Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 41 (2000), No. 2, 411-416.

On Rhombic Dodecahedra

Károly Bezdek*

Eötvös University, Institute of Mathematics II, Department of Geometry
H-1088 Budapest, Rákóczi út 5, Hungary
e-mail: kbezdek@ludens.elte.hu
and
Cornell University, Department of Mathematics
Malott Hall, Ithaca, NY 14853-7901, USA
e-mail: bezdek@math.cornell.edu

Abstract. In this note we prove that the intrinsic i-volume of any d-dimensional zonotope generated by $d+1$ (resp. d) line segments and containing a d-dimensional unit ball in $\mathbf{E^d}$ is at least as large as the intrinsic i-volume of the d-dimensional regular zonotope generated by $d+1$ line segments having inradius 1, where $i = 1, \ldots, d-1, d$.

MSC 2000: 52C17, 52A40, 52B60, 52A38
Keywords: zonotopes, parallelohedra, rhombic dodecahedra, lattice sphere packings

0. Introduction

According to a well-known theorem of Gauss [1] the density of any lattice packing of unit spheres in the 3-dimensional Euclidean space $\mathbf{E^3}$ is at most $\frac{\pi}{\sqrt{12}} = 0.7404\ldots$ and equality holds for the lattice packing in which the unit spheres are centered at the points $(a\sqrt{2}, b\sqrt{2}, c\sqrt{2})$, where a, b and c are integers and their sum is even. One can easily see that in this case the Voronoi cells are regular rhombic dodecahedra that generate a face-to-face lattice tiling of $\mathbf{E^3}$. In general, a convex d-dimensional polytope of the d-dimensional Euclidean space $\mathbf{E^d}$ that tiles $\mathbf{E^d}$ by translation is called a *parallelohedron*. Venkov [6] and later independently, McMullen [3] proved that any d-dimensional parallelohedron admits (uniquely) a face-to-face lattice tiling of $\mathbf{E^d}$. Putting these results together one can claim that the volume of any 3-dimensional parallelohedron of inradius at least 1 is at least as

* The author was partially supported by the Hung. Nat. Sci. Found. (OTKA), grant no. T029786 and by the Combinatorial Geometry Project of the Research Found. FKFP0151/1999.

large as the volume of a regular rhombic dodecahedron of inradius 1. Recall (see for example [5]) that there are 5 combinatorial types of parallelohedra in $\mathbf{E^3}$ namely, affine cubes, hexagonal prisms, rhombic dodecahedra, elongated dodecahedra and truncated octahedra (Figure 1). (Also, recall that a rhombic dodecahedron whose faces are congruent rhombi

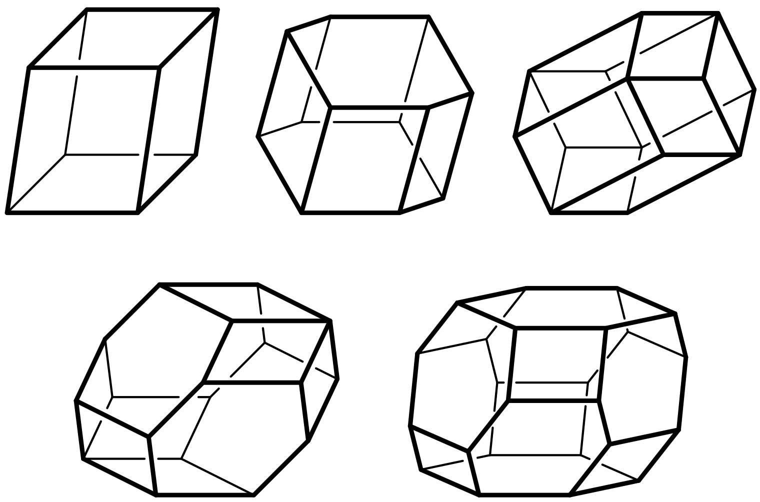

Figure 1

and whose vertex figures are regular polygons is called a regular rhombic dodecahedron.) Finally, we mention the well-known fact (see [5]) that all five parallelohedra in $\mathbf{E^3}$ are zonotopes (that is they are vector sums of line segments). We raise the following problem, an affirmative answer to which would obviously imply the above classical result of Gauss.

