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Modular Elements in Congruence Lattices of G-sets

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Abstract. An element x in a lattice $\langle L;\ \vee, \wedge\rangle$ is called *modular* if $(x \vee y) \wedge z = (x \wedge z) \vee y$ whenever $y, z \in L$ and $y \leq z$. We describe modular elements in congruence lattices of G-sets.

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Introduction

Let A be a non-empty set, G a group, and φ a homomorphism from G into the full transformation group of A. Then A may be considered as a unary algebra with the set G of operations where an operation $g \in G$ is defined by the rule $g(a) = (\varphi(g))(a)$ for every $a \in A$. These algebras are called *G-sets.* Some basic information about G-sets and, in particular, about their congruences may be found, for example, in [1]. In [3], we described G-sets with several natural properties of their congruences, in particular, G-sets whose congruence lattice is distributive, modular, arguesian, semimodular, etc.

Studying congruences of G-sets appears to be sufficiently natural per se. Our main motivation, however, comes from a different source. Recent results by M. V. Volkov and the author show that, for some wide classes of semigroup varieties, the structure of subvariety lattices can be described in terms of congruence lattices of certain G-sets (see, in particular, [5, 6]). In particular, it is the case for overcommutative semigroup varieties (that is, varieties containing the variety of all commutative semigroups). These results together with the mentioned results of [3] permit to obtain a series of results concerning identities in lattices of overcommutative varieties and related questions (see [4]).

Here we consider a new restriction to congruences on G-sets. Recall that an element x in a lattice $\langle L;\, \vee, \wedge\rangle$ is called *modular* if

$$(x \vee y) \wedge z = (x \wedge z) \vee y$$

whenever $y, z \in L$ and $y \leq z$. In the present paper we describe modular elements in congruence lattices of G-sets. We think that this result certainly may be applied for a description of modular elements in the lattice of overcommutative semigroup varieties.

The paper consists of three sections. Section 1 contains preliminary information from [3] about congruences on G-sets. At the conclusion of this section, we formulate the main result of the work. It is proved in Sections 2 and 3.

1. Preliminaries

The congruence lattice of a G-set A is denoted by $Con(A)$. A G-set A is said to be *transitive* if, for any pair of elements $a, b \in A$, there exists $g \in G$ such that $g(a) = b$. It is well known (see, for example, [1, Lemma 4.20]) that if A is a transitive G-set then the lattice $Con(A)$ is isomorphic to an interval of the subgroup lattice of G (more precisely, $Con(A) \cong [Stab_G(a), G]$ where a is an arbitrary element in A and $Stab_G(a) = \{g \in G \mid g(a) = a\}$).

A transitive G-subset of a G-set A is called an *orbit* of A. Clearly, any G-set is a disjoint union of its orbits. In view of the remark in the previous paragraph, it is natural to investigate the lattice $Con(A)$ modulo the congruence lattices of the orbits of A.

Let A be a G-set and μ a congruence on A. We say that μ *isolates* an orbit B of A if B is the union of μ-classes. We say that μ *connects* (*collapses*) orbits B and C of A if $B \neq C$ and there are elements $b \in B$ and $c \in C$ with $b\,\mu\,c$ (respectively, if $B \neq C$ and $x\,\mu\,y$ whenever $x, y \in B \cup C$). If $\mathcal{M}$ is a non-singleton family of orbits of A then we say that μ *connects* (*collapses*) $\mathcal{M}$ if μ connects (collapses) any pair of different orbits of $\mathcal{M}$. We call μ *greedy* if it collapses any pair of orbits it connects. By μ^* we denote the following binary relation on the set of all orbits of A: $B\,\mu^*\,C$ if and only if either $B = C$ or μ connects B and C. Clearly, μ^* is an equivalence relation. Let $GCon(A)$ denote the set of all greedy congruences of A. We denote by $Orb(A)$ the set of all orbits of A and by $Eq(X)$ the lattice of equivalence relations on a set X. The following lemma clarifies the structure of $GCon(A)$.

