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Double Coverings and Reflexive Abelian Hypermaps

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Abstract. We describe a general theory of hypermaps on surfaces, possibly non-orientable or with boundary. This includes techniques for constructing hypermaps as products and as double coverings, and for representing hypermaps as maps using homomorphisms between extended triangle groups. As a corollary we obtain a classification of the 16 reflexible hypermaps with abelian automorphism groups.

1. Introduction

Our aim is to describe some general techniques which we will use in [3] to classify the reflexible hypermaps (on orientable or non-orientable surfaces) with the same automorphism groups as the regular platonic solids, and in [4] to classify the rotary hypermaps of genus 2. These are combinatorial techniques, which we will interpret both topologically and algebraically.

One such technique, described in Section 2, is the construction of a product $\mathcal{H}_1 \times \mathcal{H}_2$ of two hypermaps $\mathcal{H}_1$ and $\mathcal{H}_2$. This is a generalisation of Wilson's parallel product of maps [23]; it allows us to construct double coverings of a hypermap $\mathcal{H}$ by forming its product $\mathcal{H} \times \mathcal{B}$ with a hypermap $\mathcal{B}$ having two blades. There are seven such hypermaps $\mathcal{B}$: we classify them in Section 5, and describe the seven associated double coverings in Section 6. As a by-product of this, we also obtain in Section 5 a classification of the reflexible hypermaps with abelian automorphism groups; there are sixteen of them, all formed by taking products of 2-blade hypermaps $\mathcal{B}$.

Another technique we will use is Walsh's bijection W [22] from hypermaps to bipartite maps; Corn and Singerman explained W algebraically for orientable hypermaps in [9], and

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in Section 7 we extend their ideas to all hypermaps by interpreting W and other similar representations of hypermaps in terms of homomorphisms between extended triangle groups.

The basic definitions we require are given in Section 3, but first in Section 2 we give a combinatorial description of a simple example of our methods; this is in order to provide motivation and illustration for the rather more abstract ideas in later sections, which can be seen as generalising this example.

Many of the ideas in this paper are based on the first author's Ph.D. thesis [2]; we wish to thank the Research and Development Joint Research Centre of the Commission of the European Communities for financially supporting this research, and the Mathematics Departments at the Universities of Coimbra and Aveiro for their hospitality and financial support for subsequent collaboration. This paper is dedicated to the memory of our colleague and friend Lynne James, whose work on hypermaps had a great influence on our research.

2. An example

Let $\mathcal{T}$ be the tetrahedron, a reflexible hypermap of type $(3,2,3)$ – that is, a reflexible map of type $\{3,3\}$ – with automorphism group $\operatorname{Aut}\mathcal{T} \cong S_4$ (including orientation-reversing automorphisms). We will construct a second reflexible hypermap $\mathcal{T}'$ with $\operatorname{Aut}\mathcal{T}' \cong S_4$. We regard $\mathcal{T}$ as drawn on the sphere S^2, and take a double covering T^2 of S^2 branched at the centres of the four faces of $\mathcal{T}$. To form T^2 we take two copies $\mathcal{T}_1$ and $\mathcal{T}_2$ of $\mathcal{T}$, cut each $\mathcal{T}_i$ along the geodesics joining the face- centres, and join $\mathcal{T}_1$ and $\mathcal{T}_2$ edge-to-edge along these six cuts (see Figure 1, where the vertices of $\mathcal{T}_1$ and $\mathcal{T}_2$ are coloured black and white respectively).

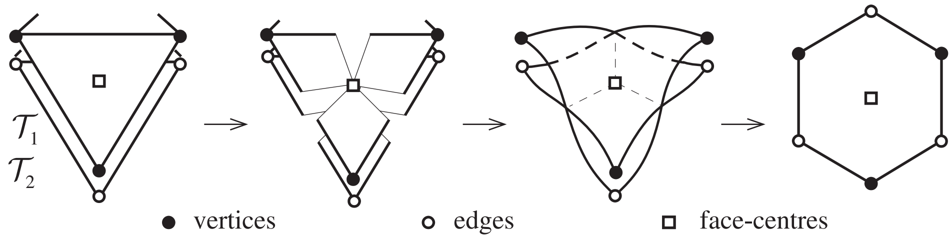


Figure 1. Construction of T^2

The resulting surface T^2 is a torus, inscribed with a map $\mathcal{W}$ of type $\{6,3\}$ (that is, a hypermap of type $(3,2,6)$) which is a branched double covering of $\mathcal{T}$ with 8 vertices, 12 edges and 4 hexagonal faces (see Figure 2); $\mathcal{W}$ is the reflexible map $\{3+3,3\} = \{6,3\}_4 = \{6,3\}_{2,0}$ in the notation of Coxeter and Moser [10, Ch.8], who give further examples of such double coverings in their §8.8.

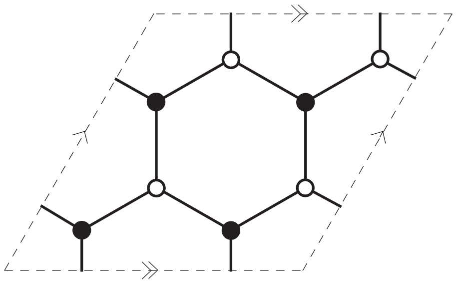


Figure 2. The map $\mathcal{W}$

Note that $\mathcal{W}$ is bipartite, the two sets of vertices (black and white) corresponding to the two copies $\mathcal{T}_i$ of $\mathcal{T}$. The automorphism group $\mathrm{Aut}\,\mathcal{W}$ of $\mathcal{W}$ is a direct product $\mathrm{Aut}\,\mathcal{T} \times C_2 \cong S_4 \times C_2$: the first factor represents the simultaneous action of $\mathrm{Aut}\,\mathcal{T}$ on the two copies $\mathcal{T}_i$ of $\mathcal{T}$, while the second factor transposes pairs of points $p_i \in \mathcal{T}_i$ $(i = 1, 2)$ covering the same point $p \in \mathcal{T}$ (this corresponds to the rotation of Figure 2 by an angle π about the centre of the parallelogram). Likewise, we will see in Section 6 that $\mathcal{W}$ can be regarded as the product $\mathcal{T} \times \mathcal{B}^{\hat{0}}$ of $\mathcal{T}$ with a certain 2-blade hypermap $\mathcal{B}^{\hat{0}}$ having two blades, one coloured black and one white.

In [22], Walsh introduced a bijection W between hypermaps $\mathcal{H}$ and bipartite maps $\mathcal{M}$ on the same surface: the black vertices, white vertices and faces of $\mathcal{M} = W(\mathcal{H})$ correspond to the hypervertices, hyperedges and hyperfaces of $\mathcal{H}$. Since our map $\mathcal{W}$ is bipartite, $\mathcal{W} = W(\mathcal{T}')$ for a hypermap $\mathcal{T}'$ on T^2 with 4 hypervertices, 4 hyperedges and 4 hyperfaces, shown in Figure 3.

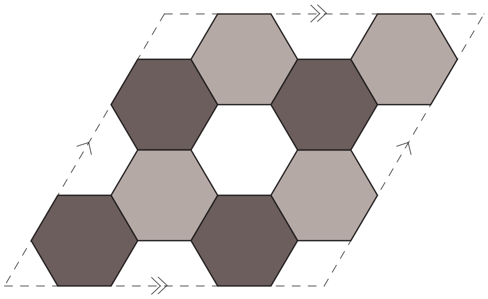


Figure 3. The hypermap $\mathcal{T}'$

$\mathcal{T}'$ has type $(3, 3, 3)$, with $\mathrm{Aut}\,\mathcal{T}' \cong \mathrm{Aut}\,\mathcal{T} \cong S_4$ (the subgroup of $\mathrm{Aut}\,\mathcal{W}$ fixing the vertex-colours of $\mathcal{W}$). Since $\mathrm{Aut}\,\mathcal{T}'$ acts transitively on the 24 blades, $\mathcal{T}'$ is reflexible.

Just as $\mathcal{T}$ corresponds to the representation of S_4 as an image of the extended triangle group

$$\Delta(3, 2, 3) = \langle R_0, R_1, R_2 \mid R_i^2 = (R_1R_2)^3 = (R_2R_0)^2 = (R_0R_1)^3 = 1\rangle$$

(in fact, $S_4 \cong \Delta(3, 2, 3)$), $\mathcal{T}'$ corresponds to the representation of S_4 as an image of $\Delta(3, 3, 3)$. Indeed, Figure 3 shows that $\mathcal{T}'$ is the quotient of the universal hypermap of type $(3, 3, 3)$, drawn in $\mathbf{R}^2$, by a subgroup $2T$ of index 4 in the translation subgroup $T \cong \mathbf{Z}^2$ of $\Delta(3, 3, 3)$, namely the kernel of the epimorphism $\Delta(3, 3, 3) \to S_4$; Figure 3 shows a fundamental region for $2T$. (See [9] for a general discussion of universal hypermaps.)

