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Homogeneous Parabolic Surfaces in $\mathbf{R}^4$

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1. Introduction

Parabolic surfaces in $\mathbf{R}^4$ bear their name because they satisfy parabolic partial differential equations. In this paper we shall determine and classify all parabolic non-ruled surfaces in $\mathbf{R}^4$ which are homogeneous in the sense of equiaffine geometry. In the course of the general discussion it will also come out that there are no compact parabolic surfaces, thus answering a question which has been open for some time.

The method for homogeneity will be a combination of differential geometry and Lie group theory. The differential geometry for parabolic surfaces has been established in [9], [10]. See also [11], [12] and, for the Lie group background, [13]. The main results here are the classification theorems B and C in Section 5 and 6: There will result nine classes of representatives, some of them depending on one or two real parameters. In the noncommutative case all surfaces are algebraic, in the commutative case most are not.

For elliptic and hyperbolic surfaces [14], [15], [16] recently developed a general unified theory, including the new approach by [7] and earlier works of Burstin/Mayer and W. Klingenberg (see the references there). For the flat nonparabolic cases, [15] determined the homogeneous copies.

2. Parabolic surfaces in affine four-space

A starting point for the affine differential geometry of surfaces in $\mathbf{R}^4$ is a conformal class of metrics invented by Burstin/Mayer. Let $x : M \to \mathbf{R}^4$ be a C^∞-immersion of a two-dimensional manifold M into $\mathbf{R}^4$, the latter considered as an affine space, equipped with the standard determinant form, denoted by brackets. Choosing a local base field $\mathcal{U} = (U, V)$ on M, a representative of the conformal class is given by

$$G_{\mathcal{U}}(X,Y) := \frac{1}{2}\left([d_U x, d_V x, d_X(d_U x), d_Y(d_V x)] + [d_U x, d_V x, d_Y(d_U x), d_X(d_V x)]\right)$$

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where X, Y are argument vector fields on M. The symbol d denotes the differential of $\mathbf{R}^{m+2}$-valued functions on M. The essential feature of $G_{\mathcal{U}}$ is its conformal invariance under any change of the frame field $\mathcal{U}$. In particular, the rank of $G_{\mathcal{U}}$ is a purely affine invariant. If the rank is 2, one has the *elliptic* (i.e. definite) and the *hyperbolic* (i.e. indefinite) case. If the rank is 1, one has the *parabolic* case. Sometimes we write G instead of $G_{\mathcal{U}}$ and call it the *pre-metric.*

The non-existence of compact elliptic surfaces proved in [12] also holds in the parabolic case:

Theorem A. *There are no compact parabolic surfaces in* $\mathbf{R}^4$.

In particular we cannot hope for compact groups connected with our homogeneous parabolic surfaces. The classification will show this again.

Proof. Choose an auxiliary Euclidean metric $\langle\ ,\ \rangle$ in $\mathbf{R}^4$, say the standard one. Consider a point $p_0 \in M$ where the squared distance function $f : M \to \mathbf{R}$, $f(p) := \langle x(p), x(p)\rangle$ reaches its maximum, and choose a local parametrization around p_0 such that the u-lines integrate the unique zero directions of the pre-metric. Then in the whole neigborhood of p_0 we have $[x_u, x_v, x_{uu}, x_{uv}] = [x_u, x_v, x_{uu}, x_{vv}] = 0$, $[x_u, x_v, x_{uv}, x_{vv}] \neq 0$. This implies the dependency $x_{uu} \equiv 0 \bmod (x_u, x_v)$ there. Now the gradient of f is 0 and its Hessian is negative semidefinite at p_0. In particular *at* p_0:

$$f_u = 2\langle x, x_u\rangle = 0, \quad f_v = 2\langle x, x_v\rangle = 0, \quad f_{uu} = 2\langle x_u, x_u\rangle + 2\langle x, x_{uu}\rangle \leq 0.$$

Together with the dependency, this implies $\langle x_u, x_u\rangle \leq 0$ at p_0, a contradiction to the rank 2 of x. □

Nevertheless, the parabolic surfaces form a large class of surfaces; see [9, Sect. 7] for a detailed result. The differential geometry for parabolic surfaces doesn't fit in any usual scheme. The first question is: When we have no Riemannian metric and no canonical connection etc., how can differential geometry be set going?

The key here is a sort of *Gauss map*: The zero directions of the pre-metric G induce a line subbundle of the tangent bundle TM. The vector parts of the lines in $\mathbf{R}^4$ define a two parameter family of points in the projective space $\mathbf{P}^3$ associated to $\mathbf{R}^4$. So we obtain a Gauss image of the surface in $\mathbf{P}^3$. As a mild additional assumption, we assume that the Gauss map is locally diffeomorphic. Equivalent with this is that the local integral curves of the line subbundle are asymptotic lines which have no point of inflection (so ruled surfaces are excluded by this). The Gauss-image in $\mathbf{P}^3$ turns out to be hyperbolically curved. One family of asymptotic lines on it corresponds to those on the original surface while the second one may serve to define another independent line subbundle of TM. Both line bundles then span TM and are defined in an affine invariant manner. For the details see [9]. Like there, local parametrizations $x(u,v)$ of a parabolic surface shall be adapted to this situation, in particular they will satisfy a parabolic equation of the form

$$x_{uu} = \chi x_u + \beta x_v, \qquad \beta \text{ without zeros,} \tag{2.1}$$

expressing that the u-lines are asymptotic lines of the surface without inflection points. Two such *distinguished* parametrizations, say $x(u,v)$ and $x^*(u^*, v^*)$, are connected by a *transition* of the form $u = f(u^*), v = g(v^*)$, with C^∞-functions f, g of one variable.

Once we have the two tangent line bundles and their local integral curves the procedure is continued by *half-invariant differentiation* as used by Bol [1] in his work on projective differential geometry. This includes certain differential operators along the distinguished parameter lines which are defined in the following way:

A *half-invariant* a of weight $(m, n) \in \mathbf{Z} \times \mathbf{Z}$ is a locally defined function of the distinguished parameters u, v with the transition behaviour $a^* = f'^m g'^n a$ under a change of distinguished parameters. Assume that λ, resp. μ are two scalar half-invariants of weights $(-1, n_0)$, resp. $(m_0, -1)$ with no zeros. Then the operations

$$a_{\mathbf{1}} := a_u + m(\ln|\lambda|)_u, \qquad a_{\mathbf{2}} := a_v + n(\ln|\mu|)_v,$$

produce from a half-invariant a of weight (m, n) two new half-invariants $a_{\mathbf{1}}$, resp. $a_{\mathbf{2}}$ of weights $(m+1, n)$, resp. $(m, n+1)$. Associated to such *fundamental* half-invariants λ, μ are the quantities

$$\sigma := \frac{1}{2}(\ln|\lambda|)_{uv}, \qquad \tau := \frac{1}{2}(\ln|\mu|)_{uv},$$

which are important for integrability conditions.

The crucial point is the finding of fundamental half-invariants λ, μ which are defined in an invariant manner. In Eqn. (2.1), the β is half-invariant of weight $(2, -1)$. If we take M as oriented, the v-lines can be oriented in such a way that $\beta > 0$. The orientation of the u-lines is then fixed by the orientation of M. Keeping this, forces the transition functions to obey $f' > 0,\ g' > 0$. Then half-invariant differentiation makes also sense for weights which are reals (instead of integers), and we can choose

$$\lambda := \beta^{-1/2}, \qquad \mu := \beta,$$

the so called *β-system*, which will be used henceforth (if nothing else is said).

Remark. It is possible to understand half-invariant differentiation in the language of connections. For a complex analogue, see [2] and [4].

