5\ldots$. Therefore there must be exactly three points in each of the three sectors.

e) Evaluating $\phi(0.5635, R)$ it turns out that it is less than $\frac{1}{2}\phi(R,R)$. □

Let us introduce the following function

$$F(d_i, d_{i+1}, s_{i+2}) = \arccos \frac{d_i^2 + d_{i+1}^2 - s_{i+2}^2}{2d_i d_{i+1}},$$

which gives the central angle of the sector P_iOP_{i+1} under the assumption that the length of the side P_iP_{i+1} is s_{i+2}. If $d_i = d_{i+1} = \sqrt{3}/3$ and $s_{i+2} = 1$, then $F(d_i, d_{i+1}, s_{i+2}) = 120°$. Let us further note that the $\phi(d_i, R)$ are monotonically increasing functions of the d_i.

Suppose that two sides of $\triangle P_1P_2P_3$ are longer than 1. Then we move the vertex, say P_1, where these two sides meet toward O, while keeping all others points fixed, until one of the sides becomes 1.

Lemma 7. *This operation is admissible, that is, the mutual distances remain at least* 1 *and the* 12 *points fit in* $C(R)$.

Proof. To see this draw circles of radius 1/2 around the 12 points. During the movement of P_1 no circles can overlap. Clearly, as we move P_1 toward O we only have to make sure that there is no circle which blocks the one around P_1 before one of the sides shrinks to 1. Assume, on the contrary, that there is a circle around a point P which blocks the circle around P_1. Then $d(P,O) \le \sqrt{d^2+1} = 1.1725\ldots$. Suppose that P is in the sector determined by P_1 and P_2. One of d_1 and d_2 is at least 0.56. We distinguish three cases. If $d(P,O) \le 1.1$ and $d_1 \ge 0.56$, then total angle of the three sectors determined by the three points in $C(S)$ is at least $2\phi(d,d) + \phi(0.56, 1.1) + 2\phi(R,R) = 219°.03\ldots + 64°.849\ldots + 77°.08\ldots = 360°.96\ldots$. If $d_1 < 0.56$, then $d_2 \ge 0.56$, thus the total angle of the three sectors is $2\phi(d,d) + \phi(0.56,R) + 2\phi(R,R) + \phi(0.43, 1.1) = 219°.03\ldots + 18°.57\ldots + 77°.08\ldots + 65°.52\ldots = 379°.08\ldots$ Finally, if $d(P,O) > 1.1$, then we will examine the 9 sectors determined by the points in the annulus $S < \rho \le R$. First we note the following.

If $d_i \leq 0.5635$, then P_i does not contribute to the angle of the sector in which it lies, that is, $2\phi(0.5635, R) < \phi(R, R)$. By Lemma 2 there are two points, P_i, P_{i+1} for which $d_i + d_{i+1} \geq 2\sqrt{3}/3$. The total angle of the two sectors they are in is $2\phi(d_1, R) + 2\phi(d_2, R)$. Supposing that $d_i + d_{i+1} = 2\sqrt{3}/3$ and $2\sqrt{3}/3 - 0.5635 \leq d_i, d_{i+1}$ the minimum of the total angle occurs when one of d_i and d_{i+1} is $2\sqrt{3}/3 - 0.5635$, namely $46°.61\ldots$. Thus, the sum of the central angles of the 9 sectors is as follows, $6\phi(R, R) + 2\phi(1.1, R) + 46°.61\ldots = 231°.28\ldots + 82°.54\ldots + 46°.61\ldots = 360°.043\ldots$. Therefore there cannot be a circle which blocks P_1 as we move it toward O, so we can shrink one side to 1. From now on we always assume that at least two sides of $\triangle P_1P_2P_3$ are 1. $\square$

Suppose we have a triangle $\triangle P_1P_2P_3$ with fixed sides s_i. We will call a position of this triangle admissible if $F(d_i, d_{i+1}, s_{i+2}) \geq 2\phi(R, R) + \phi(d_i, R) + \phi(d_{i+1}, R)$ for all i. Notice that if $d_i = \sqrt{3}/3$ and all three sides are of unit length, then there is equality in each of the above inequalities. If any of these conditions fails, then we will call that position of the triangle not admissible.

First we will consider the case when $s_i = 1$ for all i.