We have constructed $\mathcal{T}'$ from a branched double covering of $\mathcal{T}$ so that $\mathrm{Aut}\,\mathcal{T}' \cong \mathrm{Aut}\,\mathcal{T}$, and our aim in the rest of this paper is to generalise this construction. We will see in Section 5 that there are just seven hypermaps $\mathcal{B}$ with two blades, giving rise to seven general methods of forming double coverings $\mathcal{H} \times \mathcal{B} \to \mathcal{H}$ of hypermaps $\mathcal{H}$, described in Section 6. Similarly we will see in Section 7 that Walsh's bijection W is one of many such transformations of hypermaps induced by homomorphisms between extended triangle groups. For further examples and applications of these results, see [2, 3, 4].

3. Hypermaps

3.1. Definitions

We will base our definition of a hypermap $\mathcal{H}$ on that given by James in [13]. This definition, closely related to the coloured triangulations of Bracho and Montejano [1], the crystallizations of Ferri, Gagliardi and others [11, 12], the graph-encoded maps of Lins [16] and the combinatorial maps of Vince [20, 21], is an extension to hypermaps of an algebraic approach to maps introduced by Tutte [19] and further developed by Bryant and Singerman [5]. It also generalises the algebraic theory of oriented hypermaps introduced by Cori in [6] and surveyed by Cori and Machì in [7] (see also Section 4.2). It has the two advantages of allowing group-theoretic methods to be introduced efficiently, and of allowing the underlying surface $\mathcal{S}$ of $\mathcal{H}$ to be non-compact, non-orientable, or with boundary; our only restriction is that $\mathcal{S}$ should be connected.

When $\mathcal{S}$ is without boundary we define a *hypermap* $\mathcal{H}$ to be an imbedding (without crossings) of a connected trivalent graph $\mathcal{G}$ (possibly with multiple edges, but no loops), such that each *face* (connected component of $\mathcal{S} \setminus \mathcal{G}$) is homeomorphic to an open disc, together with a labelling of the faces with labels $0, 1$ and 2 so that each edge of $\mathcal{G}$ is incident with two faces carrying different labels. An *i-face* (a face labelled i) is called a *hypervertex*, *hyperedge* or *hyperface* of $\mathcal{H}$ as $i = 0, 1$ or 2 respectively, and a vertex of $\mathcal{G}$ is called a *blade* of $\mathcal{H}$.

It is useful to extend these definitions to the case where $\mathcal{S}$ has boundary $\partial\mathcal{S} \neq \emptyset$. We now allow $\mathcal{G}$ (still connected and trivalent) to have *free edges*: these are homeomorphic to the closed interval $[0, 1]$, with only one end-point a vertex of $\mathcal{G}$, the other being free. We require that $\mathcal{G} \cap \partial\mathcal{S}$ is the set of all such free end-points (so $\partial\mathcal{S}$ contains no vertices of $\mathcal{G}$), and that each face meeting $\partial\mathcal{S}$ is homeomorphic to a half-disc. Figure 4 illustrates a hypermap, which we shall denote by $\mathcal{B}^{\hat{0}}$, with $\mathcal{S}$ a closed disc; there are two blades and five edges (four free); $\mathcal{B}^{\hat{0}}$ has two hypervertices, one hyperedge and one hyperface.

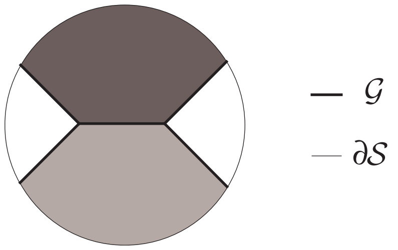


Figure 4. The hypermap $\mathcal{B}^{\hat{0}}$

For later use in constructing branched coverings, we will choose an arbitrary point c in each face F of a hypermap $\mathcal{H}$, called the *centre* of F; the only restriction is that c should lie in $F \cap \partial\mathcal{S}$ if this is non-empty.

3.2. Algebraic hypermaps

Given a hypermap $\mathcal{H}$, each edge of $\mathcal{G}$ can be assigned the label i complementary to the two labels $j, k \neq i$ of its incident faces; such an edge is called an *i-edge* of $\mathcal{H}$. Thus each blade is incident with a unique i-edge for $i = 0, 1$ and 2, so we can define permutations r_0, r_1 and r_2 of the set Ω of blades of $\mathcal{H}$: r_i transposes pairs of blades incident with the same i-edge, and fixes blades incident with a free i-edge. In Figure 5 for example (based on the hypermap $\mathcal{B}^{\hat{0}}$ in Figure 4), r_0 transposes the two blades while r_1 and r_2 fix them.

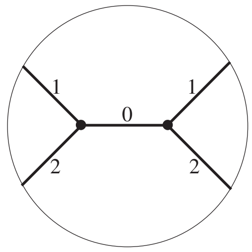


Figure 5. Edge-labelling of $\mathcal{B}^{\hat{0}}$

Clearly $r_i^2 = 1$ for each i, and the connectedness of $\mathcal{G}$ implies that r_0, r_1 and r_2 generate a transitive group G of permutations of Ω. We call $(G, \Omega, r_0, r_1, r_2)$ the *algebraic hypermap* $\mathrm{Alg}(\mathcal{H})$ associated with $\mathcal{H}$ (see [9] for a similar definition where $\mathcal{S}$ is orientable and without boundary). Each hypermap $\mathcal{H}$ determines a transitive permutation representation $\pi : \Delta \to G \leq S^\Omega$ of the group

$$\Delta = \langle R_0, R_1, R_2 \mid R_0^2 = R_1^2 = R_2^2 = 1 \rangle \cong C_2 * C_2 * C_2$$

(a free product of three cyclic groups $\langle R_i \rangle$ of order 2), given by $R_i \mapsto r_i$. We call the subgroup $H \leq \Delta$ fixing a blade a *hypermap subgroup*; these are determined by $\mathcal{H}$ up to conjugacy, and $\mathcal{G}$ can be regarded as the Schreier coset graph for H in Δ with respect to the generators R_i, with free edges replacing loops.

3.3. Equivalence of definitions

This process is reversible: given any transitive permutation representation π of Δ, one can reconstruct $\mathcal{G}$ as the Schreier coset graph of a point-stabiliser $H \leq \Delta$. To reconstruct $\mathcal{H}$ from $\mathcal{G}$, we take a 2-simplex σ_α for each vertex α of $\mathcal{G}$, and label its vertices $0, 1$ and 2 (in any order); we then join the centre c_α of σ_α, by an edge labelled i, to the mid-point of the side jk of σ_α opposite the vertex i, for $i = 0, 1$ and 2. Whenever vertices $\alpha \neq \beta$ of $\mathcal{G}$ are joined by an i-edge, we join σ_α to σ_β by identifying their sides jk so that vertices j and k and mid-points match up; the resulting surface $\mathcal{S} = \bigcup_\alpha \sigma_\alpha$ carries a graph homeomorphic to $\mathcal{G}$, with vertices c_α and with edges labelled $0, 1$ and 2. This is the required hypermap $\mathcal{H}$, the i-faces being the faces incident with edges labelled j and k. (See Figure 6 for the construction of $\mathcal{H} = \mathcal{B}^{\hat{0}}$.)

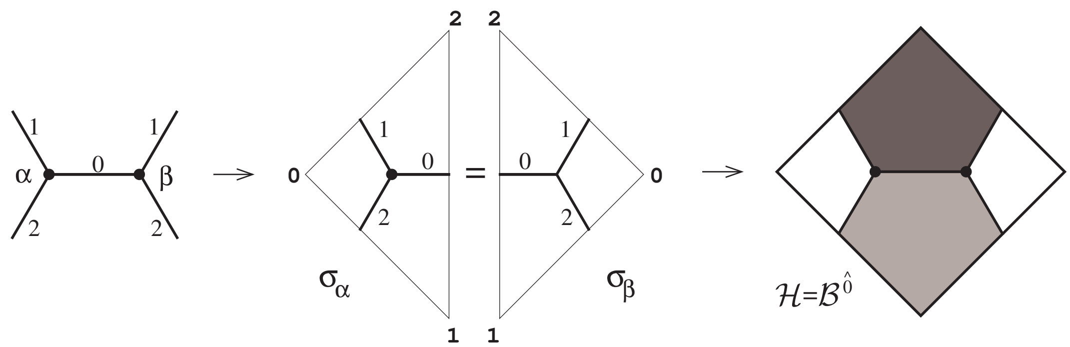


Figure 6. Construction of $\mathcal{B}^{\hat{0}}$

For this reason, it is often convenient to ignore the formal distinctions between the representation π, the algebraic hypermap $\mathrm{Alg}(\mathcal{H})$, the edge-labelled graph $\mathcal{G}$, and the hypermap $\mathcal{H}$: we shall use whichever concept is most suitable for the particular context.