The half-invariant differentiation in the β-system now induces two further line bundles which are *transversal* to the surface and thus can take the role of a first and second normal, namely spanned by the vectors

$$h := x_{\mathbf{12}}, \qquad b := x_{\mathbf{22}}.$$

Observe that the position vector x is half-invariant of weight $(0, 0)$, i.e. fully invariant. The structure equations for the moving frame $x_{\mathbf{1}}, x_{\mathbf{2}}, h, b$ have the form [9, Sect. 5]

$$\begin{aligned}
x_{\mathbf{11}} &= \alpha x_{\mathbf{1}} + \beta x_{\mathbf{2}} \\
x_{\mathbf{12}} &= x_{\mathbf{21}} \;=\; h \\
x_{\mathbf{22}} &= b \\
h_{\mathbf{1}} &= (\alpha_{\mathbf{2}} - 2\sigma)x_{\mathbf{1}} + \beta_{\mathbf{2}} x_{\mathbf{2}} + \alpha h + \beta b \\
h_{\mathbf{2}} &= \gamma x_{\mathbf{1}} + \varepsilon x_{\mathbf{2}} + \varphi h \\
b_{\mathbf{1}} &= \gamma x_{\mathbf{1}} + (\varepsilon + 2\tau) x_{\mathbf{2}} + \varphi h \\
b_{\mathbf{2}} &= b_{21} x_{\mathbf{1}} + b_{22} x_{\mathbf{2}} + b_{23} h + b_{24} b.
\end{aligned} \tag{2.2}$$

We now specialize to the equiaffine situation and thus have at our disposal the volume form $[\ldots]$ on $\mathbf{R}^4$. We may then form the additional quantity

$$D := [x_u, x_v, x_{uv}, x_{vv}] = [x_{\mathbf{1}}, x_{\mathbf{2}}, h, b],$$

which is half-invariant of weight $(2,4)$. As a consequence of the structure equations (2.2), it is differentiated according to

$$D_{\mathbf{1}} = 2\alpha D, \qquad D_{\mathbf{2}} = 2\varphi D.$$

Next we may form a moving frame with *canonically defined* vectors spanning our four line bundles:

$$\begin{array}{rcl} g_1 &:=& \beta^{-2/5}|D|^{-1/10}x_{\mathbf{1}} \\ g_2 &:=& \beta^{1/5}|D|^{-1/5}x_{\mathbf{2}} \\ g_3 &:=& \beta^{-1/5}|D|^{-3/10}h \\ g_4 &:=& \beta^{2/5}|D|^{-2/5}b \end{array} \qquad [g_1, g_2, g_3, g_4] = \operatorname{sign}(D),$$

the last equation following from the definition of D.

Such a unique moving frame is quite essential for the homogeneity question. Namely, if the immersion is the orbit of a Lie subgroup $\Gamma \subseteq \mathbf{SA}$ (= special affine group of $\mathbf{R}^4$) then the dimension of Γ must be 2. Because otherwise the isotropy group at a fixed orbit point would be infinite. But all elements of the isotropy group fix the distinguished frame at that point, so coincide with the identity.

Therefore we will define a parabolic surface to be homogeneous if it is the orbit of a two-dimensional Lie subgroup of $\mathbf{SA}$. We only consider oriented parabolic surfaces whose asymptotic lines have no points of inflection. The last requirement exactly rules out homogeneous parabolic ruled surfaces.

3. The differential geometric reduction

As a first step for the homogeneity question we draw from the differential geometry necessary conditions which reduce the number of cases for the possible Lie group representations.

So assume that x is a local parametrization for a parabolic orbit as defined. One main consequence is that each scalar full invariant (i.e. of weight $(0,0)$) must be constant along the orbit. Most of the quantities entering the structure equations are only half-invariant. But it is easy to generate full invariants by forming quotients. For example A, C, B, F, S, defined as follows, are fully invariant:

$$\alpha = A\beta^{2/5}|D|^{1/10},\ \gamma = C\beta^{-2/5}|D|^{2/5},\ \varepsilon = B\beta^{1/5}|D|^{3/10},\ \varphi = F\beta^{-1/5}|D|^{1/5},\ \sigma = S\beta^{1/5}|D|^{3/10}.$$

We now need the following lemma on general half-invariant differentiation w.r.t. fundamental half-invariants λ, μ:

3.1. Lemma. *If a scalar half-invariant a of weight (m,n) has no zeros and satisfies*

$$\left(\frac{a_{\mathbf{1}}}{a}\right)_{\mathbf{2}} = 2c\sigma, \qquad c = \text{const.}$$

then, by a suitable positive transition, we can reach

$$|a| = |\lambda|^{c-m}.$$

Proof. Since $\frac{a_{\mathbf{1}}}{a}$ has weight $(1,0)$, we have

$$\left(\frac{a_{\mathbf{1}}}{a}\right)_{\mathbf{2}} = \left(\frac{a_{\mathbf{1}}}{a}\right)_v, \qquad a_{\mathbf{1}} = a_u + m(\ln|\lambda|)_u a.$$

From this and the assumption one infers

$$\left(\ln\left(|a|\ |\lambda|^{m-c}\right)\right)_{uv} = 0,$$

so $|a|\ |\lambda|^{m-c} = U(u)V(v)$ for some positive functions U, V of one variable. Under transition of the distinguished parameters, $|a|\ |\lambda|^{m-c}$ takes a factor of the form $|f'|^r|g'|^s$. Thus by the choice

$$f^{-1}(u) = \int U(u)^{1/r}\, du, \qquad g^{-1}(v) = \int V(v)^{1/s}\, dv,$$

we can render $|a|\ |\lambda|^{m-c}$ to 1 in the new parameters. □

Remark. Under the same assumptions on a one deduces from the interchanging rule for mixed half-invariant derivatives:

$$\left(\frac{a_{\mathbf{1}}}{a}\right)_{\mathbf{2}} - \left(\frac{a_{\mathbf{2}}}{a}\right)_{\mathbf{1}} = 2m\sigma - 2n\tau. \tag{3.1}$$

We now return to calculate in the β-system.

3.2. Lemma. *If A, F are constant then, by a suitable positive transition, we can reach $|D| = \beta^6$.*

Proof. We have

$$\frac{|D|_{\mathbf{1}}}{|D|} = 2\alpha, \qquad \frac{|D|_{\mathbf{2}}}{|D|} = 2\varphi.$$

On the other hand, by the definition of A, F:

$$\alpha_{\mathbf{2}} = \frac{1}{10}A\beta^{2/5}\frac{|D|^{1/10}}{|D|}|D|_{\mathbf{2}} = \frac{1}{5}\alpha\varphi, \qquad \varphi_{\mathbf{1}} = \frac{1}{5}F\beta^{-1/5}\frac{|D|^{1/5}}{|D|}|D|_{\mathbf{1}} = \frac{2}{5}\alpha\varphi,$$

thus

$$\left(\frac{|D|_{\mathbf{1}}}{|D|}\right)_{\mathbf{2}} = \frac{2}{5}\alpha\varphi, \qquad \left(\frac{|D|_{\mathbf{2}}}{|D|}\right)_{\mathbf{1}} = \frac{4}{5}\alpha\varphi.$$

With (3.1) this gives $\alpha\varphi = -50\sigma$, hence $\left(\frac{|D|_{\mathbf{1}}}{|D|}\right)_{\mathbf{2}} = -20\sigma$, so from Lemma 3.1, after suitable transition, $|D| = (\beta^{-1/2})^{-12} = \beta^6$. □

If we use a parametrization with $|D| = \beta^6$, our quantities specialize to

$$\begin{aligned} \alpha &= A\beta \\ \gamma &= C\beta^2 \\ \varepsilon &= B\beta^2, \\ \varphi &= F\beta, \\ \sigma &= S\beta^2 \end{aligned} \qquad \begin{aligned} g_1 &= \beta^{-1}x_{\mathbf{1}} \\ g_2 &= \beta^{-1}x_{\mathbf{2}} \\ g_3 &= \beta^{-2}h \\ g_4 &= \beta^{-2}b. \end{aligned}$$

Then also (from the derivatives of D):

$$\beta_u = \frac{2}{5}A\beta^2, \qquad \beta_v = \frac{1}{5}F\beta^2.$$

The structure equations for $x_{\mathbf{1}}, x_{\mathbf{2}}, h, b$ will now be converted to those for the g_i. Observe that the g_i are fully invariant, so the derivatives for $_{\mathbf{1},\mathbf{2}}$ coincide with the ordinary derivatives for $_{u,v}$. For a compact writing, we introduce the formal row

$$Y := (g_1, g_2, g_3, g_4),$$

define its derivatives entry-wise, and then can state the new structure equations in the form:

$$\begin{aligned} Y_u &= \beta Y \widetilde{H}, \quad \widetilde{H} := \begin{pmatrix} \frac{4}{5}A & 0 & \frac{1}{5}AF - 2S & C \\ 1 & -\frac{2}{5}A & 0 & B - 4S \\ 0 & 1 & \frac{2}{5}A & F \\ 0 & 0 & 1 & -\frac{4}{5}A \end{pmatrix} \\ Y_v &= \beta Y \widetilde{E}, \quad \widetilde{E} := \begin{pmatrix} -\frac{1}{5}F & 0 & C & \frac{2}{25}(AF^2 - 10FS + 10AC) \\ 0 & -\frac{2}{5}F & B & \frac{1}{5}(-2AB + 5C) \\ 1 & 0 & \frac{2}{5}F & B + 8S \\ 0 & 1 & 0 & \frac{1}{5}F \end{pmatrix}. \end{aligned} \tag{3.2}$$