Lemma 1.1. ([3, Lemma 1.1 and Proposition 1.2]) *Let A be a G-set and $Orb(A) = \{A_i \mid i \in I\}$. The set $GCon(A)$ forms a sublattice of the lattice $Con(A)$. The lattice $GCon(A)$ is isomorphic to a subdirect product of the lattices $Eq(Orb(A))$ and $Con(A_i)$ where i runs over I. The corresponding embedding f of $GCon(A)$ into $Eq(Orb(A)) \times \prod\limits_{i\in I} Con(A_i)$ is given by the rule: if $\alpha \in GCon(A)$ then $f(\alpha) = (\alpha^*; \ldots, \alpha_i, \ldots)$ where α_i is the restriction of α to the orbit A_i.* □

We need also information about non-greedy congruences on a G-set.

Lemma 1.2. ([3, proof of Proposition 1.3]) *Let A be a G-set. Suppose that a congruence μ on A connects but does not collapse the orbits A_i and A_j of A. Put $\mu_i = \mu|_{A_i}$ and $\mu_j = \mu|_{A_j}$. Then the G-sets A_i/μ_i and A_j/μ_j are isomorphic. More precisely, the following mapping $\mu^*_{i,j} : A_i/\mu_i \longmapsto A_j/\mu_j$ is an isomorphism: if M_i is a μ_i-class, then $M_j = \mu^*_{i,j}(M_i)$ is the μ_j-class such that $x\,\mu\,y$ for some $x \in M_i$ and $y \in M_j$.* □

It is evident that if, in notation of Lemma 1.2, $M_j = \mu^*_{i,j}(M_i)$ then $x\,\mu\,y$ whenever $x, y \in M_i \cup M_j$. One can note also that the isomorphisms $\mu^*_{i,j}$ and $\mu^*_{j,i}$ are mutually inverse.

Let μ be a non-greedy congruence on A and $\mathcal{M} = \{M_i \mid i \in J\}$ a non-singleton μ^*-class such that μ does not collapse $\mathcal{M}$. For any congruence ξ on A and $i \in J$, let $\xi_i = \xi|_{M_i}$. Let us fix some $j \in J$. By Lemma 1.2, $M_i/\mu_i \cong M_j/\mu_j$ for any $i \in J$. Let $\mu^\#_{i,j}$ be the isomorphism from $Con(M_i/\mu_i)$ onto $Con(M_j/\mu_j)$ induced by the isomorphism $\mu^*_{i,j}$ from M_i/μ_i onto M_j/μ_j. For a congruence χ on A with $\chi \supseteq \mu_i$, we put $\chi_{i,j,\mu} = \mu^\#_{i,j}(\chi_i/\mu_i)$. One can note that $\mu^*_{j,j}$ and $\mu^\#_{j,j}$ are identical mappings and that $\chi_{j,j,\mu} = \chi_j/\mu_j$.

The main result of the work is the following

Theorem. *Let A be a G-set. A congruence α on A is a modular element in the lattice $Con(A)$ if and only if the following holds:*

a) *α either isolates every orbit of A or collapses some non-singleton set $\mathcal{P}$ of orbits of A and isolates every orbit of A outside $\mathcal{P}$;*

b) *the restriction of α to any orbit of A is a modular element in the congruence lattice of this orbit;*

c) *if β and γ are congruences on A with $\beta \subset \gamma$ and $\mathcal{B} = \{B_i \mid i \in J\}$ is a non-singleton β^*-class such that α isolates every orbit in $\mathcal{B}$ while each of the congruences β and γ does not collapse $\mathcal{B}$ then there is a $j \in J$ such that*

$$\Big(\bigvee_{i\in J} \alpha_{i,j,\beta}\Big) \wedge \gamma_{j,j,\beta} = \bigvee_{i\in J}(\alpha_{i,j,\beta} \wedge \gamma_{j,j,\beta}). \tag{1}$$

It is easy to verify that if α is a modular element in $Con(A)$ then the equality (1) actually holds for every $j \in J$.