3.4. Valency and type

The *valency* v of an i-face F is the length of the corresponding cycle of $r_j r_k$ on Ω; the edges of F are alternately labelled j and k, and there are $2v$ or $v+1$ of them as F is disjoint from $\partial\mathcal{S}$ or not. We say that $\mathcal{H}$ has *type* (l_0, l_1, l_2) if l_i is the order of $r_j r_k$, that is, the least common multiple of the valencies of the i-faces. A hypermap of type (l_0, l_1, l_2) can be regarded as a transitive permutation representation of the extended triangle group

$$\Delta(l_0, l_1, l_2) = \langle R_0, R_1, R_2 \mid R_i^2 = (R_1R_2)^{l_0} = (R_2R_0)^{l_1} = (R_0R_1)^{l_2} = 1 \rangle.$$

(Note that $\Delta = \Delta(\infty, \infty, \infty)$, where we regard a relation $(R_jR_k)^{\infty} = 1$ as vacuous.)

A hypermap $\mathcal{H}$ with $l_1 = 1$ or 2 is called a *map* [5, 15, 19]; the usual way of representing $\mathcal{H}$ is to contract each hypervertex to a point (now called a *vertex* of $\mathcal{H}$), and each hyperedge to a line-segment (now called an *edge*), as in Figure 7. The hyperfaces of a map are usually called *faces*.

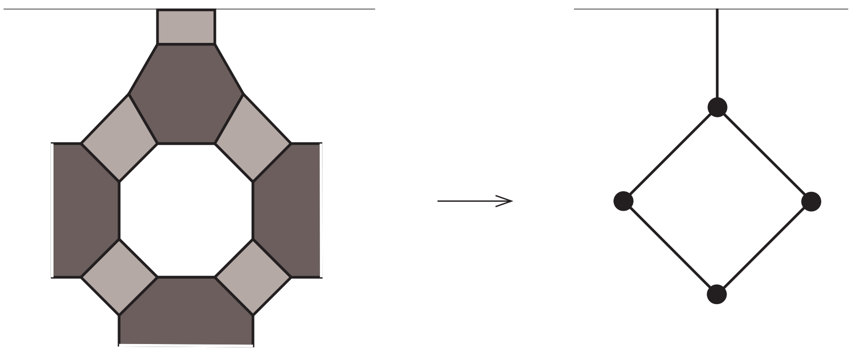


Figure 7. A map

3.5. Automorphisms and coverings

The *automorphism group* $\mathrm{Aut}\,\mathcal{H}$ of a hypermap $\mathcal{H}$ is the group of permutations of Ω commuting with G; it is isomorphic to $N_\Delta(H)/H$, where $N_\Delta(H)$ denotes the normaliser of H in Δ. We say that $\mathcal{H}$ is *reflexible* (the term *regular* is sometimes used, for instance in [21]) if $\mathrm{Aut}\,\mathcal{H}$ acts transitively on Ω; this is equivalent to G acting regularly on Ω, and hence to H being normal in Δ. When $\mathcal{H}$ is reflexible we have

$$\mathrm{Aut}\,\mathcal{H} \cong \Delta/H \cong G.$$

(See Figure 5, where $\mathrm{Aut}\,\mathcal{B}^{\hat{0}} \cong C_2$, for example.)

A *covering*, or morphism $\mathcal{H}_1 \to \mathcal{H}_2$ between two hypermaps is a morphism between their associated permutation representations of Δ, that is, a function $\Omega_1 \to \Omega_2$ between the sets of blades which commutes with the actions of Δ. Coverings correspond to inclusions $H_1 \leq H_2$

between hypermap subgroups, and they induce coverings $\mathcal{G}_1 \to \mathcal{G}_2$ and $\mathcal{S}_1 \to \mathcal{S}_2$ between the underlying graphs $\mathcal{G}_i$ and surfaces $\mathcal{S}_i$ (we allow branch-points and pairwise identifications along boundary-segments). An automorphism of $\mathcal{H}$ is simply a bijective covering of $\mathcal{H}$ by itself.

3.6. Orientability and colourings

A hypermap $\mathcal{H}$ (or more precisely its underlying surface $\mathcal{S}$) is without boundary if and only if $\mathcal{G}$ has no free edges, that is, no R_i fixes a blade. By the torsion theorem for free products (see Theorem 1.6 in §IV.1 of [17]), the conjugates of R_0, R_1 and R_2 are the only non-identity elements of finite order in Δ, so this is equivalent to H being torsion-free. Similarly, $\mathcal{H}$ is orientable and without boundary if and only if its blades can be 2-coloured with each R_i $(i = 0, 1, 2)$ transposing the colours; the two monochrome sets of blades are those around which the orientation induces the cyclic permutations (012) and (210) of the edge-labels.

We say that $\mathcal{H}$ is *i-face bipartite* if Ω can be 2-coloured with R_i transposing the colours and R_j, R_k preserving them $(j, k \neq i)$. Equivalently, the i-faces can be assigned two colours, so that each i-edge joins faces of different colours (thus an i-edge cannot be free, and cannot join an i-face to itself). A 0-face bipartite map or hypermap is often simply called "bipartite" (see Figures 2 and 4, for example). Similarly, $\mathcal{H}$ is *i-edge bipartite* if Ω can be 2-coloured with R_i preserving colours and R_j, R_k transposing them; equivalently, the i-edges can be 2-coloured so that neighbouring i-edges (joined by a j- or k-edge) have different colours (thus a j- or k-edge cannot be free or join an i-edge to itself). If $\mathcal{H}$, of type (l_0, l_1, l_2), is i-face or i-edge bipartite, then l_j and l_k are both even, but not conversely.

3.7. Operations

An *operation* on hypermaps is a transformation of hypermaps induced by an outer automorphism θ of Δ (inner automorphisms induce isomorphisms of hypermaps). If $\mathcal{H}$ has hypermap subgroup H, and is associated with a permutation representation $\pi : \Delta \to S^{\Omega}$, then $\mathcal{H}^{\theta}$ has hypermap subgroup H^{θ}, and is associated with the representation $\theta^{-1} \circ \pi : \Delta \to S^{\Omega}$. Machì [18] has described a group $S \cong S_3$ of six operations on oriented hypermaps without boundary; these correspond to permuting the face-labels $\{0, 1, 2\}$, and extend to six operations on all hypermaps, induced by permuting the three generators R_i of Δ. James [13] has classified all operations on hypermaps, showing that they form an infinite group $\mathrm{Out}\,\Delta \cong PGL_2(\mathbf{Z})$ containing Machì's group S. Operations preserve algebraic properties of hypermaps such as reflexibility, automorphism group, coverings, number of blades, etc., but in general they do not preserve topological properties such as Euler characteristic, orientability, boundary or type. For a survey on operations on maps and hypermaps, see [14].

4. Products of hypermaps

If $\mathcal{H}_1 = (G_1, \Omega_1, r_0, r_1, r_2)$ and $\mathcal{H}_2 = (G_2, \Omega_2, s_0, s_1, s_2)$ are algebraic hypermaps, with hypermap subgroups $H_1, H_2 \leq \Delta$, then there is a natural action of Δ on $\Omega = \Omega_1 \times \Omega_2$. This action is transitive if and only if the pair $\mathcal{H}_1$ and $\mathcal{H}_2$ are *disjoint*, meaning that $\Delta = H_1 H_2$. (When each Ω_i is finite, this is equivalent to the condition that the permutation characters

$g \mapsto |\{\, \alpha \in \Omega_i \mid \alpha g = \alpha \,\}|$ of Δ on Ω_1 and Ω_2 have only the principle irreducible complex character in common; hence the name "disjoint".)

When $\mathcal{H}_1$ and $\mathcal{H}_2$ are disjoint, the transitive action of Δ on Ω corresponds to a third hypermap, with hypermap subgroup $H = H_1 \cap H_2$, which we will call the *disjoint product* $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$ of $\mathcal{H}_1$ and $\mathcal{H}_2$. (This is an extension to hypermaps of the parallel product of orientable maps introduced by Wilson in [23].) The inclusions $H \leq H_1, H_2$ induce coverings $\mathcal{H} \to \mathcal{H}_1, \mathcal{H}_2$, and any hypermap which covers $\mathcal{H}_1$ and $\mathcal{H}_2$ must also cover $\mathcal{H}$.