The integrability condition of the new structure equations result from the comparison of Y_{uv}, Y_{vu}. This yields, using the derivatives of β from above:

$$[\widetilde{E}, \widetilde{H}] = \frac{2}{5}A\widetilde{E} - \frac{1}{5}F\widetilde{H},$$

where the brackets of two matrices denotes their commutator. Expressed with the entries of $\widetilde{E}, \widetilde{H}$ this says:

$$\begin{array}{ll} \text{(a)} & 4A(AF^2 - 10FS + 10AC) = 20CF - 5(AF - 10S)(B + 8S) \\ \text{(b)} & 4A(-2AB + 5C) = -10F(B - 4S) + 25BF - 2(AF^2 - 10FS + 10AC) \\ \text{(c)} & 3AB + 32AS = F^2 \\ \text{(d)} & AF = -50S. \end{array} \tag{3.3}$$

Adjoining to the system of the g_i the position vector function x, we obtain no additional integrability conditions, because the derivatives of x are $x_u = \beta g_1,\ x_v = \beta g_2$, and one easily deduces $x_{uv} = \beta^2 g_3,\ x_{vu} = \beta^2 g_3$, The following is of general interest for handling affine situations within the linear framework of vector spaces: We represent points, resp. vectors of the space $\mathbf{R}^m$ by elements of $\mathbf{R}^{m+1}$ with last entry 1, resp. 0, both written as columns.

$$\text{point of } \mathbf{R}^m: \quad \left(\begin{array}{c} z_1 \\ \vdots \\ z_m \\ \hline 1 \end{array}\right), \qquad \text{vector of } \mathbf{R}^m: \quad \left(\begin{array}{c} z_1 \\ \vdots \\ z_m \\ \hline 0 \end{array}\right).$$

An affine mapping $\alpha : \mathbf{R}^m \to \mathbf{R}^m$ is of the form $\alpha(z) = Lz + l$ where $L : \mathbf{R}^m \to \mathbf{R}^m$ is linear and l fixed in $\mathbf{R}^m$ (z, l written as columns and L written as $(m \times m)$-matrix). We then represent α, L by $(m+1) \times (m+1)$-matrices:

$$\alpha \text{ by: } \quad \mathbf{L} := \left(\begin{array}{ccc|c} & & & l_1 \\ & L & & \vdots \\ & & & l_m \\ \hline 0 & \dots & 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} L & l \\ \hline 0 & 1 \end{array}\right), \quad L \text{ by: } \quad \left(\begin{array}{ccc|c} & & & 0 \\ & L & & \vdots \\ & & & 0 \\ \hline 0 & \dots & 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} L & 0 \\ \hline 0 & 1 \end{array}\right),$$

where we also noted suitable short-hand versions. L is called the *linear part*, l the *translation part* of $\mathbf{L}$.

By this imbedding, affine mappings in $\mathbf{R}^m$ can be treated as linear mappings of $\mathbf{R}^{m+1}$. This is in particular true for the composition and also for all Lie-group objects. For example the affine group of $\mathbf{R}^m$ becomes the subgroup of $\mathbf{GL}(m+1)$ consisting of the $(m+1) \times (m+1)$-matrices $\left(\begin{array}{c|c} L & l \\ \hline 0 & 1 \end{array}\right)$ with regular left upper corner L, and its Lie-algebra will consist of the matrices $\left(\begin{array}{c|c} L & l \\ \hline 0 & 0 \end{array}\right)$ with arbitrary left upper corner L. Also the determinants and traces carry over appropriately, and the same is true for the exponential map. The special affine group $\mathbf{SA}$ becomes now as Lie-algebra the last mentioned matrices with trace 0 of the left upper corner. Observe that the affine mappings of $\mathbf{R}^m$ now are sitting in the affine hyperplane of the set of $(m+1)$-square matrices with last entry 1, while the elements of the corresponding vector hyperplane have a 0 there instead of 1.

Back to our system $Y = (g_1, g_2, g_3, g_4)$, we may interprete the entries g_i really as column vectors, and the matrix

$$\mathbf{Y} := \left(\begin{array}{c|c} Y & x \\ \hline 0 & 1 \end{array}\right)$$

as a representation for the vector system consisting of the g_i *and* x. Its derivatives are calculated as

$$\begin{aligned} \mathbf{Y}_u &= \beta \mathbf{Y} \widetilde{\mathbf{H}}, \qquad \text{where} \qquad \widetilde{\mathbf{H}} := \left(\begin{array}{c|c} \widetilde{H} & e_1 \\ \hline 0 & 0 \end{array}\right) \\ \mathbf{Y}_v &= \beta \mathbf{Y} \widetilde{\mathbf{E}}, \qquad \text{where} \qquad \widetilde{\mathbf{E}} := \left(\begin{array}{c|c} \widetilde{E} & e_2 \\ \hline 0 & 0 \end{array}\right), \end{aligned} \tag{3.4}$$

e_i denoting the standard base columns of $\mathbf{R}^4$. The integrability condition for $\mathbf{Y}$ is the same as that for Y, namely

$$[\widetilde{\mathbf{E}}, \widetilde{\mathbf{H}}] = \frac{2}{5} A \widetilde{\mathbf{E}} - \frac{1}{5} F \widetilde{\mathbf{H}}.$$

Essentially, this $\mathbf{Y}$ is a parametrization of the group of our orbit x: Assume that a fixed point of the orbit with parameters u_0, v_0 is the origin: $x(u_0, v_0) = 0$, and moreover that the $g_i(u_0, v_0)$ form the standard base e_i of $\mathbf{R}^4$. (The general position differs from this only by a fixed affine map of determinant ± 1.) Then $\mathbf{Y}(u, v)$ represents the affine map in $\mathbf{R}^4$ transporting the origin to the point $x(u, v)$ and the base vectors e_i to the vectors $g_i(u, v)$. Tangent vectors to the group at u_0, v_0 are thus given by $\mathbf{Y}_u(u_0, v_0) = \beta(u_0, v_0) \widetilde{\mathbf{H}}$ and $\mathbf{Y}_v(u_0, v_0) = \beta(u_0, v_0) \widetilde{\mathbf{E}}$ or simply by $\widetilde{\mathbf{H}}$, $\widetilde{\mathbf{E}}$. These are then generators for the Lie algebra of the orbit.

In view of the integrability conditions we distinguish two main cases:

Case (I): $A = 0$.
By (3.3) (c) also $F = 0$, and the integrability condition becomes:

$$[\widetilde{\mathbf{E}}, \widetilde{\mathbf{H}}] = 0.$$

We write

$$\mathbf{E} := \widetilde{\mathbf{E}}, \qquad \mathbf{H} := \widetilde{\mathbf{H}}. \tag{3.5}$$

Case (II): $A \neq 0$.
The integrability condition suggests to replace $\widetilde{\mathbf{E}}$, $\widetilde{\mathbf{H}}$ by

$$\mathbf{E} := \frac{2}{5} A \widetilde{\mathbf{E}} - \frac{1}{5} F \widetilde{\mathbf{H}}, \qquad \mathbf{H} := \frac{5}{2A} \widetilde{\mathbf{H}}. \tag{3.6}$$

Then

$$[\mathbf{E}, \mathbf{H}] = \mathbf{E}.$$

The change from $\widetilde{\mathbf{E}}$, $\widetilde{\mathbf{H}}$ to $\mathbf{E}, \mathbf{H}$ corresponds to a reparametrization (to non-distinguished parameters) by

$$u = \frac{5}{2A} u^* - \frac{1}{5} F v^*, \qquad v = \frac{2}{5} A v^*.$$

In fact we then have

$$\mathbf{Y}_{u^*} = \beta \mathbf{Y} \mathbf{H}, \qquad \mathbf{Y}_{v^*} = \beta \mathbf{Y} \mathbf{E}$$

and henceforth use this parametrization writing u, v instead of u^*, v^* (if nothing else is said).

Collecting both cases we can write

$$[\mathbf{E}, \mathbf{H}] = \varkappa \mathbf{E}, \qquad \varkappa := \begin{cases} 0 & \text{in Case (I)} \\ 1 & \text{in Case (II).} \end{cases}$$

So (I) will be the *commutative case* and (II) the *noncommutative case.*

4. The algebraic reduction

The possible generators $\mathbf{E}, \mathbf{H}$ for the Lie algebra are given by the above formulas.