Lemma 8. *If the three points in $C(S)$ form a regular triangle $\triangle P_1P_2P_3$ of side length 1, then the only admissible position is when O is at the center of this triangle.*

Proof. Let d_1 be the largest of the d_i, then $d_1 \geq \sqrt{3}/3$. Consider the position in which $d_2 = d_3$ and fix d_1. Note that in this position d_2 is a function of d_1. Notice that the $F(d_1, d_i, 1)$ are monotonically decreasing functions of the d_i. Thus, if we show that in this symmetric position the two sides P_1P_2 and P_1P_3 are not admissible, then in any other position, with the same fixed d_1, one of them will not be admissible.

It is enough to consider the following function of d_1, $F(d_1, d_2, 1) - 2\phi(R, R) - \phi(d_1, R) - \phi(d_2, R)$ in the interval $[\frac{\sqrt{3}}{3}, d]$. Figure 2 shows its graph in $[\frac{\sqrt{3}}{3}, d]$. It can be seen that it is zero at $\sqrt{3}/3$ and negative elsewhere.

$$\mathrm{R} := 1.5148$$

$$\mathrm{Plot}\Big[180/\mathrm{Pi}\Big(\mathrm{ArcCos}[\frac{-\frac{3}{4}+\left(\frac{\sqrt{3}}{2}-\mathrm{x}\right)^2+\mathrm{x}^2}{2\mathrm{x}\sqrt{\frac{1}{4}+\left(\frac{\sqrt{3}}{2}-\mathrm{x}\right)^2}}] -$$

$$\Big(4\mathrm{ArcSin}[\tfrac{1}{2\mathrm{R}}] + \mathrm{ArcCos}[\frac{-\frac{3}{4}+\mathrm{R}^2+\left(\frac{\sqrt{3}}{2}-\mathrm{x}\right)^2}{2\mathrm{R}\sqrt{\frac{1}{4}+\left(\frac{\sqrt{3}}{2}-\mathrm{x}\right)^2}}] + \mathrm{ArcCos}[\tfrac{-1+\mathrm{R}^2+\mathrm{x}^2}{2\mathrm{Rx}}]\Big)\Big),$$

$$\{\mathrm{x},\ \mathrm{Sqrt}[3]/3,\ .6122\}\Big]$$

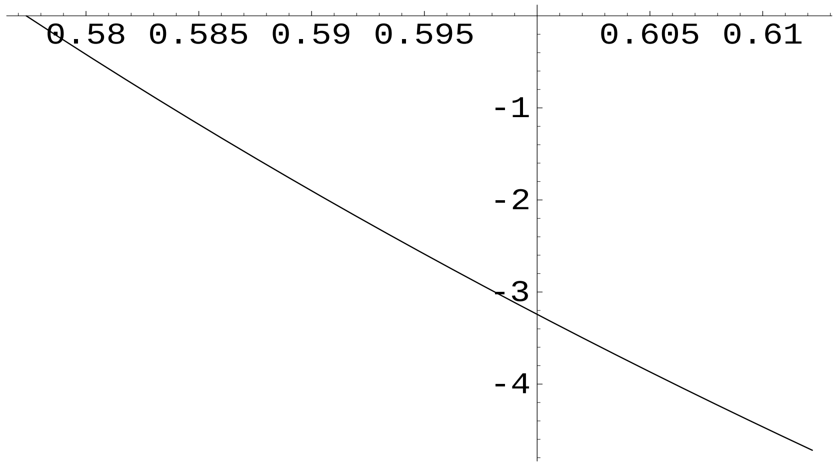

Figure 2.

Thus, we can conclude that a regular triangle of unit side length can only be placed in $C(S)$ exactly one way, namely, when O is at the center of the triangle. $\square$

Now, we return to the general case letting one side of $\triangle P_1P_2P_3$ be greater than 1.

Lemma 9. *If one of the sides, say P_1P_3, is longer than 1, then the triangle $\triangle P_1P_2P_3$ cannot be admissible in any position.*

Proof. Suppose that $d_1 \geq d_3$. Notice that $d_2 \leq d_1, d_3$. Otherwise $d_1 \geq \left(1 + d_2^2 - 2d_2\sqrt{1 - s_2^2/4}\right)^{1/2}$ which is longer than it would be if $\triangle P_1P_2P_3$ were a regular unit triangle. Therefore the side P_1P_2 would not be admissible unless $d_2 = \sqrt{3}/3$ by Lemma 8.