Clearly, a lattice L is modular if and only if every $x \in L$ is a modular element in L. This permits us to easily deduce from Theorem the description of G-sets with modular congruence lattices given by [3, Corollary 2.4].

We note also that the lattice of equivalence relations on a non-empty set A can be considered as the congruence lattice of some G-set. Indeed, we may consider A as a G-set where G is the singleton group. Clearly, any equivalence relation on A is a congruence of this G-set and the lattices $Eq(A)$ and $Con(A)$ coincide. Thus, our Theorem extends the description of modular elements in lattices of equivalence relations obtained in [2].

Sections 2 and 3 are devoted, respectively, to the proof of necessity and sufficiency of Theorem.

2. Necessity

We start with the following

Lemma 2.1. *Let A be a G-set. If a congruence α on A is a modular element in the lattice $Con(A)$ then α is greedy.*

Proof. Suppose that α is not greedy. Then it connects but does not collapse some pair of orbits A_i and A_j of A. Let us consider the following three congruences on A:

$\alpha_{i,j}$: $x\,\alpha_{i,j}\,y$ if and only if either $x = y$ or $x, y \in A_i$ and $x\,\alpha\,y$ or $x, y \in A_j$ and $x\,\alpha\,y$;

β: $x\,\beta\,y$ if and only if either $x = y$ or $x, y \in A_i$ and $x\,\alpha\,y$ or $x, y \in A_j$;

γ: $x\,\gamma\,y$ if and only if either $x = y$ or $x, y \in A_i$ or $x, y \in A_j$.

Since α connects the orbits A_i and A_j, if the restriction of α to A_i is the universal relation on this orbit then α collapses A_i and A_j. Hence there are elements $x, y \in A_i$ such that $(x, y) \notin \alpha$. On the other hand, since α connects A_i and A_j, there is an element $x' \in A_j$ such that $x\,\alpha\,x'$. Furthermore, since A_i is an orbit, there is $g \in G$ with $y = g(x)$. We have $g(x)\,\alpha\,g(x')$ and $g(x') \in A_j$. Hence $x\,\alpha\,x'\,\beta\,g(x')\,\alpha\,g(x) = y$. Thus, $(x, y) \in \alpha \vee \beta$. Furthermore, it is evident that $x\,\gamma\,y$. Therefore, $(x, y) \in (\alpha \vee \beta) \wedge \gamma$. Since α is a modular element in $Con(A)$ and $\beta < \gamma$, we have $(x, y) \in (\alpha \wedge \gamma) \vee \beta$. But it is evident that $(\alpha \wedge \gamma) \vee \beta = \alpha_{i,j} \vee \beta = \beta$. We have $x\,\beta\,y$. Since $x, y \in A_i$, this implies $x\,\alpha\,y$. A contradiction. □

Now, let us verify the necessity of Theorem. Let A be a G-set and α a modular element in $Con(A)$. By Lemma 2.1, α is greedy. Since $GCon(A)$ is a sublattice in $Con(A)$ (see Lemma 1.1), α is a modular element in $GCon(A)$ too. It is clear that if a lattice L is a subdirect product of lattices $\{L_\lambda \mid \lambda \in \Lambda\}$ then an element $x \in L$ is modular if and only if, for any $\lambda \in \Lambda$, the projection of x to L_λ is a modular element in L_λ. By Lemma 1.1, we immediately have that the condition b) of Theorem holds and that the equivalence relation α^* is a modular element in the lattice $Eq(Orb(A))$. Results of the article [2] easily imply that an equivalence relation ρ on a set X is a modular element in the lattice $Eq(X)$ if and only if either ρ is the equality relation or ρ has exactly one non-singleton class. We see that the condition a) of Theorem holds. It remains to verify the condition c). Here we need the following

Lemma 2.2. *Let A be a G-set, B_i and B_j different orbits of A and β and γ congruences on A such that $\beta \subseteq \gamma$ and each of these congruences connects but does not collapse the orbits B_i and B_j. Then $\gamma_{i,j,\beta} = \gamma_{j,j,\beta}$.*

Proof. For $\mu \in \{\beta, \gamma\}$, we put $\mu_i = \mu|_{B_i}$ and $\mu_j = \mu|_{B_j}$.