Usually we follow the convention to write

$$\mathbf{L} = \left(\begin{array}{c|c} L & l \\ \hline 0 & \eta \end{array}\right)$$

in this arrangement of boldface, normal capital and small letters. Here $\eta = 1$, resp. $\eta = 0$, whether $\mathbf{L}$ represents an affine map, resp. a generator of a Lie-algebra. Besides these specialities of affine geometry, the generalities on linear Lie groups and their orbits are as in [13, Sect. 2].

The classification will be made w.r.t. affine similarities in $\mathbf{R}^4$. They correspond to linear similarities in $\mathbf{R}^5$, described by the transition from $\mathbf{L}$ to $\mathbf{T}^{-1}\mathbf{L}\mathbf{T}$, where

$$\mathbf{T} := \left(\begin{array}{c|c} T & t \\ \hline 0 & 1 \end{array}\right), \qquad \det(\mathbf{T}) = \det(T) \neq 0.$$

It is no restriction to assume $\det \mathbf{T} = \pm 1$. So it is natural (and avoids a subsplitting into more cases) to do the final classification of orbits w.r.t. the group $\widetilde{\mathbf{SA}}$ of affine transformations of determinant ± 1.

There are two special types of such similarities, the pure *translational* ones for which $T = I$ (= unit matrix) and the pure *linear* ones for which $t = 0$.

Sometimes it is possible to simplify the translation part of an $\mathbf{L}$ with $\eta = 0$ by a pure translation similarity. Consider for this the equation

$$\left(\begin{array}{c|c} I & t \\ \hline 0 & 1 \end{array}\right)^{-1} \left(\begin{array}{c|c} L & l \\ \hline 0 & 0 \end{array}\right) \left(\begin{array}{c|c} I & t \\ \hline 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} L & Lt + l \\ \hline 0 & 0 \end{array}\right).$$

So the new translation part is $l_n = Lt + l$, hence all l_n lying in the affine space $l + \operatorname{im} L$ are possible. For example, $l_n = 0$ is possible if L is regular. If L has block form, then these possibilities for l_n can be discussed block-wise. We shall call this a *translation reduction.* This can be applied to more than one $\mathbf{L}$ simultaneously (with due regard, that only one translation part t is at our disposal).

We often meet the following situation: If T is a matrix rendering the linear part L of $\mathbf{L}$ to L_0, $T^{-1}LT = L_0$ then, with the corresponding pure linear $\mathbf{T}$, the $\mathbf{L}$ is rendered to $\left(\begin{array}{c|c} L_0 & l_0 \\ \hline 0 & 1 \end{array}\right)$ with $l_0 := T^{-1}l$. We say for this that the translation part of $\mathbf{L}$ is *taken along* with such a $\mathbf{T}$.

For any Lie-algebra $\Gamma^\bullet$ there is defined the *group of self-similarities* $\mathcal{F}_\Gamma$, consisting of those $\mathbf{T}$ whose similarities transport the Lie-algebra as a whole onto itself. Expressed with generators $\mathbf{E}, \mathbf{H}$ for our two dimensional Lie-algebras, the group $\mathcal{F}_\Gamma$ is given by the $\mathbf{T}$ for which there exist real $\alpha_1, \alpha_2, \beta_1, \beta_2$ with

$$\begin{array}{rcl} \mathbf{T}^{-1}\mathbf{E}\mathbf{T} &=& \alpha_1\mathbf{E} + \alpha_2\mathbf{H} \\ \mathbf{T}^{-1}\mathbf{H}\mathbf{T} &=& \beta_1\mathbf{E} + \beta_2\mathbf{H} \end{array}, \qquad \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} \neq 0.$$

(Of course the variables $\alpha_1, \alpha_2, \beta_1, \beta_2$ here have nothing to do with the coefficient functions in the structure equations.)

The geometric role of $\mathcal{F}_\Gamma$ is that it decides on initial points, whether their orbits are similar or not. This is the same situation as in [13, Sect. 6]. In fact it is somewhat easier here because we have a unique moving frame. This replaces Lemma 6.1 there, and the consequences drawn there are valid appropriately. In particular, each orbit determines its group uniquely.

In the course of the classification, it is useful first to normalize the linear parts and afterwards to look how the translation parts can be normalized. We do that mostly with the following *three step procedure*:

(i) Transform the linear parts of generators to normal forms.

(ii) Take the translation parts of the original generators along with the transitions of step (i).

(iii) Apply a translation reduction to these generators.

Observe that part (i) is quite non standard because the simultaneous normal form problem of two or more vector space endomorphisms is generally unsolved. In the present situation we get through by the results of [13].

We have the two main cases from above:

(I) $A = 0$ with $[\mathbf{E}, \mathbf{H}] = 0$ (commutative case)

(II) $A \neq 0$ with $[\mathbf{E}, \mathbf{H}] = \mathbf{E}$ (noncommutative case).

The algebraic reduction will be done separately for these cases during the proofs of the main theorems. (Sometimes it is convenient to interchange notationally the role of $\mathbf{T}$ and $\mathbf{T}^{-1}$.)

5. The commutative case

Theorem B. *The non-ruled homogeneous parabolic surfaces of $\mathbf{R}^4$ with commutative groups are classified according to the following five cases, specifying normal forms for the generators $\mathbf{E}, \mathbf{H}$ of the corresponding Lie algebra and implicit equations resp. parametrizations for the orbits. In all cases, $\mathbf{E} = \mathbf{H}^2$ and $c > 0$:*

(I.1)

$$\mathbf{H} = \left(\begin{array}{cccc|c} 0 & & & & 1 \\ 1 & 0 & & & 0 \\ & 1 & 0 & & 0 \\ & & 1 & 0 & 0 \\ \hline & & & & 0 \end{array}\right), \qquad \begin{aligned} x_3 &= x_1 x_2 - \frac{1}{3} x_1^3 \\ x_4 &= \frac{1}{2} x_2^2 - \frac{1}{12} x_1^4. \end{aligned}$$

(I.2)

$$\mathbf{H} = \left(\begin{array}{cc|cc|c} 1 & & & & \\ 1 & 1 & & & \\ \hline & & -1 & -\sqrt{2} & \\ & & \sqrt{2} & -1 & \\ \hline & & & & 0 \end{array}\right), \qquad x(u,v) = \begin{pmatrix} c e^{u+v} \\ c e^{u+v}(u + 2v) \\ e^{-u-v} \cos(\sqrt{2}(u - 2v)) \\ e^{-u-v} \sin(\sqrt{2}(u - 2v)) \end{pmatrix}$$

(I.3.a)

$$\mathbf{H} = \left(\begin{array}{cc|cc|c} 0 & & & & 1 \\ & 2 & & & 0 \\ \hline & & -1 & -\sqrt{3} & \\ & & \sqrt{3} & -1 & \\ \hline & & & & 0 \end{array}\right), \qquad x(u,v) = \begin{pmatrix} u \\ ce^{2(u+2v)} \\ e^{-u-2v}\cos(\sqrt{3}(u-2v)) \\ e^{-u-2v}\sin(\sqrt{3}(u-2v)) \end{pmatrix}$$

(I.3.b)

$$\mathbf{H} = \left(\begin{array}{cc|cc|c} p+\sqrt{2}q & & & & \\ & p-\sqrt{2}q & & & \\ \hline & & -p & -\sqrt{2} & \\ & & \sqrt{2} & -p & \\ \hline & & & & 0 \end{array}\right), \quad x(u,v) = \begin{pmatrix} ce^{\left(p+\sqrt{2}q\right)u+\left(p^2+4\sqrt{2}pq+2\right)v} \\ e^{\left(p-\sqrt{2}q\right)u+\left(p^2-4\sqrt{2}pq+2\right)v} \\ e^{-pu-(p^2+2)v}\cos(\sqrt{2}u) \\ e^{-pu-(p^2+2)v}\sin(\sqrt{2}u) \end{pmatrix}$$

$$q := \sqrt{1-p^2}, \quad 0 \le p < 1, \quad p \ne \sqrt{2/3}$$

(I.4)

$$\mathbf{H} = \left(\begin{array}{cc|cc|c} \omega & -p & & & \\ p & \omega & & & \\ \hline & & -\omega & -q & \\ & & q & -\omega & \\ \hline & & & & 0 \end{array}\right), \qquad x(u,v) = c \cdot \begin{pmatrix} e^{\omega u+p'v}\cos\left(pu+\frac{p}{\omega}v\right) \\ e^{\omega u+p'v}\sin\left(pu+\frac{p}{\omega}v\right) \\ e^{-(\omega u+p'v)}\cos\left(qu-\frac{q}{\omega}v\right) \\ e^{-(\omega u+p'v)}\sin\left(qu-\frac{q}{\omega}v\right) \end{pmatrix}.$$

$$\omega := \frac{1}{2}\sqrt{2}, \quad q := \sqrt{1-p^2}, \quad 0 < p \le \omega$$

$$p' := \frac{1}{2} - p^2$$

Any such homogeneous parabolic surface is $\widetilde{\mathbf{SA}}$*-equivalent to exactly one of the representatives listed above. In particular, the occuring parameters* p, c *are classifying.*