The Rhombic Dodecahedral Conjecture. The surface area of any 3-dimensional parallelohedron of inradius at least 1 in $\mathbf{E^3}$ is at least as large as $12\sqrt{2} = 16.9705\ldots$ the surface area of a regular rhombic dodecahedron of inradius 1.

In order to phrase the main result of this note properly we have to recall the following. Let $K \subset \mathbf{E^d}$ be a convex body (i.e. a compact convex set with nonempty interior in $\mathbf{E^d}$). Let ω_i denote the i-dimensional volume of the unit i-ball, $0 \le i \le d$. Then the *intrinsic i-volume* $V_i(K)$ of K can be defined via Steiner's formula

$$\mathrm{Vol}_d(K + \rho B^d) = \sum_{i=0}^{d} \omega_i V_{d-i}(K)\rho^i,$$

where $\rho > 0$ is an arbitrary positive real number and ρB^d denotes the closed ball of radius ρ centered at the origin $\mathbf{o}$ of $\mathbf{E^d}$ and $K + \rho B^d$ denotes the vector sum of the convex bodies K and ρB^d with d-dimensional volume $\mathrm{Vol}_d(K + \rho B^d)$. It is well-known (see for example [4]) that $\mathrm{Vol}_d(K)$ is the d-dimensional volume of K, $2\mathrm{Vol}_{d-1}(K)$ is the surface area of K and $\frac{2\omega_{d-1}}{d\omega_d}\mathrm{V}_1(K)$ is equal to the mean width of K. (Moreover, $V_0(K) = 1$.) Finally, the d-dimensional zonotope Z generated by $d+1$ line segments in $\mathbf{E^d}$ is called *regular* if Z

can be generated by the segments connecting the center of a regular d-dimensional simplex with its vertices.

Theorem. *The intrinsic i-volume of any d-dimensional zonotope generated by $d+1$ (resp., d) line segments and containing a d-dimensional unit ball in $\mathbf{E^d}$ is at least as large as the intrinsic i-volume of the d-dimensional regular zonotope generated by $d+1$ line segments having inradius 1, where $i = 1, \ldots, d-1, d$.*

The following is an immediate corollary that on the one hand, generalizes a result of Linhart [2] on the inradii of rhombic dodecahedra on the other hand, supports an affirmative answer to the Rhombic Dodecahedral Conjecture.

Corollary. *The mean width (resp., surface area, volume) of any rhombic dodecahedron containing a ball of radius 1 in $\mathbf{E^3}$ is at least as large as the mean width (resp., surface area, volume) of the regular rhombic dodecahedron of inradius 1.*

1. Proof of the Theorem

The following lemma plays a key role in our proof of the theorem. The special case of the lemma having four equal line segments in $\mathbf{E^3}$ has been proved by Linhart [2] several years ago. Our method of the proof presented below is different from the method introduced for $d = 3$ in [2].

Lemma. *The inradius of any d-dimensional zonotope generated by $d+1$ line segments of total length $s > 0$ in $\mathbf{E^d}$, $d \geq 1$ is at most as large as the inradius of the d-dimensional regular zonotope generated by $d+1$ line segments of total length s.*