Let $P\,\gamma_{j,j,\beta}\,Q$, $p \in P$ and $q \in Q$. Since $\gamma_{j,j,\beta} = \gamma_j/\beta_j$ and $\beta_j \subseteq \gamma_j$, we have $p\,\gamma\,q$. Let now $P' = \beta^*_{j,i}(P)$, $Q' = \beta^*_{j,i}(Q)$, $p' \in P'$ and $q' \in Q'$. Then $p'\,\beta\,p\,\gamma\,q\,\beta\,q'$ whence $p'\,\gamma\,q'$. This implies that $(P', Q') \in \gamma_i/\beta_i$. Since $P = \beta^*_{i,j}(P')$ and $Q = \beta^*_{i,j}(Q')$, we have $P\,\gamma_{i,j,\beta}\,Q$. Thus, $\gamma_{j,j,\beta} \subseteq \gamma_{i,j,\beta}$.

Let now $P\,\gamma_{i,j,\beta}\,Q$. Put $P' = \beta^*_{j,i}(P)$ and $Q' = \beta^*_{j,i}(Q)$. Then $(P', Q') \in \gamma_i/\beta_i$. Let $p' \in P'$ and $q' \in Q'$. Since $\beta_i \subseteq \gamma_i$, we have $p'\,\gamma\,q'$. Let now $p \in P$ and $q \in Q$. Since $P = \beta^*_{i,j}(P')$ and $Q = \beta^*_{i,j}(Q')$, we have $p\,\beta\,p'\,\gamma\,q'\,\beta\,q$ whence $p\,\gamma\,q$. This implies that $(P, Q) \in \gamma_j/\beta_j = \gamma_{j,j,\beta}$. Thus, $\gamma_{i,j,\beta} \subseteq \gamma_{j,j,\beta}$ whence $\gamma_{i,j,\beta} = \gamma_{j,j,\beta}$. □

Let us return to the proof of necessity. Let β and γ be congruences on A with $\beta \subset \gamma$ and $\mathcal{B} = \{B_i \mid i \in J\}$ a non-singleton β^*-class such that α isolates every orbit of $\mathcal{B}$ while each of the congruences β and γ does not collapse $\mathcal{B}$. We need to verify that the equality (1) holds for some $j \in J$. Let us prove that, in fact, this equality is valid for an arbitrary $j \in J$. Indeed, let us fix $j \in J$. Clearly, the right part of the equality (1) is contained in the left one. Thus, it remains to check that

$$\Big(\bigvee_{i\in J} \alpha_{i,j,\beta}\Big) \wedge \gamma_{j,j,\beta} \leq \bigvee_{i\in J} (\alpha_{i,j,\beta} \wedge \gamma_{j,j,\beta}).$$