Proof. We first continue the necessary conditions. The existence will come out easily at the end. The generators of the commutative cases are generally from (3.2) – (3.5):

$$\mathbf{E} = \left(\begin{array}{cccc|c} 0 & 0 & C & 0 & 0 \\ 0 & 0 & B & C & 1 \\ 1 & 0 & 0 & B & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right), \qquad \mathbf{H} = \left(\begin{array}{cccc|c} 0 & 0 & 0 & C & 1 \\ 1 & 0 & 0 & B & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The main observation is

$$\mathbf{E} = \mathbf{H}^2.$$

This considerably simplifies the classification of the pair $\mathbf{E}, \mathbf{H}$ in this case. In fact it suffices to render $\mathbf{H}$ to a normal form for the following reason:

There are no other elements in the span of $\mathbf{E}, \mathbf{H}$ whose squares are also in that span (besides the multiples of $\mathbf{H}$). This follows from the fact, that the powers $\mathbf{H}, \mathbf{H}^2, \mathbf{H}^3, \mathbf{H}^4$ are calculated to be linearly independent. Thus the relation $\mathbf{E} = \mathbf{H}^2$ distinguishes the pair $\mathbf{E}, \mathbf{H}$ up to a nonvanishing factor of $\mathbf{H}$. This is clearly invariant under similarities $\mathbf{T}$: After applying $\mathbf{T}$, the new distinguished generators are just $\mathbf{T}^{-1}\mathbf{E}\mathbf{T}, \mathbf{T}^{-1}\mathbf{H}\mathbf{T}$ (with one scalar factor free for the second one).

Analogously the self-similarities are here those $\mathbf{T}$ for which there is a $\beta \neq 0$ such that $\mathbf{T}^{-1}\mathbf{H}\mathbf{T} = \beta\mathbf{H}$.

The linear part H has the characteristic polynomial

$$\chi_H(\xi) = \xi^4 - B\xi - C.$$

The fine classification now runs according to the possible zeros of χ_H.

Case (I.1): χ_H has a zero of multiplicity 4, i.e. $B = C = 0$.

Then H has already Jordan normal form, and it is not possible to simplify further the translation part of $\mathbf{H}$. In fact, replacing the 1 in the translation part by 0 would lead to a matrix which is not similar to $\mathbf{H}$, and also not proportional similar to $\mathbf{H}$; see [13, Lemma 3.3]. So the original form of $\mathbf{H}$ is already a good affine normal form. Of course, there will exist nontrivial similarities leaving this normal form invariant.

The self-similarities are calculated to become

$$\mathbf{T} = \left(\begin{array}{cccc|c} \lambda & & & & t_{15} \\ \lambda t_{15} & \lambda^2 & & & t_{25} \\ \lambda t_{25} & \lambda^2 t_{15} & \lambda^3 & & t_{35} \\ \lambda t_{35} & \lambda^2 t_{25} & \lambda^3 t_{15} & \lambda^4 & t_{45} \\ \hline & & & & 1 \end{array}\right), \qquad \det(\mathbf{T}) \neq 0.$$

with free parameters λ, etc. such that $\det T \neq 0$. It is obvious that any two points can be transformed to each other by a suitable $\mathbf{T}$ of this type with $|\det \mathbf{T}| = 1$. So all orbits are similar to each other, and we may pick one, say that with initial point 0. Its parametrization (in $\mathbf{R}^5$) is

$$(u, v) \longmapsto \exp(u\mathbf{H}) \exp(v\mathbf{E}) \left(\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ \hline 1 \end{array}\right) = \left(\begin{array}{c} u \\ \frac{1}{2}u^2 + v \\ \frac{1}{6}u^3 + uv \\ \frac{1}{24}u^4 + \frac{1}{2}u^2 v + \frac{1}{2}v^2 \\ \hline 1 \end{array}\right).$$

By eliminating u, v the implicit equations of this orbit (in $\mathbf{R}^4$) result as announced.

Case (I.2): χ_H has a real double zero and a pair of genuine conjugate complex zeros.

The discriminant of χ_H is $27B^4 + 256C^3 = 0$. We parametrize the set of these matrices H by setting $C = -3\gamma^4$, $B = 4\gamma^3$, where γ varies in $\mathbf{R} \setminus \{0\}$. So γ is the real double zero, the other two being $(-1 \pm i\sqrt{2})\gamma$.

The linear part H has Jordan normal form γH_0, where $H_0 := J(2,1) + N(-1, \sqrt{2})$, in the notation of [13, p. 138–141]. An additional translation similarity renders the translation part of $\mathbf{H}$ to zero, because H_0 is regular. So we obtain a new generator which replaces $\mathbf{H}$ and may be called $\mathbf{H}$ again for simplicity. Finally we may divide out the γ and thus obtain the $\mathbf{H}$ as announced.

The procedure, to give new improved generators the old names, will be followed in the future without explicit mention.

The self-similarities belonging to the new generators are

$$\mathbf{T} = \left(\begin{array}{cc|cc|c} t_{11} & 0 & & & \\ t_{21} & t_{11} & & & \\ \hline & & t_{33} & t_{34} & \\ & & -t_{34} & t_{33} & \\ \hline & & & & 1 \end{array}\right), \qquad \det(\mathbf{T}) \neq 0.$$

Two initial points in $\mathbf{R}^4$ with coordinates x_i, y_i can be transformed to each other by such a $\mathbf{T}$ with $|\det(\mathbf{T})| = 1$ iff $x_1^2(x_3^2 + x_4^2) = y_1^2(y_3^2 + y_4^2)$, which must have a value c^2 with $c > 0$ (by the rank 1 condition on the pre-metric). This c is classifying. As initial point we may choose $(c, 0, 1, 0)^\top$ in $\mathbf{R}^4$ and calculate its orbit as announced.

Case (I.3): χ_H has exactly two real zeros and a pair of genuine conjugate complex zeros.

The problem is that these zeros are not expressible by B, C in a reasonable manner. So we parametrize the set of these matrices H by the zeros themselves, more precisely by the two real ones called α, β: Observe that for some real r, s there must be an identity:

$$\chi_H(\xi) = \xi^4 - B\xi - C = (\xi - \alpha)(\xi - \beta)(\xi^2 + r\xi + s), \qquad r^2 - 4s < 0.$$

Solving the corresponding coefficient equalities for r, s, B, C gives:

$$B = (\alpha + \beta)(\alpha^2 + \beta^2), \qquad C = -\alpha\beta(\alpha^2 + \alpha\beta + \beta^2), \qquad 3\alpha^2 + 2\alpha\beta + 3\beta^2 > 0. \tag{5.1}$$

In fact these conditions together with $\alpha \neq \beta$ are necessary and sufficient for the present zero behaviour of χ_H. These zeros become $\alpha, \beta, -\frac{1}{2}\left(\alpha + \beta \pm i\sqrt{3\alpha^2 + 2\alpha\beta + 3\beta^2}\right)$. It is still more convenient to replace α, β by two other parameters p, q with

$$\begin{array}{rcl} \alpha + \beta & = & 2p \\ \alpha - \beta & = & 2\sqrt{2}q \end{array}, \qquad \text{i.e.} \qquad \begin{array}{rcl} \alpha & = & p + \sqrt{2}q \\ \beta & = & p - \sqrt{2}q \end{array},$$

where $q \neq 0$. The inequality in (5.1) is then automatically satisfied because $8(p^2 + q^2) = 3\alpha^2 + 2\alpha\beta + 3\beta^2$.