If $\mathcal{H}_i$ has type (l_i, m_i, n_i) then $\mathcal{H}$ has type (l, m, n), where l is the least common multiple of l_1 and l_2, etc. If θ is an operation on hypermaps, then $\mathcal{H}^\theta = \mathcal{H}_1^\theta \times \mathcal{H}_2^\theta$. If $\overline{\mathcal{H}}_1$ is any covering hypermap of $\mathcal{H}_1$, also disjoint from $\mathcal{H}_2$, then $\overline{\mathcal{H}}_1 \times \mathcal{H}_2$ is a covering of $\mathcal{H}_1 \times \mathcal{H}_2$.

If $\mathcal{H}_1$ and $\mathcal{H}_2$ are both reflexible then each H_i is normal in Δ, so H is normal and hence $\mathcal{H}$ is reflexible. Since $\Delta/H \cong (\Delta/H_1) \times (\Delta/H_2) \cong G_1 \times G_2$, we then have

$$\mathcal{H} = (G_1 \times G_2, \Omega_1 \times \Omega_2, t_0, t_1, t_2)$$

where $t_i = (r_i, s_i)$ for $i = 0, 1, 2$; in particular,

$$\mathrm{Aut}\,\mathcal{H} \cong G_1 \times G_2 \cong \mathrm{Aut}\,\mathcal{H}_1 \times \mathrm{Aut}\,\mathcal{H}_2.$$

5. Reflexible abelian hypermaps

Many examples of double coverings of a hypermap $\mathcal{H}$ can be formed by taking disjoint products of $\mathcal{H}$ with 2-blade hypermaps $\mathcal{B}$. In this section we will classify these hypermaps $\mathcal{B}$, and more generally determine all *reflexible abelian* hypermaps – those reflexible hypermaps with an abelian automorphism group.

A hypermap $\mathcal{B}$ has two blades if and only if its hypermap subgroup B has index 2 in Δ. All such subgroups are normal, so $\mathcal{B}$ is reflexible, with $\mathrm{Aut}\,\mathcal{B} \cong \Delta/B \cong C_2$ which is abelian. More generally, any hypermap $\mathcal{H}$ is reflexible and abelian if and only if its hypermap subgroup H contains the derived group Δ' of Δ. Since $\Delta \cong C_2 * C_2 * C_2$, we have $\Delta/\Delta' = \Delta^{\mathrm{ab}} \cong C_2 \times C_2 \times C_2$, so $|\Delta : \Delta'| = 8$ and there are sixteen subgroups $H \geq \Delta'$, corresponding to the sixteen subgroups of Δ^{ab}: there is one subgroup of index 1, seven each of indices 2 or 4, and one of index 8. Hence there are sixteen reflexible abelian hypermaps $\mathcal{H}$, of which one has 1 blade, seven each have 2 or 4 blades, and one has 8 blades. In each case $\mathrm{Aut}\,\mathcal{H} \cong \Delta/H$ is an elementary abelian 2-group of rank $0, 1, 2$ or 3 respectively.

Since the subgroups $H \geq \Delta'$ can all be formed as intersections (including the empty intersection Δ) of subgroups B of index 2, these sixteen hypermaps $\mathcal{H}$ can all be formed by taking products of 2-blade hypermaps $\mathcal{B}$ (provided we regard the trivial hypermap, with one blade, as the empty product).

In each of the following four sections we give a table describing the reflexible abelian hypermaps $\mathcal{H}$, as $\mathcal{H}$ has $|\Omega| = 1, 2, 4$ or 8 blades respectively. The columns give a symbol for $\mathcal{H}$ (and in some cases a representation for $\mathcal{H}$ as a product), the hypermap subgroup H, topological illustrations of $\mathcal{H}$ (with i-faces black, grey and white for $i = 0, 1$ and 2) and of the bipartite Walsh map $W(\mathcal{H})$, the underlying surface $\mathcal{S}$, the type of $\mathcal{H}$, and finally a description of the hypermaps $\overline{\mathcal{H}}$ covering $\mathcal{H}$ (those with hypermap subgroup $\overline{H} \leq H$).

5.1. $|\Omega| = 1$, $\mathrm{Aut}\,\mathcal{H} = 1$

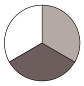
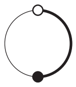
$\mathcal{H}$	H	$\mathcal{H}$	$W(\mathcal{H})$	S	type	$\overline{\mathcal{H}}$
$\mathcal{A}$	Δ			disc	$(1,1,1)$	any hypermap

Table 1. The 1-blade hypermap $\mathcal{A}$

In this case $\mathcal{H}$ is the trivial hypermap $\mathcal{A}$ with one blade, associated with the hypermap subgroup $H = \Delta$.

5.2. $|\Omega| = 2$, $\mathrm{Aut}\,\mathcal{H} \cong C_2$

There are $2^3 - 1 = 7$ epimorphisms $\Delta \cong C_2 * C_2 * C_2 \to C_2$, their kernels giving the seven subgroups B of index 2 in Δ. Let $\langle \ldots \rangle^{\Delta}$ denote normal closure in Δ, let $\{0,1,2\} = \{i,j,k\}$, and let w_i denote the image of an element $w \in \Delta$ under the epimorphism $\Delta \to \mathbf{Z}$, $R_i \mapsto 1$, $R_j, R_k \mapsto 0$. Then the subgroups B are:

$$\Delta^{\hat{i}} = \{\, w \in \Delta \mid w_i \equiv 0 \bmod(2) \,\} = \langle R_j, R_k \rangle^{\Delta},$$
$$\Delta^{i} = \{\, w \in \Delta \mid w_j \equiv w_k \bmod(2) \,\} = \langle R_i, R_j R_k \rangle^{\Delta},$$
$$\Delta^{+} = \{\, w \in \Delta \mid w_0 + w_1 + w_2 \equiv 0 \bmod(2) \,\} = \langle R_0 R_2, R_1 R_2 \rangle^{\Delta},$$

where $i = 0,1,2$. The last group Δ^+, the "even subgroup" of Δ, is a free group of rank 2, with a pair of free generators R_0R_2 and R_1R_2 corresponding to the rotations α and σ around hyperedges and hypervertices in the theory of oriented hypermaps [6, 7, 9]. Associated with these seven subgroups B, we have the following seven 2-blade hypermaps $\mathcal{B}$:

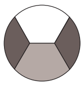
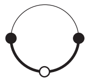
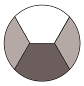
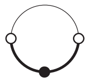
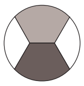
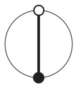
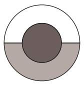
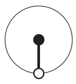
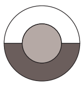
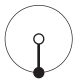
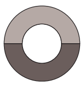
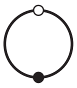
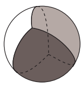
$\mathcal{H}$	H	$\mathcal{H}$	$W(\mathcal{H})$	S	type	$\overline{\mathcal{H}}$
$\mathcal{B}^{\hat{0}}$	$\Delta^{\hat{0}}$			disc	$(1,2,2)$	0-face bipartite
$\mathcal{B}^{\hat{1}}$	$\Delta^{\hat{1}}$			disc	$(2,1,2)$	1-face bipartite
$\mathcal{B}^{\hat{2}}$	$\Delta^{\hat{2}}$			disc	$(2,2,1)$	2-face bipartite
$\mathcal{B}^{0}$	Δ^{0}			disc	$(1,2,2)$	0-edge bipartite
$\mathcal{B}^{1}$	Δ^{1}			disc	$(2,1,2)$	1-edge bipartite
$\mathcal{B}^{2}$	Δ^{2}			disc	$(2,2,1)$	2-edge bipartite
$\mathcal{B}^{+}$	Δ^{+}			sphere	$(1,1,1)$	orientable

Table 2. The 2-blade hypermaps $\mathcal{B}$

5.3. $|\Omega| = 4$, $\mathbf{Aut}\,\boldsymbol{\mathcal{H}} \cong \boldsymbol{C_2} \times \boldsymbol{C_2}$

There are seven epimorphisms $\Delta \to C_2 \times C_2$, and their kernels are:

$$\Delta^{\hat{i}\hat{j}} = \{\, w \in \Delta \mid w_i \equiv w_j \equiv 0 \bmod(2) \,\} = \langle (R_iR_j)^2, R_k \rangle^{\Delta} = \Delta^{\hat{i}} \cap \Delta^{\hat{j}},$$

$$\Delta^{+\hat{i}} = \{\, w \in \Delta \mid w_i \equiv 0,\ w_j \equiv w_k \bmod(2) \,\} = \langle R_jR_k, (R_iR_j)^2 \rangle^{\Delta} = \Delta^{+} \cap \Delta^{\hat{i}},$$

$$\Delta^{012} = \{\, w \in \Delta \mid w_i \equiv w_j \equiv w_k \bmod(2) \,\} = \langle R_iR_jR_k \rangle^{\Delta} = \Delta^{i} \cap \Delta^{j} = \Delta^{0} \cap \Delta^{1} \cap \Delta^{2}.$$