Suppose that $d_2 = d_3$. Then $d_2\sqrt{3}/3$ or one of P_1P_2 and P_2P_3 would not be admissible. Moreover, as $d_1 \leq d$, it follows that $d_2 > 0.56$. If we can show that in this symmetric position both sides P_1P_2 and P_1P_3 are not admissible, then by the same monotonicity argument as in Lemma 8, $\triangle P_1P_2P_3$ is not admissible in any other position.

For every d_2 in $[0.5635, \sqrt{3}/3]$ there is an $f(d_2)$ defined by $6\phi(R,R) + 4\phi(d_2, R) + 2\phi(f(d_2), R) = 360°$, which means that $\sqrt{3}/3 \leq d_1 \leq f(d_2)$. For d_2 in $[0.56, 0.5635]$ define $f(d_2) = d$. Assume that $\triangle P_1'P_2P_3$ is a regular triangle of unit side length, then $\sqrt{3}/3 \leq d_1' < d_1$. However, from the previous lemma we already know that the side $P_1'P_2$ is not admissible unless $d_1' = d_2 = \frac{\sqrt{3}}{3}$. Therefore P_1P_2 is not admissible.

Now we have only to show that P_1P_3 is not admissible as well. Notice that for a fixed d_2 the $\angle P_1OP_2 + \angle P_1OP_3 = const.$ Therefore $\frac{\partial}{\partial d_1}F(d_1, d_2, s_2) = -\frac{\partial}{\partial d_1}F(d_1, d_2, 1)$. Using this fact simple calculus shows that

$$\frac{\frac{\partial}{\partial d_1}F(d_1, d_2, s_2)}{\frac{\partial}{\partial d_1}\phi(d_1, R)} = \left(\frac{(2d_1R)^2 - (d_1^2 + R^2 - 1)^2}{(2d_1d_2)^2 - (d_1^2 + d_2^2 - 1)^2}\right)^{1/2} \frac{1 - d_2^2 + d_1^2}{R^2 - 1 - d_1^2}.$$

It can be easily seen that if d_1 is in $[\sqrt{3}/3, d]$ and d_2 in $[0.56, \sqrt{3}/3]$, then both the numerators and denominators are monotone functions of d_1 as well as d_2. Thus, plugging in concrete values for d_1 and d_2 we can see that the expression is larger than one. So $F(d_1, d_2, s_2)$

is increasing faster that $\phi(d_1, R)$. We already know that $F(d_1', d_2, 1) \leq 2\phi(R, R) + \phi(d_1', R) + \phi(d_2, R)$ with equality only if $d_2 = \sqrt{3}/3$. Thus, if $F(f(d_2), d_2, s_2) - 2\phi(R, R) - \phi(f(d_2), R) - \phi(d_2, R) < 0$, then the same relation holds everywhere in $[d_1', f(d_2)]$. Figure 3 shows the graph of the above function for $d_2 \in [0.5635, \sqrt{3}/3]$ drawn by *Mathematica*. It can be seen that it is zero at $d_2 = \sqrt{3}/3$ and negative elsewhere.

$$\mathrm{R} := 1.5148$$

$$\mathrm{F[x\_]} := -\mathrm{RCos}\left[2\mathrm{ArcCos}\left[\tfrac{-1+\mathrm{R}^2+\mathrm{x}^2}{2\mathrm{Rx}}\right] + 6\mathrm{ArcSin}\left[\tfrac{1}{2\mathrm{R}}\right]\right] - \sqrt{\left(1 - \mathrm{R}^2 + \mathrm{R}^2\mathrm{Cos}\left[2\mathrm{ArcCos}\left[\tfrac{-1+\mathrm{R}^2+\mathrm{x}^2}{2\mathrm{Rx}}\right] + 6\mathrm{ArcSin}\left[\tfrac{1}{2\mathrm{R}}\right]\right]^2\right)}$$