A Jordan normal form of the linear part H may start with $\operatorname{diag}(p+\sqrt{2}q, p-\sqrt{2}q)$ as left upper block, followed by the normal block corresponding to the two complex eigenvalues. As to the translation part of $\mathbf{H}$, we have to distinguish two subcases:

Subcase (I.3.a): The left upper block has rank 1, i.e. $p^2 = 2q^2$ or equivalently $\alpha\beta = 0$.

We take the α as 0. The three step procedure leads to a new translation part which can be chosen as $(p,0,0,0)^\top$. Since $q = -p/\sqrt{2}$, the imaginary parts of the complex eigenvalue become $\pm\sqrt{3}p$. So we may divide $\mathbf{H}$ by p, eventually interchange the two complex eigenvalues and arrive at the new generator, as announced. The elements of the similarity group become

$$\mathbf{T} = \left(\begin{array}{cc|cc|c} 1 & & & & t_{15} \\ & t_{22} & & & 0 \\ \hline & & t_{33} & t_{34} & \\ & & -t_{34} & t_{33} & \\ \hline & & & & 1 \end{array}\right), \qquad \det(\mathbf{T}) \neq 0.$$

Two initial points can be transformed to each other by such a $\mathbf{T}$ with $|\det(\mathbf{T})| = 1$ iff $x_2^2(x_3^2+x_4^2)^2 = y_2^2(y_3^2+y_4^2)^2$, which must have a value c^2 with $c > 0$ (by the rank 1 condition on the pre-metric). This c is classifying. As initial point we may choose $(0,c,1,0)^\top$ in $\mathbf{R}^4$ and calculate its orbit as announced.

Subcase (I.3.b): The left upper block has rank 2, i.e. $p^2 \neq 2q^2$ or equivalently $\alpha\beta \neq 0$.

Then the whole translation part of $\mathbf{H}$ can be rendered to zero. We arrange the two real zeros by $\alpha > \beta$, i.e. $q > 0$. By finally dividing the whole $\mathbf{H}$ by $\sqrt{p^2+q^2}$ we can assume $p^2+q^2 = 1$ and thus arrive at $\mathbf{H}$, as announced.

Two values of p lead to similar proportional generators $\mathbf{H}$ iff they have the same absolute value. So $0 \leq p < 1$ (the latter from $\alpha \neq \beta$) and $p \neq \sqrt{2/3}$. Then the p is separating between similarity classes. The self-similarities are

$$\mathbf{T} = \left(\begin{array}{cc|cc|c} t_{11} & & & & \\ & t_{22} & & & \\ \hline & & t_{33} & t_{34} & \\ & & -t_{34} & t_{33} & \\ \hline & & & & 1 \end{array}\right), \qquad \det(\mathbf{T}) \neq 0.$$

Two initial points can be transformed to each other by such a $\mathbf{T}$ with $|\det(\mathbf{T})| = 1$ iff on them $x_1^2x_2^2(x_3^2+x_4^2)^2$ has the same value c^2 with $c > 0$ (by the rank 1 condition on the pre-metric). This c is classifying. As initial point we may choose $(c,1,1,0)^\top$ in $\mathbf{R}^4$ and calculate its orbit as announced, where we used $\mathbf{E}+2p\mathbf{H}$ instead of $\mathbf{E}$ because the expressions become simpler.

Case (I.4): χ_H has four genuine conjugate complex zeros.

The problem is again that the zeros of χ_H are not expressible by B, C in a reasonable manner. So we parametrize the set of these matrices H differently. The four zeros are of the form $r \pm is, -r \pm it$ with $2r^2 = s^2+t^2$, because χ_H has no term of order 2 and 3. We may assume $s > 0$, $t > 0$. B and C are then uniquely expressible by r, s, t. The translation part of $\mathbf{H}$ can be rendered to zero since H is regular. Dividing the real Jordan normal form corresponding to these four distinct eigenvalues by $\sqrt{2}r$ and renaming we arrive at $\mathbf{H}$, as announced.

Only the interchange of p, q leads to similar proportional generators $\mathbf{H}$, so we may assume $0 < p \leq q$, i.e.

$$0 < p \leq \omega, \qquad q := \sqrt{1-p^2}.$$

Then the p is separating between similarity classes.

The self-similarities are

$$\mathbf{T} = \left(\begin{array}{cc|cc|c} t_{11} & t_{12} & & & \\ -t_{12} & t_{11} & & & \\ \hline & & t_{33} & t_{34} & \\ & & -t_{34} & t_{33} & \\ \hline & & & & 1 \end{array}\right), \quad \text{and if } p = q \text{ in addition:} \quad \mathbf{T} = \left(\begin{array}{cc|cc|c} & & t_{13} & t_{14} & \\ & & t_{14} & -t_{13} & \\ \hline t_{31} & t_{32} & & & \\ t_{32} & -t_{31} & & & \\ \hline & & & & 1 \end{array}\right)$$

with $\det(\mathbf{T}) \neq 0$. The equivalence of initial points is described by $(x_1^2 + x_2)(x_3^2 + x_4^2) = c^4$ with $c > 0$ (by the rank 1 condition on the pre-metric). This c is classifying. As initial point we may choose $(c, 0, c, 0)^\top$ in $\mathbf{R}^4$ and calculate its orbit as announced.

In all five cases the existence of the surfaces follows from a calculation of the pre-metric at the initial point and by observing that it is diagonal and has $G_{22} \neq 0$ under the given conditions. Ruled surfaces do not occur because, using this pre-metric, one easily sees that x_{uu} is a linear combination $\chi x_u + \beta x_v$ at $(u, v) = (0, 0)$ with $\beta \neq 0$ there. □

6. The noncommutative case

Theorem C. *The non-ruled homogeneous parabolic surfaces of $\mathbf{R}^4$ with noncommutative groups are classified according to the following four cases, specifying normal forms for the generators $\mathbf{E}, \mathbf{H}$ of the corresponding Lie algebra and implicit equations for the orbits, where always $c > 0$:*

(II.1)

$$\mathbf{E} = \left(\begin{array}{cc|cc|c} 0 & & & & \\ 1 & 0 & & & \\ \hline & & 0 & & 1 \\ & & 1 & 0 & 0 \\ \hline & & & & 0 \end{array}\right), \quad \mathbf{H} = \left(\begin{array}{cc|cc|c} 2 & & & & \\ & 1 & & & \\ \hline & & -1 & & \\ & & & -2 & \\ \hline & & & & 0 \end{array}\right), \quad \begin{array}{l} 4096 x_1^2 (x_3^2 - 2x_4)^2 = c^4 \\ (x_1 x_3 - x_2)^4 = c^4 x_1^2 \end{array}$$

(II.2)

$$\mathbf{E} = \left(\begin{array}{cccc|c} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & 1 & 0 & \\ \hline & & & & 0 \end{array}\right), \quad \mathbf{H} = \frac{1}{2}\left(\begin{array}{cccc|c} 3 & & & & \\ & 1 & & & \\ & & -1 & & \\ & & & -3 & \\ \hline & & & & 0 \end{array}\right), \quad \begin{array}{l} 3x_1^2 x_4 - 3x_1 x_2 x_3 + x_2^3 = c x_1 \\ (2x_1 x_3 - x_2^2)^3 = -\dfrac{9}{8} c^2 x_1^2 \end{array}$$

(II.3)

$$\mathbf{E} = \left(\begin{array}{ccc|c|c} 0 & & & & 1 \\ 1 & 0 & & & 0 \\ & 1 & 0 & & 0 \\ \hline & & & 0 & \\ \hline & & & & 0 \end{array}\right), \quad \mathbf{H} = \left(\begin{array}{ccc|c|c} -1 & & & & \\ & -2 & & & \\ & & -3 & & \\ \hline & & & 6 & \\ \hline & & & & 0 \end{array}\right), \quad \begin{array}{l} (x_1^3 - 3x_1 x_2 + 3x_3)^2 = \\ = \dfrac{32}{81}(x_1^2 - 2x_2)^3, \\ (x_1^2 - 2x_2)^6 x_4^2 = c^2 \end{array}$$

(II.4)

$$\mathbf{E} = \left(\begin{array}{ccc|c|c} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ \hline & & & 0 & 1 \\ \hline & & & & 0 \end{array}\right), \quad \mathbf{H} = \frac{1}{3}\left(\begin{array}{ccc|c|c} 4 & & & & \\ & 1 & & & \\ & & -2 & & \\ \hline & & & -3 & \\ \hline & & & & 0 \end{array}\right), \quad \begin{array}{l} 147x_1^2x_4^2 - 294x_1x_2x_4 + \\ +256x_1x_3 + 19x_2^2 = 0, \\ (x_1x_4 - x_2)^8 = c^6x_1^2. \end{array}$$

Any such homogeneous parabolic surface is $\widetilde{\mathbf{SA}}$-equivalent to exactly one of the representatives listed above. In particular, the parameter c is classifying.