Let P and Q be β_j-classes such that $(P,Q) \in (\bigvee_{i \in J} \alpha_{i,j,\beta}) \wedge \gamma_{j,j,\beta}$. Then $P \, \gamma_{j,j,\beta} \, Q$ and there are β_j-classes $P_0, P_1, \ldots, P_n$ such that $P = P_0$, $P_n = Q$ and, for any $s = 0, 1, \ldots, n-1$, there is $i_s \in J$ such that $P_s \, \alpha_{i_s,j,\beta} \, P_{s+1}$. Put $P'_0 = \beta^*_{j,i_0}(P_0)$ and $P'_1 = \beta^*_{j,i_0}(P_1)$. By the definition of the mapping β^*_{j,i_0}, we have $(P'_0, P'_1) \in \alpha_{i_0}/\beta_{i_0}$. This means that if $p'_0 \in P'_0$ and $p'_1 \in P'_1$ then $(p'_0, p'_1) \in \alpha_{i_0} \vee \beta_{i_0}$. Besides that, if $p_0 \in P_0$ and $p'_0 \in P'_0$ then $p_0 \, \beta \, p'_0$. Furthermore, put $P''_1 = \beta^*_{j,i_1}(P_1)$ and $P'_2 = \beta^*_{j,i_1}(P_2)$. Then $(P''_1, P'_2) \in \alpha_{i_1}/\beta_{i_1}$, and therefore, $(p''_1, p'_2) \in \alpha_{i_1} \vee \beta_{i_1}$ whenever $p''_1 \in P''_1$ and $p'_2 \in P'_2$. Furthermore, since $\beta^*_{i_1,j}(P''_1) = P_1 = \beta^*_{i_0,j}(P'_1)$, we have that if $p_1 \in P_1$, $p'_1 \in P'_1$ and $p''_1 \in P''_1$ then $p'_1 \, \beta \, p_1 \, \beta \, p''_1$ whence $p'_1 \, \beta \, p''_1$. Furthermore, put $P''_2 = \beta^*_{j,i_2}(P_2)$ and $P'_3 = \beta^*_{j,i_2}(P_3)$. Then $p'_2 \, \beta \, p''_2$ and $(p''_2, p'_3) \in \alpha_{i_2} \vee \beta_{i_2}$ whenever $p'_2 \in P'_2$, $p''_2 \in P''_2$ and $p'_3 \in P'_3$. Continuing this process, we construct $P''_3, P'_4, P''_4, \ldots, P'_{n-1}, P''_{n-1}$ and P'_n such that $p'_3 \, \beta \, p''_3$, $(p''_3, p'_4) \in \alpha_{i_3} \vee \beta_{i_3}$, $\ldots$, $p'_{n-1} \, \beta \, p''_{n-1}$ and $(p''_{n-1}, p'_n) \in \alpha_{i_{n-1}} \vee \beta_{i_{n-1}}$ whenever $p'_3 \in P'_3$, $p''_3 \in P''_3$, $p'_4 \in P'_4$, $\ldots$, $p'_{n-1} \in P'_{n-1}$, $p''_{n-1} \in P''_{n-1}$ and $p'_n \in P'_n$. Further, if $p'_n \in P'_n$ and $p_n \in P_n$ then $p'_n \, \beta \, p_n$. Now let us fix arbitrary elements $p \in P_0$ and $q \in P_n$. The considerations given above show that $(p,q) \in \alpha \vee \beta$. In addition, since $P_0 \, \gamma_{j,j,\beta} \, P_n$ and $\beta \subseteq \gamma$, we have $p \, \gamma \, q$. Thus, $(p,q) \in (\alpha \vee \beta) \wedge \gamma$.