Proof. Let Z be an arbitrary d-dimensional zonotope generated by $d+1$ line segments of total length $s > 0$ in $\mathbf{E^d}$. Without loss of generality we may assume that Z is generated by the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{d+1}$ of total length s that positively span $\mathbf{E^d}$. Thus,

$$Z = \{\mathbf{z} \in \mathbf{E^d} \mid \mathbf{z} = \sum_{i=1}^{d+1} \lambda_i \mathbf{v}_i, 0 \leq \lambda_i \leq 1, 1 \leq i \leq d+1\}, \tag{1}$$

$$\sum_{i=1}^{d+1} ||\, \mathbf{v}_i|| = s, \tag{2}$$

$$\mathbf{o} \in \mathrm{int}[\mathrm{conv}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{d+1}\}], \tag{3}$$

where $||\ldots||, \mathbf{o}, \mathrm{int}[\ldots]$, and $\mathrm{conv}\{\ldots\}$ stand for the norm of a vector, the origin of $\mathbf{E^d}$, the interior of a set in $\mathbf{E^d}$ and for the convex hull of a set in $\mathbf{E^d}$ (Figure 2). Now Z is centrally symmetric. Moreover, it follows from the above construction that the pairs of opposite facets of Z are

$$\{F_{ij}^i, F_{ij}^j\}, 1 \leq i < j \leq d+1, \tag{4}$$

where

$$F_{ij}^{i} = \mathbf{v}_i + F_{ij}, F_{ij}^{j} = \mathbf{v}_j + F_{ij} \tag{5}$$

with F_{ij} being equal to the $(d-1)$-dimensional parallelotope generated by the vectors $\mathcal{V}_{ij} = \{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_{d+1}\} \setminus \{\mathbf{v}_i, \mathbf{v}_j\}$, that is,

$$F_{ij} = \{\mathbf{z} \in \mathbf{E^d} \mid \mathbf{z} = \sum_{\mathbf{v}_k \in \mathcal{V}_{ij}} \lambda_k \mathbf{v}_k, 0 \leq \lambda_k \leq 1\}. \tag{6}$$

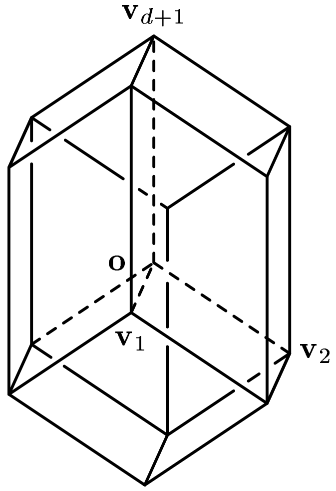

Figure 2

Thus,

$$\operatorname{dist}(F_{ij}^{i}, F_{ij}^{j}) \leq \operatorname{dist}(\mathbf{v}_i, \mathbf{v}_j) = ||\mathbf{v}_i - \mathbf{v}_j|| \text{ for all } 1 \leq i < j \leq d+1, \tag{7}$$

where $\operatorname{dist}(\dots, \dots)$ stands for the distance between two sets (resp., two points) in $\mathbf{E^d}$. Now let $V = \operatorname{conv}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{d+1}\}$. Then (7) implies that the diameter of the insphere of Z is at most as large as the minimum edge length of the d-dimensional simplex V, that is, it is at most $\min\{||\mathbf{v}_i - \mathbf{v}_j|| \mid 1 \leq i < j \leq d+1\}$. As a result in order to finish the proof of the lemma it is sufficient to show that among the d-dimensional simplices $V = \operatorname{conv}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{d+1}\}$ satisfying (2) and (3) in $\mathbf{E^d}$ the regular one with center $\mathbf{o}$ has the largest possible shortest edge length. This we prove as follows.

Obviously, there is an extremal d-dimensional simplex $V^* = \operatorname{conv}\{\mathbf{v}_1^*, \mathbf{v}_2^*, \dots, \mathbf{v}_{d+1}^*\}$ satisfying

$$\sum_{i=1}^{d+1} ||\mathbf{v}_i^*|| = s, \tag{2*}$$

$$\mathbf{o} \in \operatorname{int}[\operatorname{conv}\{\mathbf{v}_1^*, \mathbf{v}_2^*, \dots, \mathbf{v}_{d+1}^*\}] \tag{3*}$$