Corresponding to these seven subgroups $C \leq \Delta$ we have the following seven 4-blade hypermaps $\mathcal{C}$:

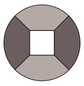
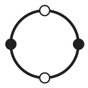
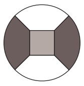
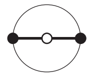
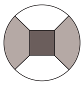
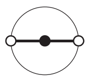
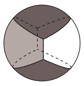
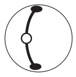
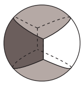
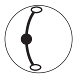
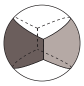
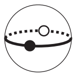
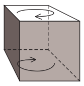
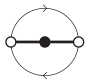
$\mathcal{H}$	H	$\mathcal{H}$	$W(\mathcal{H})$	S	type	$\overline{\mathcal{H}}$
$\mathcal{B}^{\hat{0}\hat{1}}$	$\Delta^{\hat{0}\hat{1}}$			disc	$(2,2,2)$	0-face bipartite 1-face bipartite
$\mathcal{B}^{\hat{0}\hat{2}}$	$\Delta^{\hat{0}\hat{2}}$			disc	$(2,2,2)$	0-face bipartite 2-face bipartite
$\mathcal{B}^{\hat{1}\hat{2}}$	$\Delta^{\hat{1}\hat{2}}$			disc	$(2,2,2)$	1-face bipartite 2-face bipartite
$\mathcal{B}^{+\hat{0}}$	$\Delta^{+\hat{0}}$			sphere	$(1,2,2)$	orientable 0-face bipartite
$\mathcal{B}^{+\hat{1}}$	$\Delta^{+\hat{1}}$			sphere	$(2,1,2)$	orientable 1-face bipartite
$\mathcal{B}^{+\hat{2}}$	$\Delta^{+\hat{2}}$			sphere	$(2,2,1)$	orientable 2-face bipartite
$\mathcal{B}^{012}$	Δ^{012}			projective plane	$(2,2,2)$	i-edge bipartite for $i = 0,1,2$

Table 3. The 4-blade reflexible abelian hypermaps $\mathcal{C}$

In the case of $\mathcal{B}^{012}$, the coverings $\overline{\mathcal{H}}$ are those whose blades can be 4-coloured so that R_0, R_1 and R_2 induce the three double-transpositions of the four colours; by identifying pairs of colours in each of the three possible ways, one easily sees that this is equivalent to $\overline{\mathcal{H}}$ being i-edge bipartite for each $i = 0, 1$ and 2.

5.4. $|\Omega| = 8$, $\mathbf{Aut}\,\boldsymbol{\mathcal{H}} \cong \boldsymbol{C_2} \times \boldsymbol{C_2} \times \boldsymbol{C_2}$

The last of the reflexible abelian hypermaps is the 8-blade hypermap $\mathcal{D}$ corresponding to the derived group

$$\Delta' = \{\, w \in \Delta \mid w_0 \equiv w_1 \equiv w_2 \bmod(2) \,\} = \langle (R_0R_1)^2, (R_1R_2)^2, (R_2R_0)^2 \rangle^{\Delta} = \Delta^{\hat{0}} \cap \Delta^{\hat{1}} \cap \Delta^{\hat{2}}.$$

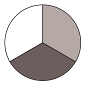
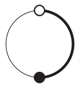
$\mathcal{H}$	H	$\mathcal{H}$	$W(\mathcal{H})$	S	type	$\overline{\mathcal{H}}$
$\mathcal{D}$	$\Delta^{\hat{0}\hat{1}\hat{2}}$			sphere	$(2,2,2)$	orientable and i-face bipartite for $i=0,1,2$

Table 4. The 8-blade reflexible abelian hypermap $\mathcal{D}$

5.5. Coverings

The inclusions between the various subgroups $H \geq \Delta'$ are those shown in Figure 8:

$$\begin{tikzcd}[column sep=small] & \Delta \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ \Delta^{\hat{i}} \arrow[dr, dash] & \Delta^{\hat{j}} \arrow[d, dash] & \Delta^{k} \arrow[dl, dash] \\ & \Delta^{\hat{i}\hat{j}} \arrow[d, dash] & \\ & \Delta' & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & \Delta \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ \Delta^{+} \arrow[dr, dash] & \Delta^{i} \arrow[d, dash] & \Delta^{\hat{i}} \arrow[dl, dash] \\ & \Delta^{+\hat{i}} \arrow[d, dash] & \\ & \Delta' & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & \Delta \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ \Delta^{0} \arrow[dr, dash] & \Delta^{1} \arrow[d, dash] & \Delta^{2} \arrow[dl, dash] \\ & \Delta^{012} \arrow[d, dash] & \\ & \Delta' & \end{tikzcd}$$

Figure 8. Inclusions

Hence $\mathcal{B}^{\hat{i}\hat{j}}$ covers $\mathcal{B}^{\hat{i}}, \mathcal{B}^{\hat{j}}$ and $\mathcal{B}^{k}$; similarly $\mathcal{B}^{+\hat{i}}$ covers $\mathcal{B}^{+}, \mathcal{B}^{i}$ and $\mathcal{B}^{\hat{i}}$, while $\mathcal{B}^{012}$ covers $\mathcal{B}^{0}, \mathcal{B}^{1}$ and $\mathcal{B}^{2}$. All of these hypermaps cover $\mathcal{A}$ and are covered by $\mathcal{D}$.

6. Double coverings

We now describe the double coverings $\mathcal{H} \times \mathcal{B} \to \mathcal{H}$ induced by the seven 2-blade hypermaps $\mathcal{B}$. Each such $\mathcal{B}$ is reflexible, so if $\mathcal{H}$ is reflexible then so is $\mathcal{H} \times \mathcal{B}$.

The group $\operatorname{Out}\Delta \cong PGL_2(\mathbf{Z})$ of all operations on hypermaps, determined by James in [13], acts on the quotient group $\Delta/\Delta' \cong C_2 \times C_2 \times C_2$ by permuting the three factors $\langle R_i\Delta'\rangle \cong C_2$ (since the conjugates of the generators R_i are the only elements of order 2 in Δ). It permutes them transitively, so it has three orbits on the set of subgroups B of index 2 in Δ, namely $\{\,\Delta^{\hat{i}} \mid i = 0,1,2\,\}, \{\,\Delta^{i} \mid i = 0,1,2\,\}$ and $\{\,\Delta^{+}\,\}$. Consequently $\operatorname{Out}\Delta$ has orbits $\{\,\mathcal{B}^{\hat{i}} \mid i = 0,1,2\,\}, \{\,\mathcal{B}^{i} \mid i = 0,1,2\,\}$ and $\{\,\mathcal{B}^{+}\,\}$ on the 2-blade hypermaps $\mathcal{B}$, and it is sufficient to consider one representative from each orbit, namely $\mathcal{B}^{\hat{0}}, \mathcal{B}^{0}$ and $\mathcal{B}^{+}$. Indeed, Machì's subgroup $S \cong S_3$ of label-permuting operations (see Section 3.7) also has these orbits, so we can obtain all the 2-blade hypermaps and their associated double coverings from these three by permuting the labels $i = 0,1$ and 2 of the faces.

We will apply these double coverings to a hypermap $\mathcal{H} = (G, \Omega, r_0, r_1, r_2)$ of type (l,m,n), with hypermap subgroup $H \leq \Delta$. We will assume throughout that $\mathcal{H}$ and $\mathcal{B}$ are disjoint (equivalently $H \not\leq B$), so that $\mathcal{H} \times \mathcal{B}$ is connected; in the cases where $\mathcal{H}$ and $\mathcal{B}$ are not disjoint (that is, $\mathcal{H}$ covers $\mathcal{B}$), the constructions we will describe would give rise to two disjoint copies of $\mathcal{H}$.

6.1. $\mathcal{B} = \mathcal{B}^{\hat{i}}$

By applying a permutation of the face-labels we can assume that $i = 0$. We will assume that $\mathcal{H}$ and $\mathcal{B}^{\hat{0}}$ are disjoint (equivalently $H \not\leq \Delta^{\hat{0}}$), so $\mathcal{H}$ is not bipartite. As an algebraic hypermap, the double covering $\mathcal{H}^{\hat{0}} := \mathcal{H} \times \mathcal{B}^{\hat{0}}$ has the form $(G \times C_2, \Omega \times C_2, s_0, s_1, s_2)$ where $C_2 = \{\pm 1\}, s_0 = (r_0, -1), s_1 = (r_1, 1)$ and $s_2 = (r_2, 1)$.