Since α is a modular element in the lattice $Con(A)$, we have $(p,q) \in (\alpha \wedge \gamma) \vee \beta$. Hence there are elements $r_0, r_1, \ldots, r_m \in A$ such that $r_0 = p$, $r_m = q$ and, for any $s = 0, 1, \ldots, m-1$, either $(r_s, r_{s+1}) \in \alpha \wedge \gamma$ or $r_s \, \beta \, r_{s+1}$. Let A_i be the orbit of A such that $r_i \in A_i$ $(i = 0, 1, \ldots, m)$. Recall that $r_0 = p \in P_0 = P$ and $r_m = q \in P_n = Q$. Since P and Q are β_j-classes, we see that $A_0 = A_m = B_j \in \mathcal{B}$. By the hypothesis, α isolates any orbit of $\mathcal{B}$, and therefore, $\alpha \wedge \gamma$ also does. This means that, for any $s = 0, 1, \ldots, m-1$, if $A_s \neq A_{s+1}$ then $r_s \, \beta \, r_{s+1}$, and therefore, $A_0, A_1, \ldots, A_m \in \mathcal{B}$. Suppose that $A_0 = A_1 = \cdots = A_{k_0}$ (all these orbits coincide with B_j) and either $k_0 = m$ or $A_{k_0} \neq A_{k_0+1}$. Then $(r_0, r_{k_0}) \in (\alpha_j \wedge \gamma_j) \vee \beta_j$. Recall that the β_j-class containing $r_0 = p$ is P. Let us denote by R_1 the β_j-class containing r_{k_0}. Clearly, $(P, R_1) \in (\alpha_j \wedge \gamma_j)/\beta_j = (\alpha_j/\beta_j) \wedge (\gamma_j/\beta_j) = \alpha_{j,j,\beta} \wedge \gamma_{j,j,\beta}$. Put $R'_0 = P$ and $R'_1 = R_1$. Furthermore, suppose that $k_0 < m$. Let $A_{k_0+1} = A_{k_0+2} = \cdots = A_{k_1}$ and either $k_1 = m$ or $A_{k_1} \neq A_{k_1+1}$. Since $A_{k_0+1} \in \mathcal{B}$, we have that $A_{k_0+1} = A_{k_0+2} = \cdots = A_{k_1} = B_{i_1}$ for some $i_1 \in J$. Clearly, $(r_{k_0+1}, r_{k_1}) \in (\alpha_{i_1} \wedge \gamma_{i_1}) \vee \beta_{i_1}$. We denote by R_2 the β_{i_1}-class containing r_{k_0+1} and by R_3 the β_{i_1}-class containing r_{k_1}. Then $(R_2, R_3) \in (\alpha_{i_1} \wedge \gamma_{i_1})/\beta_{i_1} = (\alpha_{i_1}\beta_{i_1}) \wedge (\gamma_{i_1}/\beta_{i_1})$. Recall that $r_{k_0} \, \beta \, r_{k_0+1}$. Therefore, $(\alpha \wedge \gamma)^*_{i_1,j}(R_2) = R'_1$. Put $R'_2 = (\alpha \wedge \gamma)^*_{i_1,j}(R_3)$. Then $(R'_1, R'_2) \in \alpha_{i_1,j,\beta} \wedge \gamma_{i_1,j,\beta}$. Using Lemma 2.2, we have $(R'_1, R'_2) \in \alpha_{i_1,j,\beta} \wedge \gamma_{j,j,\beta}$. Continuing this process, we obtain β_j-classes $R'_3, \ldots, R'_t$ such that $R'_t = Q$ and $(R'_2, R'_3) \in \alpha_{i_2,j,\beta} \wedge \gamma_{j,j,\beta}$, $(R'_3, R'_4) \in \alpha_{i_3,j,\beta} \wedge \gamma_{j,j,\beta}$, $\ldots$, $(R'_{t-1}, R'_t) \in \alpha_{i_{t-1},j,\beta} \wedge \gamma_{j,j,\beta}$ for some $i_2, i_3, \ldots, i_{t-1} \in J$. Therefore, $(P,Q) \in \bigvee_{i \in J} (\alpha_{i,j,\beta} \wedge \gamma_{j,j,\beta})$, and we are done.

3. Sufficiency

Let α be a congruence on a G-set A and suppose the conditions a)–c) of Theorem hold. We are going to verify that α is a modular element in the lattice $Con(A)$. Let us fix congruences β and γ on A with $\beta \subseteq \gamma$. It is clear that $(\alpha \wedge \gamma) \vee \beta \subseteq (\alpha \vee \beta) \wedge \gamma$. It remains to verify that $(\alpha \vee \beta) \wedge \gamma \subseteq (\alpha \wedge \gamma) \vee \beta$. Evidently, we may (and will) assume that $\beta \subset \gamma$. Suppose that $x, y \in A$ and $(x,y) \in (\alpha \vee \beta) \wedge \gamma$. We need to check that