with the largest possible value of $\min\{||\mathbf{v}_i^* - \mathbf{v}_j^*|| \mid 1 \leq i < j \leq d+1\}$. Suppose that V^* is not a regular simplex. Let $m > 0$ be the length of the shortest edge of V^*. Then V^* must have a vertex say, $\mathbf{v}_k^*$ with some edges having length equal to m and some edges having length $> m$. As the total number of edges meeting at $\mathbf{v}_k^*$ is d there exists a hyperplane H_k

of $\mathbf{E^d}$ passing through $\mathbf{o}$ as well as $\mathbf{v}_k^*$ such that it separates the edges of $\mathbf{v}_k^*$ of length m from the edges of $\mathbf{v}_k^*$ of length $> m$. Let H_k^+ be the closed half-space of $\mathbf{E^d}$ bounded by H_k that contains all the edges of $\mathbf{v}_k^*$ of length $> m$. Now it is clear that if we rotate the vertex $\mathbf{v}_k^*$ about the origin $\mathbf{o}$ towards H_k^+ with initial tangent vector being perpendicular to H_k by a small angle, then all the edges of $\mathbf{v}_k^*$ will have length $> m$. Repeating this transformation at some other vertices of V^* we can increase $\min\{||\mathbf{v}_i^* - \mathbf{v}_j^*|| \mid 1 \leq i < j \leq d+1\}$ without changing the norms of the vectors $\mathbf{v}_1^*, \mathbf{v}_2^*, \dots, \mathbf{v}_{d+1}^*$, a contradiction. Thus V^* must be a regular d-dimensional simplex of $\mathbf{E^d}$. Finally, let F_i^* be the facet of V^* opposite to the vertex $\mathbf{v}_i^*$ and let $U_i^* = \operatorname{conv}(F_i^* \cup \{\mathbf{o}\}), 1 \leq i \leq d+1$. If $I = \{1, 2, \dots, d+1\}$ and $v^* = \operatorname{Vol}_{d-1}(F_1^*) = \operatorname{Vol}_{d-1}(F_2^*) = \ldots = \operatorname{Vol}_{d-1}(F_{d+1}^*)$, then it is easy to see that

$$\sum_{j \in I \setminus \{i\}} \operatorname{Vol}_d(U_j^*) \leq \frac{1}{d} \cdot ||\mathbf{v}_i^*|| \cdot v^* \text{ for all } i \in I. \tag{8}$$

Thus, (8) implies in a straightforward way that

$$d \cdot \sum_{j \in I} \operatorname{Vol}_d(U_j^*) \leq \frac{1}{d} \cdot s \cdot v^* \tag{9}$$

that is

$$\frac{d^2}{v^*} \cdot \operatorname{Vol}_d(V^*) \leq s \tag{10}$$

with equality if and only if $\mathbf{o}$ is the center of the regular d-dimensional simplex V^*. Hence, if $\mathbf{o}$ were not the center of the regular d-dimensional simplex V^*, then using (10) we could move $\mathbf{o}$ to the center of V^* thereby shortening the total length $\sum_{i=1}^{d+1} ||\mathbf{v}_i^*||$ of the spanning vectors of V^*, a contradiction. This completes the proof of the lemma. □

Now we turn to the proof of the theorem. We distinguish two cases.

Case (1): The d-dimensional zonotope in question is generated by $d+1$ line segments. The proof is by induction on d. Clearly, the theorem holds for $d = 2$. So, we may assume that $d \geq 3$ and the theorem holds in any Euclidean space of dimension less than d. Then let Z be an arbitrary d-dimensional zonotope generated by $d+1$ line segments and containing a d-dimensional unit ball in $\mathbf{E^d}$. Without loss of generality we may assume that Z is generated by the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{d+1}$ of $\mathbf{E^d}$ satisfying (1) and (3). Recall the following elegant formula for the intrinsic i-volume $V_i(Q)$ of a d-dimensional convex polyhedron Q in $\mathbf{E^d}$ (see for example [4]):

$$V_i(Q) = \sum_{F^{(i)}} \gamma(F^{(i)}, Q) \cdot \operatorname{Vol}_i(F^{(i)}), \tag{11}$$

where the summation is over all i-dimensional faces $F^{(i)}$ of Q and $\gamma(F^{(i)}, Q)$ denotes the normalized exterior angle of Q at the face $F^{(i)}$. We split the proof of the theorem in two subcases according to the values of i.