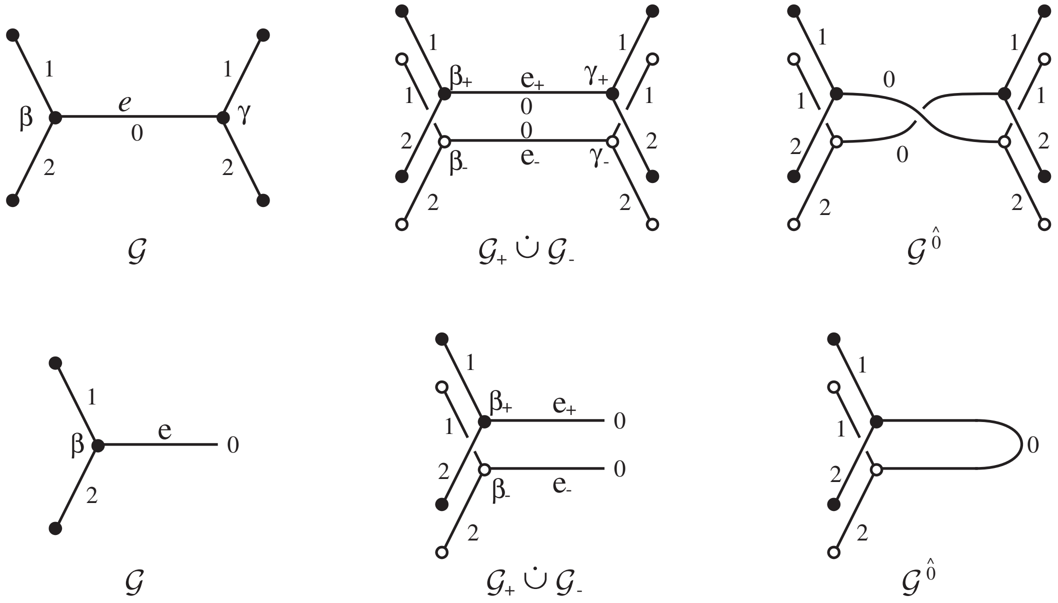


Figure 9. Construction of $\mathcal{G}^{\hat{0}}$

The construction of the underlying graph $\mathcal{G}^{\hat{0}}$ of $\mathcal{H}^{\hat{0}}$, illustrated in Figure 9, is based on the fact that R_0 transposes the two blades of $\mathcal{B}^{\hat{0}}$ while R_1 and R_2 fix them. We take two copies $\mathcal{G}_+$ and $\mathcal{G}_-$ of the underlying graph $\mathcal{G}$ of $\mathcal{H}$. Whenever a 0-edge e of $\mathcal{G}$ joins disjoint blades β, γ of $\mathcal{H}$ we cut the corresponding 0-edges e_+ and e_- of $\mathcal{G}_+$ and $\mathcal{G}_-$, and rejoin them as in Figure 9(a) so that they now join β_+ to γ_- and β_- to γ_+. Similarly, if e is any free 0-edge in $\mathcal{G}$, incident with a blade β of $\mathcal{H}$, we join e_+ to e_- thus creating a single 0-edge joining β_+ to β_-, as in Figure 9(b). The resulting edge-labelled graph $\mathcal{G}^{\hat{0}}$ is connected, for otherwise it would have two components, giving rise to a 2-colouring of the hypervertices of $\mathcal{H}$, against our assumption that $\mathcal{H}$ is not bipartite. We can therefore take $\mathcal{H}^{\hat{0}}$ to be the hypermap corresponding to $\mathcal{G}^{\hat{0}}$.

The natural isomorphisms $\mathcal{G}_+ \cong \mathcal{G} \cong \mathcal{G}_-$ induce a double covering $\mathcal{G}^{\hat{0}} \to \mathcal{G}$ which, in turn, induces the double covering $\mathcal{H}^{\hat{0}} \to \mathcal{H}$. One can see that $\mathcal{H}^{\hat{0}}$ is bipartite (that is, 0-face bipartite) by colouring the vertices of $\mathcal{G}_+$ and $\mathcal{G}_-$ respectively black and white (as in Figure 9), and observing that in $\mathcal{G}^{\hat{0}}$ the colours are transposed by R_0 but preserved by R_1 and R_2.

We can reinterpret the above construction of $\mathcal{H}^{\hat{0}}$ in terms of surface topology by taking two copies $\mathcal{H}_+$ and $\mathcal{H}_-$ of $\mathcal{H}$, and cutting them across each non-free 0-edge between the centres of the incident 1-face and 2-face, as shown in Figure 10.

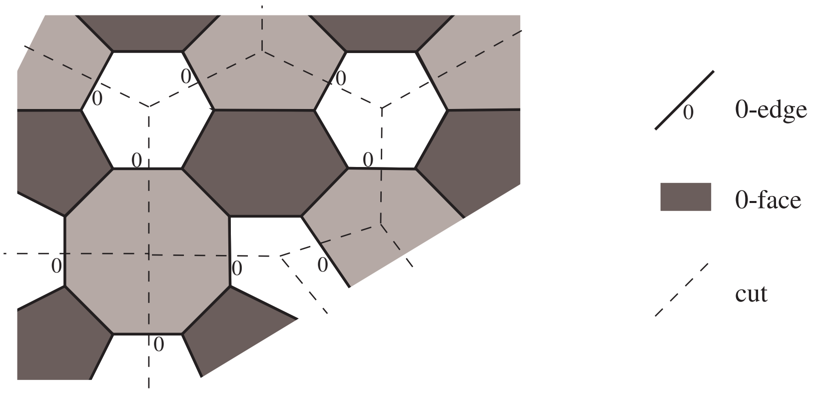


Figure 10. Construction of $\mathcal{H}^{\hat{0}}$

We then join $\mathcal{H}_+$ and $\mathcal{H}_-$ across these cuts (just as in the standard construction of the 2-sheeted Riemann surface of $\sqrt{z}$, see Figure 11), and also along boundary segments between the centres of adjacent 1-faces and 2-faces (see Figure 12).

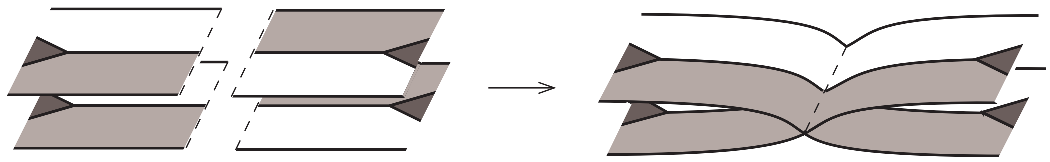


Figure 11. Joining across cuts

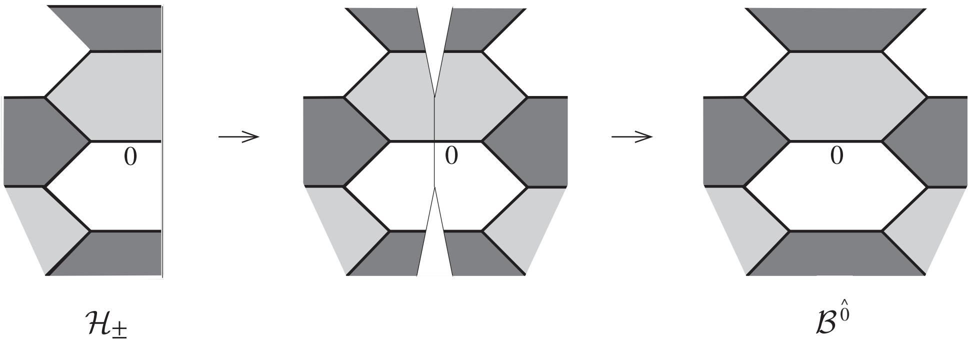


Figure 12. Joining along boundary segments

Away from the boundary, the resulting surface is a 2-sheeted branched covering of $\mathcal{H}$. The branch-points are at the centres c of those 1-faces and 2-faces F of $\mathcal{H}$ whose valency v is odd: this is because v cuts meet at c, so a rotation around c induces the permutation $(-1)^v \in C_2 \cong S_2$ of the two sheets, and this is non-trivial if and only if v is odd. Such a face F (with v odd) lifts to a 1- or 2-face of valency $2v$ in $\mathcal{H}^{\hat{0}}$, whereas when v is even it lifts to two disjoint faces of valency v. Thus $\mathcal{H}^{\hat{0}}$ has type (l, m', n') where m' is the least common multiple of m and 2, and likewise for n'.

For example, if $\mathcal{H}$ is the tetrahedron $\mathcal{T}$, a reflexible hypermap of type $(3,2,3)$, then $\mathcal{H}^{\hat{0}}$ is the reflexible hypermap $\mathcal{W}$ of type $(3,2,6)$ described in Section 1 and illustrated (as a bipartite map) in Figure 2.