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Plot[180/Pi(ArcCos[(x^2 + F[x]^2 - (2 - 2Cos[ArcCos[1/(2x)] +
    ArcCos[(x^2 - F[x]^2 + 1)/(2x)]]))/(2xF[x])] - (4ArcSin[1/(2R)] +
    ArcCos[(x^2 + R^2 - 1)/(2xR)] + ArcCos[(F[x]^2 + R^2 - 1)/(2F[x] R)])),
    {x, .5635, Sqrt[3]/3}]
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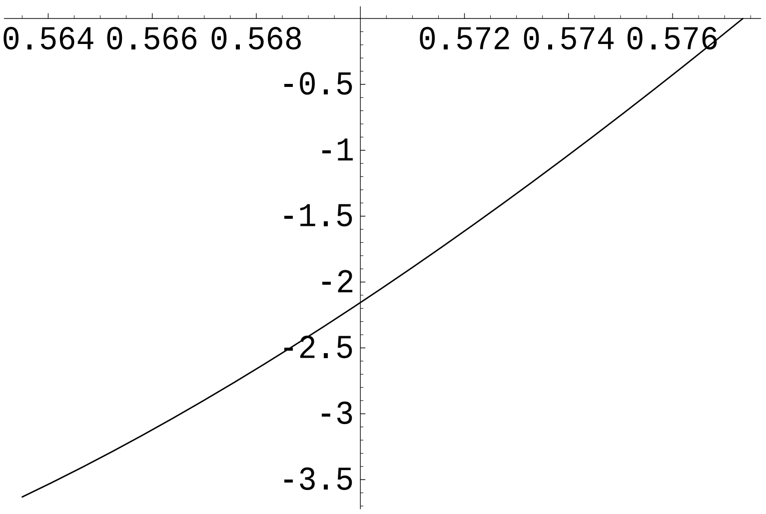

Figure 3.

If $d_2 \in [0.56, 0.5635]$ it is not hard to check that $F(d, d_2, s_2)$ takes on its maximum if $d_2 = 0.5635$ and $\phi(d_2, R)$ attains its minimum if $d_2 = 0.56$. Thus, by plugging in the actual numbers it turns out that the same relation holds as in $[0.5635, \sqrt{3}/3]$.

Therefore we can conclude that the triangle $\triangle P_1P_2P_3$ is not admissible in any position if any of the sides is longer than 1. Thus, we have finished the proof of Theorem 1. $\square$

References

[1] Bateman, P.; Erdős, P.: *Geometrical extrema suggested by a lemma of Besicovitch.* Amer. Math. Monthly **58** (1951), 306–314.

[2] Bezdek, K.: *Ausfüllungen eines Kreises durch kongruente Kreise in der hyperbolischen Ebene.* Studia Sci. Math. Hungar. **17** (1982), 353–356.

[3] Coxeter, H. S. M.; Greening, M. G.; Graham, R. L.: *Sets of points with given maximum separation (Problem E1921).* Amer. Math. Monthly **75** (1968), 192–193.

[4] Crilly, T.; Suen, S.: *An improbable game of darts.* Math. Gazette **71** (1987), 97–100.

[5] Croft, H. T.; Falconer, K. J.; Guy, R. K.: *Unsolved Problems in geometry.* Springer Verlag, New York, Berlin, Heidelberg 1991.

[6] Fodor, F.: *The densest packing of* 19 *congruent circles in a circle.* Geom. Dedicata, to appear.

[7] Goldberg, M.: *Packing of* $14, 16, 17$ *and* 20 *circles in a circle.* Math. Mag. **44** (1971), 134-139.

[8] Graham, R. L.; Lubachevsky, B. D.: *Dense packings of* $3k(k+1)+1$ *equal disks in a circle for* $k = 1, 2, 3, 4$ *and* 5. In: Proc. First Int. Conf. "Computing and Combinatorics" COCOON'95, Springer Lecture Notes in Computer Science **959** (1996), 303–312.

[9] Graham, R. L.; Lubachevsky, B. D.: *Curved hexagonal packings of equal disks in a circle.* Discrete Comp. Geom. **18** (1997), 179–194.

[10] Kravitz, S.: *Packing cylinders into cylindrical containers.* Math. Mag. **40** (1967), 65–71.

[11] Melissen, J. B. M.: *Densest packing of eleven congruent circles in a circle.* Geom. Dedicata **50** (1994), 15–25.

[12] Melissen, J. B. M.: *Packing and Covering with Circles.* Doctoral dissertation 1997.

[13] Pirl, U.: *Der Mindestabstand von* n *in der Einheitskreisscheibe gelegenen Punkten.* Math. Nachr. **40** (1969), 111–124.

[14] Reis, G. E.: *Dense packing of equal circles within a circle.* Math. Mag. **48** (1975), 33–37.

Received October 20, 1998