Subcase $1 \leq i \leq d-1$. Let Z^* be the d-dimensional regular zonotope of inradius 1 generated by the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{d+1}$ of $\mathbf{E^d}$ satisfying the corresponding versions of (1) and (3). Take a generating vector $\mathbf{u}_j$ of Z^*, $1 \leq j \leq d+1$. Let $\mathrm{Pr}_j(Z^*)$ denote the orthogonal projection of Z^* onto a hyperplane of $\mathbf{E^d}$ perpendicular to $\mathbf{u}_j$. Finally, let $u_{i-1} = \mathrm{V}_{i-1}[\mathrm{Pr}_1(Z^*)] = \mathrm{V}_{i-1}[\mathrm{Pr}_2(Z^*)] = \ldots = \mathrm{V}_{i-1}[\mathrm{Pr}_{d+1}(Z^*)]$. Now, take a generating vector $\mathbf{v}_j$ of Z, $1 \leq j \leq d+1$. Then let $Z_i^{(j)}$ denote the union ("zone") of the i-dimensional faces of Z that are parallel to $\mathbf{v}_j$. Then by induction one can easily get that

$$\sum_{F^{(i)} \in Z_i^{(j)}} \gamma(F^{(i)}, Z) \cdot \mathrm{Vol}_i(F^{(i)}) \geq ||\mathbf{v}_j|| \cdot u_{i-1} \text{ for all } 1 \leq j \leq d+1. \tag{12}$$

Thus, (11) and (12) imply that

$$i \cdot \mathrm{V}_i(Z) \geq u_{i-1} \cdot \sum_{j=1}^{d+1} ||\mathbf{v}_j||. \tag{13}$$

Hence, (13) and the lemma imply the theorem in a straightforward way.

Subcase $i = d$. As the d-dimensional volume of any d-dimensional zonotope generated by $d+1$ line segments and containing a d-dimensional unit ball in $\mathbf{E^d}$ is at least as large as $\frac{1}{d}$ times its surface area the theorem follows from the subcase $i = d-1$ in a trivial way.

Case (2): The d-dimensional zonotope in question is an affine cube.

As any d-dimensional affine cube of inradius at least one can be approximated by d-dimensional zonotopes generated by $d+1$ line segments and containing a d-dimensional unit ball in $\mathbf{E^d}$ the theorem follows from case (1).

This completes the proof of the theorem. □

References

[1] Gauss, C.F.: *Untersuchungen über die Eigenschaften der positiven ternären quadratischen Formen von Ludwig August Seber, Göttingische gelehrte Anzeigen, 1831 Juli 9.* J. reine und angew. Math. **20** (1840), 312–320.

[2] Linhart, J.: *Extremaleigenschaften der regulären 3-Zonotope.* Studia Sci. Math. Hung. **21** (1986), 181–188.

[3] McMullen, P.: *Convex bodies which tile space by translation.* Mathematika **27** (1980), 113–121.

[4] Sangwine-Yager, J.R.: *Mixed volumes.* In: Handbook of Convex Geometry, vol. A, P.M. Gruber and J.M. Wills, eds., New York 1993, 43–71.

[5] Schulte, E.: *Tilings.* In: Handbook of Convex Geometry, vol. A, P.M. Gruber and J. M. Wills, eds., New York 1993, 899–932.

[6] Venkov, B.A.: *On a class of Euclidean polytopes* (in Russian). Vestnik Leningrad. Univ. Ser. Mat. Fiz. Him. **9** (1954), 11–31.

Received December 20, 1998