6.2. $\mathcal{B} = \mathcal{B}^i$

As in Section 6.1 we can assume that $i = 0$. The construction of $\mathcal{G}^0$ is similar to that for $\mathcal{G}^{\hat{0}}$, except that now we cut and rejoin the 1-edges and 2-edges of $\mathcal{G}_+ \cup \mathcal{G}_-$, and likewise we join pairs of free 1- and 2-edges, but we do nothing to the 0-edges. In terms of the surfaces, we cut and rejoin $\mathcal{H}_+$ and $\mathcal{H}_-$ across all 1- and 2-edges (between centres of incident faces), and also join along segments of the boundary between the centre of each 0-face and the centres of adjacent 1-faces and 2-faces. Away from the boundary, $\mathcal{H}^0$ is a 2-sheeted branched covering of $\mathcal{H}$. If c is the centre of a j-face F of valency v, then $2v$ cuts meet at c if $j = 0$, so there is no branching at c; however, if $j = 1$ or 2 then v cuts meet at c, so c is a branch-point if and only if v is odd, in which case F lifts to a j-face of valency $2v$. Thus $\mathcal{H}^0$ has type (l, m', n') where m' and n' are as in Section 6.1. By colouring the vertices of $\mathcal{G}_+$ and $\mathcal{G}_-$ black and white, one can see that $\mathcal{H}^0$ is 0-edge bipartite.

It is not hard to see that if $\mathcal{H}$ is orientable and without boundary then $\mathcal{H}^0 \cong \mathcal{H}^{\hat{0}}$: one can continuously deform the cuts used for $\mathcal{H}^0$ until they coincide with those for $\mathcal{H}^{\hat{0}}$. Algebraically, this is because $H \leq \Delta^+$ with $\Delta^+ \cap \Delta^0 = \Delta^+ \cap \Delta^{\hat{0}}$ $(= \Delta^{+\hat{0}}$, see Figure 8), so that $H \cap \Delta^0 = H \cap \Delta^+ \cap \Delta^0 = H \cap \Delta^+ \cap \Delta^{\hat{0}} = H \cap \Delta^{\hat{0}}$. Similarly $\mathcal{H}^1 \cong \mathcal{H}^{\hat{1}}$ and $\mathcal{H}^2 \cong \mathcal{H}^{\hat{2}}$ for such hypermaps $\mathcal{H}$.

6.3. $\mathcal{B} = \mathcal{B}^+$

Since each R_i acts non-trivially on the two blades of $\mathcal{B}^+$, we form $\mathcal{G}^+$ from two copies of $\mathcal{G}$ by cutting and rejoining all corresponding pairs of non-free edges, and joining all corresponding pairs of free edges. Equivalently, we cut two copies of $\mathcal{H}$ across all non-free i-edges $(i = 0, 1, 2)$, and then join across the cuts and along all the boundary segments. Since $2v$ cuts meet at the centre of an i-face of valency v, there are no branch-points, and $\mathcal{H}^+$ has type (l, m, n). Clearly $\mathcal{H}^+$ has no boundary, and moreover $\mathcal{H}^+$ is orientable: one can use the cyclic orderings (012) and (210) of i-edges around the blades of the two copies of $\mathcal{H}$ to define an orientation on $\mathcal{H}^+$, which is thus the orientable double covering of $\mathcal{H}$.

6.4. Other abelian coverings

Although it is possible to give similar direct constructions for the 4- and 8-fold coverings induced by taking products with the remaining eight non-trivial reflexible abelian hypermaps, it is simpler to observe that such hypermaps are themselves products of 2-blade hypermaps, so the corresponding coverings can be obtained as compositions of two or three of the double coverings described above.

6.5. Coverings which are not disjoint products

Just as not every group extension is a direct product, not every covering between reflexible hypermaps is a disjoint product. We have concentrated on the latter class of coverings because they are widely applicable and easy to describe, but for a more general theory of coverings see [21, 23].

There are many examples of coverings which are not disjoint products, and here we will describe one infinite family. Let $\mathcal{H}_n$ be the universal hypermap of type $(2,2,n)$, a reflexible spherical map with two faces (the north and south hemispheres) separated by an equatorial circuit of n vertices and n edges, as in Figure 13. Its automorphism group is the extended triangle group

$$\mathrm{Aut}\,\mathcal{H}_n = \Delta(2,2,n) \cong D_n \times C_2,$$

with D_n (the dihedral group of order $2n$) acting as the rotation group $\mathrm{Aut}^+\mathcal{H}_n$, and the second factor generated by the reflection in the equatorial plane.

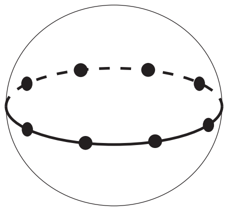


Figure 13. The hypermap $\mathcal{H}_8$

Now consider $\mathcal{H}_{2n}$. Rotation by π about the vertical axis is an automorphism of $\mathcal{H}_{2n}$, which commutes with all other automorphisms, so it generates a central subgroup $C \cong C_2$ in $\mathrm{Aut}\,\mathcal{H}_{2n}$. By identifying points equivalent under C, we obtain a double covering

$$\mathcal{H}_{2n} \to \mathcal{H}_{2n}/C \cong \mathcal{H}_n,$$

with branch-points of order 1 at the two face-centres of $\mathcal{H}_{2n}$. This corresponds to the group extension

$$1 \to C \to \Delta(2,2,2n) \to \Delta(2,2,n) \to 1$$

induced by the epimorphism $D_{2n} \to D_{2n}/C \cong D_n$.

If n is odd, this extension splits: $\mathcal{H}_{2n}$ is bipartite, and we have $D_{2n} = D_n \times C$ and $\Delta(2,2,2n) = \Delta(2,2,n) \times C$, with the first factors preserving the two vertex-colours while C transposes them. This gives rise to a disjoint product decomposition

$$\mathcal{H}_{2n} \cong \mathcal{H}_n \times \mathcal{C},$$

where $\mathcal{H}_n \cong \mathcal{H}_{2n}/C$ and $\mathcal{C} \cong \mathcal{H}_{2n}/\Delta(2,2,n) \cong \mathcal{B}^{\hat{0}}$, so $\mathcal{H}_{2n} \cong \mathcal{H}_n^{\hat{0}}$.

If n is even, however, the extension does not split, since each of the seven subgroups of index 2 in $\Delta(2,2,n)$ contains C (which now preserves vertex-colours); it follows that in this case the double covering $\mathcal{H}_{2n} \to \mathcal{H}_n$ cannot arise from any disjoint product decomposition. (Note that $\mathcal{H}_n$ is now bipartite, so it is not disjoint from $\mathcal{B}^{\hat{0}}$.)

7. The Walsh and Vince correspondences

The Walsh map $W(\mathcal{H})$ [22] of a hypermap $\mathcal{H}$ is the dual of the map formed by contracting each hyperface of $\mathcal{H}$ to a point; it is a bipartite map, the two monochrome sets of vertices (usually coloured black and white) corresponding to the hypervertices and hyperedges of $\mathcal{H}$, while the faces and edges of $W(\mathcal{H})$ correspond to the hyperfaces and 2-edges of $\mathcal{H}$. Corn and Singerman [9] have given an algebraic interpretation of W for orientable hypermaps without boundary; we shall extend their ideas to all hypermaps, and give a similar interpretation of a representation of hypermaps due to Vince [20].

Each map $\mathcal{M}$ corresponds to a conjugacy class of map subgroups $M \leq \Gamma$, where

$$\Gamma = \Delta(\infty, 2, \infty) = \langle P_0, P_1, P_2 \mid P_i^2 = (P_0P_2)^2 = 1\rangle,$$

under the epimorphism $\alpha : \Delta \to \Gamma, R_i \mapsto P_i$. In particular, the 2-blade hypermap $\mathcal{B}^{\hat{0}} = W(\mathcal{A})$ is a map, with map subgroup $\Gamma^{\hat{0}} = \alpha(\Delta^{\hat{0}})$ of index 2 in Γ. Thus $\Gamma^{\hat{0}}$ is the normal closure of P_1 and P_2 in Γ, and the Reidemeister-Schreier process [17, §II.4] shows that it has a presentation

$$\Gamma^{\hat{0}} = \langle Q_0, Q_1, Q_2 \mid Q_i^2 = 1\rangle$$

where $Q_0 = P_0P_1P_0,\ Q_1 = P_1$ and $Q_2 = P_2$, so there is an isomorphism $\phi : \Delta \to \Gamma^{\hat{0}}$ given by $R_i \mapsto Q_i$. Each hypermap $\mathcal{H}$ determines a hypermap subgroup $H \leq \Delta$; we define $\Phi(\mathcal{H})$ to be the map $\mathcal{M}$ corresponding to the map subgroup $M = \phi(H) \leq \Gamma$. It is bipartite since $M \leq \Gamma^{\hat{0}}$ (see Figure 14).

$$\begin{tikzcd}
 & & \Gamma \arrow[d, dash, "2"'] & \\
 & \Delta \arrow[r, "\phi"] \arrow[d, dash] & \Gamma^{\hat{0}} \arrow[d, dash] & \\
\mathcal{H} & H \arrow[r, "\phi"] \arrow[d, dash] & M \arrow[d, dash] & \mathcal{M} = \phi(\mathcal{H}) \\
 & 1 & 1 &
\end{tikzcd}$$

Figure 14. The homomorphism ϕ

Intuitively, one can regard $\Phi(\mathcal{H})$ as a covering of $\mathcal{B}^{\hat{0}}$ with fibre $\mathcal{H}$; we shall show that $\Phi(\mathcal{H})$ is, in fact, the Walsh map $W(\mathcal{H})$.