$$(x,y) \in (\alpha \wedge \gamma) \vee \beta. \tag{2}$$

Since $(x,y) \in (\alpha \vee \beta) \wedge \gamma$, there are elements $x_0, x_1, \ldots, x_n \in A$ such that $x_0 = x$, $x_n = y$ and, for any $i = 0,1,\ldots,n-1$, either $x_i \, \alpha \, x_{i+1}$ or $x_i \, \beta \, x_{i+1}$. Besides that $x \, \gamma \, y$. Let us denote by A_i the orbit of A containing x_i $(i = 0,1,\ldots,n)$. If $A_0 = A_1 = \cdots = A_n$ then $(x,y) \in (\alpha_0 \vee \beta_0) \wedge \gamma_0$ (where μ_0 is the restriction of μ to A_0 for any $\mu \in \{\alpha, \beta, \gamma\}$). By the condition b) of Theorem, $(x,y) \in (\alpha_0 \wedge \gamma_0) \vee \beta_0 \subseteq (\alpha \wedge \gamma) \vee \beta$, and we are done. Thus, we may assume that $A_i \neq A_{i+1}$ for some $i \in \{0,1,\ldots,n-1\}$.

Suppose at first that there is $i \in \{0,1,\ldots,n-1\}$ such that $A_i \neq A_{i+1}$ and $x_i \, \alpha \, x_{i+1}$. Let

$$j = \min_{0 \le i \le n-1} \{A_i \neq A_{i+1},\ x_i \, \alpha \, x_{i+1}\}$$

and

$$k = \max_{1 \le i \le n} \{A_{i-1} \neq A_i,\ x_{i-1} \, \alpha \, x_i\}.$$

Both the orbits A_j and A_k belong to non-singleton α^*-classes. But by the condition a) of Theorem, α^* contains at most one non-singleton class. Hence the orbits A_j and A_k lie in the same α^*-class. In other words, either $A_j = A_k$ or α connects A_j and A_k. The condition a) of Theorem implies that α is greedy. Hence either $A_j = A_k$ or α collapses A_j and A_k. The choice of the orbits A_j and A_k guarantees that either $A_0 = A_j$ or β connects the orbits A_0 and A_j as well as either $A_k = A_n$ or β connects the orbits A_k and A_n. Since $x \in A_0$ and $y \in A_n$, there are elements $x' \in A_j$ and $y' \in A_k$ such that $x \, \beta \, x'$ and $y' \, \beta \, y$. We have $x' \, \beta \, x \, \gamma \, y \, \beta \, y'$, and therefore, $x' \, \gamma \, y'$. The orbit A_j belongs to a non-singleton α^*-class, and therefore, the restriction of α to this orbit is the universal relation on A_j. In particular, this means that if $A_j = A_k$ then $x' \, \alpha \, y'$. If α collapses A_j and A_k then, evidently, $x' \, \alpha \, y'$ too. Hence $(x', y') \in \alpha \wedge \gamma$. Thus, $x \, \beta \, x'$, $(x', y') \in \alpha \wedge \gamma$ and $y' \, \beta \, y$. Therefore, $(x,y) \in (\alpha \wedge \gamma) \vee \beta$.

Thus, we may assume that, for any $i = 0,1,\ldots,n-1$, if $A_i \neq A_{i+1}$ then $x_i \, \beta \, x_{i+1}$. In other words, all the orbits $A_0, A_1, \ldots, A_n$ lie in the same β^*-class. Let us denote this β^*-class by $\mathcal{B}$. Let $\mathcal{B} = \{B_i \mid i \in J\}$. Since $A_i \neq A_{i+1}$ for some $i \in \{0,1,\ldots,n-1\}$, the class $\mathcal{B}$ is non-singleton. We need the following easy

Lemma 3.1. *Let A be a G-set, μ a congruence on A and $\mathcal{M}$ a non-singleton μ^*-class. If μ collapses some pair of orbits of $\mathcal{M}$ then μ collapses $\mathcal{M}$.*