Proof. In this case the integrability conditions (3.3) (a) – (d) can be solved for S, B, C recursively from below. Condition (a) then becomes equivalent to the equation

$$F(5F + 2A^2)(5F - 16A^2)(15F + 8A^2) = 0,$$

which has the solutions $F = \varrho A^2$ with either

$$\varrho = 0, \qquad \varrho = -\frac{2}{5}, \qquad \varrho = \frac{16}{5}, \qquad \varrho = -\frac{8}{15}.$$

Introducing this in conditions (b) – (d) expresses S, B, C by ϱ, A. So the two generators $\mathbf{E}, \mathbf{H}$ are determined by ϱ, A alone.

The fine classification follows these four values of ϱ. The corresponding values of $\mathbf{E}, \mathbf{H}$ will be displayed separately in each case. For convenience, we shall multiply the first one with $5/(2A)$. From the equation $[\mathbf{E}, \mathbf{H}] = \mathbf{E}$ it is known [13, Sect. 5] that $\mathbf{E}$ is nilpotent (as well as E) and that span($\mathbf{E}$) is the only ray in the Lie-algebra consisting of nilpotent elements. Thus the group of self-similarities here consists of all $\mathbf{T}$ such that there are $\alpha_1, \beta_1, \beta_2$ satisfying

$$\begin{array}{rcl} \mathbf{TE} & = & \alpha_1\mathbf{ET} \\ \mathbf{TH} & = & \beta_1\mathbf{ET} + \beta_2\mathbf{HT} \end{array}, \qquad \alpha_1\beta_2 \neq 0.$$

The classification of the linear parts of $\mathbf{E}, \mathbf{H}$ follows the same line as in [13, Sect. 5]. The translation parts will be handled by the three step procedure described in Section 2.

Case (II.1): $\varrho = 0$.

The two generators are originally from (3.2) – (3.4), (3.6):

$$\mathbf{E} = \left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline & & & & 0 \end{array}\right), \qquad \mathbf{H} = \left(\begin{array}{cccc|c} 2 & 0 & 0 & 0 & \frac{5}{2}A^{-1} \\ \frac{5}{2}A^{-1} & -1 & 0 & 0 & 0 \\ 0 & \frac{5}{2}A^{-1} & 1 & 0 & 0 \\ 0 & 0 & \frac{5}{2}A^{-1} & -2 & 0 \\ \hline & & & & 0 \end{array}\right).$$

The Jordan normal form of E is $J(2,0) + J(2,0)$, and the eigenvalues of H are $2, 1, -1, -2$, so the pair E, H is of type F (III.1) (with $b = 3$) in the notation of [13]. The three step procedure leads to the new generators as announced. The rank 1 condition on the pre-metric requires $x_1 \neq 0$ and $f(x) = 0$, where

$$f(x) := 64x_1^2(x_3^2 - 2x_4) - (x_1x_3 - x_2)^2.$$

The self-similarities become

$$\mathbf{T} = \left(\begin{array}{cc|cc|c} t_{11} & & & & \\ \beta_1 t_{11} & \alpha_1 t_{11} & & & \\ \hline & & \alpha_1 & & \beta_1 \\ & & \alpha_1\beta_1 & \alpha_1^2 & \frac{1}{2}\alpha_1^2 \\ \hline & & & & 1 \end{array}\right), \qquad \det(\mathbf{T}) \neq 0.$$

From this one deduces that two initial points with $x_1 \neq 0, f(x) = 0$ and $y_1 \neq 0, f(y) = 0$ are equivalent under such a $\mathbf{T}$ with $|\det(\mathbf{T})| = 1$ iff

$$\frac{(x_1x_3 - x_2)^4}{x_1^2} = \frac{(y_1y_3 - y_2)^4}{y_1^2}.$$

The value of this may be set to c^4 with $c > 0$, and this c is then classifying. (For $c = 0$ a ruled surface would arise.) As initial point we choose $x_0 := (1, 0, c, 63/128c^2)^\top$ and calculate the following parametrization of the orbit (and also its implicit equations as announced):

$$x(u,v) := \mathbf{R}^4\text{-part of } \exp(v\mathbf{E})\exp(u\mathbf{H})\left(\frac{x_0}{1}\right) = \begin{pmatrix} e^{2u} \\ ve^{2u} \\ v + ce^{-u} \\ \frac{1}{2}v^2 + cve^{-u} + \frac{63}{128}c^2e^{-2u} \end{pmatrix}.$$

Case (II.2): $\varrho = -\dfrac{2}{5}$.

The two generators are originally

$$\mathbf{E} = \left(\begin{array}{cccc|c} \frac{6}{25}A^2 & 0 & -\frac{21}{625}A^4 & \frac{3}{3125}A^5 & \frac{1}{5}A \\ \frac{1}{5}A & \frac{2}{25}A^2 & -\frac{4}{125}A^3 & -\frac{9}{625}A^4 & 1 \\ 1 & \frac{1}{5}A & -\frac{2}{25}A^2 & -\frac{6}{125}A^3 & 0 \\ 0 & 1 & \frac{1}{5}A & -\frac{6}{25}A^2 & 0 \\ \hline & & & & 0 \end{array}\right)$$

$$\mathbf{H} = \left(\begin{array}{cccc|c} 2 & 0 & -\frac{6}{25}A^2 & -\frac{9}{250}A^3 & \frac{5}{2}A^{-1} \\ \frac{5}{2}A^{-1} & -1 & 0 & -\frac{4}{25}A^2 & 0 \\ 0 & \frac{5}{2}A^{-1} & 1 & -A & 0 \\ 0 & 0 & \frac{5}{2}A^{-1} & -2 & 0 \\ \hline & & & & 0 \end{array}\right).$$

The Jordan normal form of E is $J(4,0)$, and the eigenvalues of H are $3/2, 1/2, -1/2, -3/2$, so the pair E, H is of type F (I), in the notation of [13]. The three step procedure leads to $\mathbf{E}, \mathbf{H}$ as announced. The rank 1 condition on the pre-metric requires $x_1 \neq 0$ and $f(x) = 0$, where

$$f(x) := 9(3x_1^2x_4 - 3x_1x_2x_3 + x_2^3)^2 + 8(2x_1x_3 - x_2^2)^3.$$

The self-similarities become

$$\mathbf{T} = \left(\begin{array}{cccc|c} t_{11} & & & & \\ \beta_1 t_{11} & \alpha_1 t_{11} & & & \\ \frac{1}{2}\beta_1^2 t_{11} & \alpha_1\beta_1 t_{11} & \alpha_1^2 t_{11} & & \\ \frac{1}{6}\beta_1^3 t_{11} & \frac{1}{2}\alpha_1\beta_1^2 t_{11} & \alpha_1^2\beta_1 t_{11} & \alpha_1^3 t_{11} & \\ \hline & & & & 1 \end{array}\right), \qquad \det(\mathbf{T}) \neq 0.$$

From this one can discuss the equivalence of initial points as usual. The result is similar to the vector space case F(I) in [13]. The equivalence is described by the two algebraic equations as announced, with $c > 0$ being classifying. The implicit equations are the same. Choosing the initial point $(1, 0, -(3/8c)^{2/3}, c/3)^\top$, the orbits are parametrized as follows:

$$x(u,v) = \begin{pmatrix} e^{3/2u} \\ ve^{3/2u} \\ \frac{1}{2}v^2e^{3/2u} - \frac{1}{8}\sqrt[3]{72}c^{2/3}e^{-1/2u} \\ \frac{1}{6}v^3e^{3/2u} - \frac{1}{8}\sqrt[3]{72}c^{2/3}ve^{-1/2u} + \frac{1}{3}ce^{-3/2u} \end{pmatrix}.$$

Case (II.3): $\varrho = \dfrac{16}{5}$.