Each blade β of $\mathcal{H}$ corresponds to a coset Hg $(g \in \Delta)$ of H in Δ; we let β' and β'' (coloured black and white respectively) be the blades of $\mathcal{M}$ corresponding to the cosets $M\phi(g)$ and $M\phi(g)P_0$ of M in Γ (notice that $P_0 \in \Gamma \setminus \Gamma^{\hat{0}}$). Since P_0, acting by right-multiplication, transposes these two cosets, we join β' to β'' by an edge labelled 0. Similarly P_2 sends

$M\phi(g)$ to $M\phi(g)P_2 = \phi(HgR_2)$ and $M\phi(g)P_0$ to $M\phi(g)P_0P_2 = M\phi(g)P_2P_0 = \phi(HgR_2)P_0$, so if γ denotes the blade βR_2 of $\mathcal{H}$ corresponding to the coset HgR_2 then we join β' to γ' and β'' to γ'' by edges labelled 2. Finally, P_1 sends $M\phi(g)$ to $\phi(HgR_1)$ and $M\phi(g)P_0$ to $M\phi(g)P_0P_1 = M\phi(g)Q_0P_0 = \phi(HgR_0)P_0$, so if $\delta = \beta R_1$ and $\epsilon = \beta R_0$ then we join β' to δ' and β'' to ϵ'' by edges labelled 1. This gives us the edge-labelled graph underlying the hypermap $\mathcal{M}$, and as can be seen in Figure 15(a), $\mathcal{M}$ is a 0-face bipartite hypermap, with black and white 0-faces corresponding to the hypervertices and hyperedges of $\mathcal{H}$, while the 1-faces and 2-faces of $\mathcal{M}$ correspond to the 2-edges and hyperfaces of $\mathcal{H}$. Figure 15(b) shows $\mathcal{M}$ as a bipartite map superimposed on $\mathcal{H}$; clearly $\mathcal{M} = W(\mathcal{H})$.

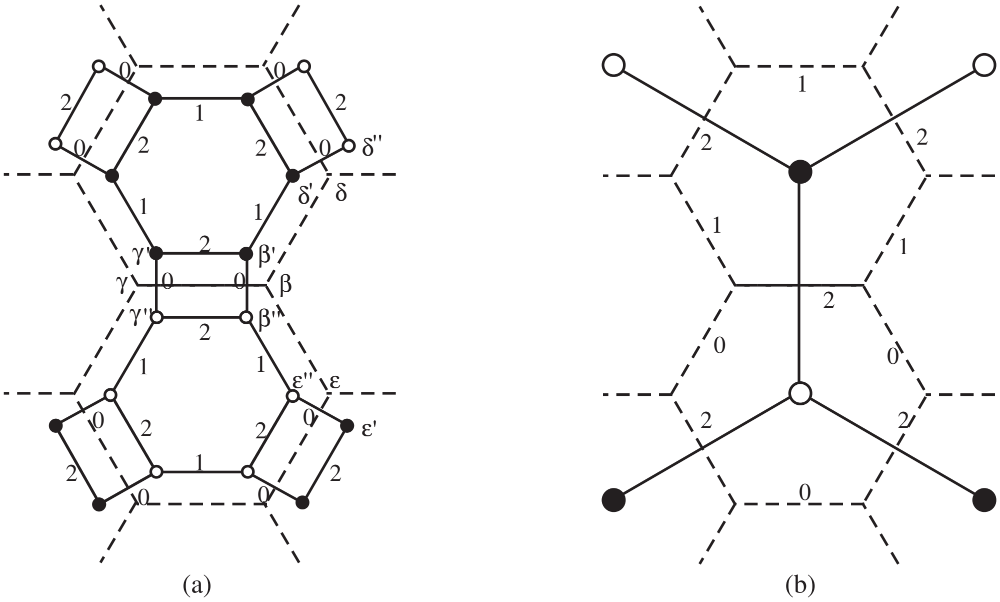


Figure 15. $\mathcal{M}$ and $\mathcal{H}$

One can reverse this process: any map $\mathcal{M}$ corresponds to a subgroup $M \leq \Gamma$, and one can define $\Phi^{-1}(\mathcal{M})$ to be the hypermap $\mathcal{H}$ corresponding to $H = \phi^{-1}(M) \leq \Delta$. If $\mathcal{M}$ is bipartite then $M \leq \Gamma^{\hat{0}}$ and $\mathcal{H} = W^{-1}(\mathcal{M})$; in particular, if $\mathcal{M}$ is reflexible then so is $\mathcal{H}$ (since if M is normal in Γ then H is normal in Δ), with $\operatorname{Aut}\mathcal{H}$ isomorphic to a subgroup of index 2 in $\operatorname{Aut}\mathcal{M}$. If $\mathcal{M}$ is not bipartite then $M \cap \Gamma^{\hat{0}}$, of index 2 in M, corresponds to the bipartite double cover $\mathcal{M}^{\hat{0}} = \mathcal{M} \times \mathcal{B}^{\hat{0}}$ of $\mathcal{M}$ described in Section 6.1, and now $\mathcal{H} = W^{-1}(\mathcal{M}^{\hat{0}})$; again, if $\mathcal{M}$ is reflexible then so is $\mathcal{H}$, but now (see Figure 16) we have

$$\operatorname{Aut}\mathcal{H} \cong \Delta/H \cong \Gamma^{\hat{0}}/(M \cap \Gamma^{\hat{0}}) \cong \Gamma/M \cong \operatorname{Aut}\mathcal{M}.$$

For example, if we take $\mathcal{M}$ to be the tetrahedron $\mathcal{T}$ (which is reflexible but not bipartite) then $\Phi^{-1}(\mathcal{M})$ is the reflexible hypermap $\mathcal{T}'$ constructed in Section 2, with $\operatorname{Aut}\mathcal{T}' \cong \operatorname{Aut}\mathcal{T}$.

Finally, we note that many other transformations and representations of hypermaps (such as all operations on hypermaps) can be explained in a similar way in terms of homomorphisms between extended triangle groups. For example Vince [20] represents a hypermap $\mathcal{H}$ as a 2-face bipartite map $V(\mathcal{H})$ formed by contracting each hypervertex of $\mathcal{H}$ to a point. (Thus $V(\mathcal{H}^{(02)}) = W(\mathcal{H})^{(02)}$ where the superscript (02) denotes the classical duality operation for

$$
\begin{tikzcd}
 & & & \Gamma \arrow[dd, no head] & \\
 & \Delta \arrow[r, "\phi"] \arrow[dd, no head] & \Gamma^{\hat{0}} \arrow[ur, no head, "2"'] \arrow[dd, no head] & & Aut\,\mathcal{M} \\
Aut\,\mathcal{H} & & & M & \\
 & H \arrow[r, "\phi"] \arrow[d, no head] & M \cap \Gamma^{\hat{0}} \arrow[ur, no head, "2"'] \arrow[d, no head] & & \\
 & 1 & 1 & &
\end{tikzcd}
$$

Figure 16.

maps and hypermaps, transposing the labels 0 and 2.) Just as W is induced by $\phi : \Delta \cong \Gamma^{\hat{0}} < \Gamma$, Vince's correspondence V is induced by an isomorphism between Δ and the subgroup $\Gamma^{\hat{2}} = \langle P_0, P_1 \rangle^{\Gamma}$ of index 2 in Γ. There is, in fact, a third subgroup of index 2 in Γ which is isomorphic to Δ, namely the normal closure Γ^1 of P_1 and P_0P_2. By choosing an isomorphism from Δ to Γ^1 we obtain a third representation of hypermaps by maps; in this case, rather than the vertices or the faces of the map, it is the Petrie polygons (closed zig-zag paths turning alternately left and right) which can be 2-coloured so that every edge meets one of each colour. For similar examples, where $\mathcal{H}$ is orientable and without boundary, see Chapter 6 of [8].

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