The two generators are originally

$$\mathbf{E} = \left(\begin{array}{cccc|c} -\frac{48}{25}A^2 & 0 & \frac{2304}{625}A^4 & -\frac{9216}{3125}A^5 & -\frac{8}{5}A \\ -\frac{8}{5}A & -\frac{16}{25}A^2 & \frac{512}{125}A^3 & -\frac{2304}{625}A^4 & 1 \\ 1 & -\frac{8}{5}A & \frac{16}{25}A^2 & -\frac{192}{125}A^3 & 0 \\ 0 & 1 & -\frac{8}{5}A & \frac{48}{25}A^2 & 0 \\ \hline & & & & 0 \end{array}\right)$$

$$\mathbf{H} = \left(\begin{array}{cccc|c} 2 & 0 & \frac{48}{25}A^2 & \frac{1536}{125}A^3 & \frac{5}{2}A^{-1} \\ \frac{5}{2}A^{-1} & -1 & 0 & \frac{272}{25}A^2 & 0 \\ 0 & \frac{5}{2}A^{-1} & 1 & 8A & 0 \\ 0 & 0 & \frac{5}{2}A^{-1} & -2 & 0 \\ \hline & & & & 0 \end{array}\right).$$

The Jordan normal form of E is $J(3,0) + J(1,0)$, and the eigenvalues of H are $-1, -2, -3, 6$, so the pair E, H is of type F (II.1) (with $b = -1$), in the notation of [13]. The three step procedure leads to $\mathbf{E}, \mathbf{H}$ as announced. The rank 1 condition on the pre-metric requires $x_4 \neq 0$ and $f(x) = 0$, where

$$f(x) := 81(-3\,x_2 x_1 + 3\,x_3 + x_1^3)^2 - 32(x_1^2 - 2\,x_2)^3.$$

The self-similarities become

$$\mathbf{T} = \left(\begin{array}{cccc|c} \alpha_1 & & & & \beta_1 \\ \alpha_1\beta_1 & \alpha_1^2 & & & \frac{1}{2}\beta_1^2 \\ \frac{1}{2}\alpha_1\beta_1^2 & \alpha_1^2\beta_1 & \alpha_1^3 & & \frac{1}{6}\beta_1^3 \\ 0 & 0 & 0 & t_{44} & 0 \\ \hline & & & & 1 \end{array}\right), \qquad \det(\mathbf{T}) \neq 0.$$

From this one can discuss the equivalence of initial points as usual. The result is again similar to the vector space case F(I) in [13]. The equivalence is described by the two algebraic equations as announced, with $c > 0$ being classifying. The implicit equations are the same. Choosing the initial point $(1, 0, (4\sqrt{2}-9)/27, c)^\top$ in $\mathbf{R}^4$, the orbits are parametrized as follows:

$$x(u,v) = \begin{pmatrix} v + e^{-u} \\ \frac{1}{2}\,v^2 + ve^{-u} \\ \frac{1}{6}\,v^3 + \frac{1}{2}v^2 e^{-u} + \frac{4\sqrt{2}-9}{27}e^{-3u} \\ ce^{6\,u} \end{pmatrix}.$$

Case (II.4): $\varrho = -\dfrac{8}{15}$.

The two generators are originally

$$\mathbf{E} = \left(\begin{array}{cccc|c} \frac{8}{25}\,A^2 & 0 & -\frac{64}{1125}\,A^4 & \frac{256}{84375}\,A^5 & \frac{4}{15}\,A \\ \frac{4}{15}\,A & \frac{8}{75}\,A^2 & -\frac{64}{3375}\,A^3 & -\frac{64}{2025}\,A^4 & 1 \\ 1 & \frac{4}{15}\,A & -\frac{8}{75}\,A^2 & -\frac{256}{3375}\,A^3 & 0 \\ 0 & 1 & \frac{4}{15}\,A & -\frac{8}{25}\,A^2 & 0 \\ \hline & & & & 0 \end{array}\right)$$

$$\mathbf{H} = \left(\begin{array}{cccc|c} 2 & 0 & -\frac{8}{25}\,A^2 & -\frac{64}{1125}\,A^3 & \frac{5}{2}\,A^{-1} \\ \frac{5}{2}\,A^{-1} & -1 & 0 & -\frac{104}{675}\,A^2 & 0 \\ 0 & \frac{5}{2}\,A^{-1} & 1 & -\frac{4}{3}\,A & 0 \\ 0 & 0 & \frac{5}{2}\,A^{-1} & -2 & 0 \\ \hline & & & & 0 \end{array}\right).$$

The Jordan normal form of E is again $J(3,0) + J(1,0)$, and the eigenvalues of H are $4/3, 1/3, -2/3, -1$, so the pair E, H is of type F (II.1) (with $b = 4/3$), in the notation of [13]. The three step procedure gives here the new generators as announced. The rank 1 condition on the pre-metric requires $x_1 \neq 0$ and $f(x) = 0$, where

$$f(x) := 19(x_1x_4 - x_2)^2 + 256x_1x_4(x_1x_4 - x_2) + 128x_1(-{x_4}^2x_1 + 2x_3).$$

The self-similarities become

$$\mathbf{T} = \left(\begin{array}{cccc|c} t_{11} & & & & 0 \\ \beta_1 t_{11} & \alpha_1 t_{11} & & & 0 \\ \frac{1}{2}\beta_1^2 t_{11} & \alpha_1\beta_1 t_{11} & \alpha_1^2 t_{11} & & 0 \\ 0 & 0 & 0 & \alpha_1 & \beta_1 \\ \hline & & & & 1 \end{array}\right), \qquad \det(\mathbf{T}) \neq 0.$$

From this one can discuss the equivalence of initial points as usual. It is described by

$$(x_1x_4 - x_2)^8 = c^6 x_1^2, \quad c > 0.$$

The value of c is then classifying. ($c = 0$ leads to a ruled surface.) The implicit equations are this together with $f(x) = 0$. Choosing the initial point $(c, 0, -147/256c, 1)^\top$, the orbits are parametrized as follows:

$$x(u,v) = \begin{pmatrix} ce^{4/3\,u} \\ cve^{4/3\,u} \\ \frac{1}{256}\,c(128v^2e^{4/3\,u} - 147e^{-2/3\,u}) \\ v + e^{-u} \end{pmatrix}.$$

The existence of these noncommutative cases is verified as in the end of Theorem B, with the distinction that the pre-metric at the initial point does not come out diagonal (but has rank 1, of course). The reason is the reparametrization performed in the end of Section 3. So one cannot see directly that the surfaces are non-ruled. However, one can go the way back and verify that, expressed in distinguished coordinates, we have the same linear dependency as in the end of Theorem B, with $\beta \neq 0$. □

Concluding remark on the Gauss images. The Gauss image of a homogeneous parabolic surface in $\mathbf{R}^4$ is a homogeneous surface in $\mathbf{P}^3$. On the present Lie group level, these Gauss images can be listed in normal form with almost no computation: The asymptotic directions along an orbit are obtained by transporting a fixed asymptotic vector $a_0 \in \mathbf{R}^4 \setminus \{0\}$ at the initial point $x_0 \in \mathbf{R}^4$ around with the generating group Γ. Thus, if Γ has the above parametrization $\gamma(u,v) := \exp(v\mathbf{E})\exp(u\mathbf{H})$, *a parametrization of the Gauss image in homogeneous coordinates is*

$$a(u,v) := \exp(vE)\exp(uH)a_0,$$

E, H the linear parts of $\mathbf{E}, \mathbf{H}$. As to the a_0, one may choose a vector $X_0 \neq 0$ from the null space of the pre-metric at x_0, say $X_0 = \mu(\partial/\partial u)_{(0,0)} + \nu(\partial/\partial v)_{(0,0)}$, and take its tangential image

$$a_0 := \mathbf{R}^4\text{-part of } (\mu\mathbf{H} + \nu\mathbf{E})\begin{pmatrix} x_0 \\ 1 \end{pmatrix}.$$

This is valid whether $\mathbf{E}, \mathbf{H}$ have normal form or not. So E, H are the generators for the group acting on the Gauss image. Taking $\mathbf{E}, \mathbf{H}, x_0$ as in Theorems B and C yields the desired list of (projectively homogeneous) Gauss images in normal form. Some of them turn out to be ruled, namely for (I.1), (I.3.b) with $p = 0$, (I.4) with $p = \omega$, and (II.1). Of course there will result a sublist of all projectively homogeneous surfaces as established by [6] in the non-ruled case and [3] in the ruled case. A reconstruction of the homogeneous parabolic surfaces in $\mathbf{R}^4$ from the presumable homogeneous Gauss images would offer several difficulties because the pre-images are very ample and require an additional integration. Moreover there would arise problems with the separation of representatives. A classification of non-ruled projectively homogeneous surfaces in $\mathbf{P}^3$ on the basis of [13] has been described recently in [5] and [8].

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