Proof. Let $\mathcal{M} = \{M_i \mid i \in J\}$. Suppose that μ collapses orbits M_j and M_k where $j, k \in J$ and $j \neq k$. Let $x, y \in \bigcup_{i \in J} M_i$. Then there are elements $x' \in M_j$ and $y' \in M_k$ such that $x \, \mu \, x'$ and $y' \, \mu \, y$. Furthermore, $x' \, \mu \, y'$ because μ collapses M_j and M_k. We have $x \, \mu \, y$. □

Thus, if β collapses some pair of orbits $B_i, B_j \in \mathcal{B}$ then $x \, \beta \, y$. This immediately implies the inclusion (2). Thus, we may assume that β does not collapse any pair of orbits of $\mathcal{B}$.

Since $\beta \subseteq \gamma$, all orbits of $\mathcal{B}$ lie in the same γ^*-class. Suppose that γ collapses some pair of orbits of $\mathcal{B}$. Then, by Lemma 3.1, γ collapses $\mathcal{B}$. In particular, $x_i \, \gamma \, x_{i+1}$ for all $i = 0,1,\ldots,n-1$. Therefore, for any $i = 0,1,\ldots,n-1$, either $(x_i, x_{i+1}) \in \alpha \wedge \gamma$ or $x_i \, \beta \, x_{i+1}$. This implies (2). Thus, we may assume that γ does not collapse any pair of orbits of $\mathcal{B}$ either.

Suppose now that there is an orbit $B_i \in \mathcal{B}$ such that the restriction of α to B_i is the universal relation on this orbit. Since $x \in A_0 \in \mathcal{B}$ and $y \in A_n \in \mathcal{B}$, there are elements $x', y' \in B_i$ such that $x \, \beta \, x'$ and $y' \, \beta \, y$. Then $x' \, \beta \, x \, \gamma \, y \, \beta \, y'$ whence $x' \, \gamma \, y'$. The choice of B_i

implies that $x' \,\alpha\, y'$. Thus, $x \,\beta\, x'$, $(x', y') \in \alpha \wedge \gamma$ and $y' \,\beta\, y$. We have $(x, y) \in (\alpha \wedge \gamma) \vee \beta$. Thus, we may assume that the restriction of α to any orbit of $\mathcal{B}$ differs from the universal relation on this orbit. In particular, this means that α isolates any orbit of $\mathcal{B}$.

Thus, it remains to consider the case when the hypothesis of the condition c) of Theorem is fulfilled. As usual, for any $i \in J$ and $\mu \in \{\alpha, \beta, \gamma\}$, we denote by μ_i the restriction of μ to B_i. By the condition c), there is $j \in J$ such that the equality (1) holds.

Each of the orbits $A_0, A_1, \ldots, A_n$ lies in $\mathcal{B}$. Suppose that $A_0 = A_1 = \cdots = A_{k_0} = B_{i_0}$ (where $i_0 \in J$) and $A_{k_0} \neq A_{k_0+1}$. Clearly, $(x_0, x_{k_0}) \in \alpha_{i_0} \vee \beta_{i_0}$. We denote by R_0 (respectively, R_1) the β_{i_0}-class containing x_0 (respectively, x_{k_0}) and put $P_0 = \beta^*_{i_0,j}(R_0)$ and $P_1 = \beta^*_{i_0,j}(R_1)$. Since $x = x_0 \in R_0$, we have $x \,\beta\, p_0$ for any $p_0 \in P_0$. Furthermore, it is clear that $P_0 \,\alpha_{i_0,j,\beta}\, P_1$. Let $A_{k_0+1} = A_{k_0+2} = \cdots = A_{k_1} = B_{i_1}$ (where $i_1 \in J$) and either $k_1 = n$ or $A_{k_1} \neq A_{k_1+1}$. Clearly, $(x_{k_0+1}, x_{k_1}) \in \alpha_{i_1} \vee \beta_{i_1}$. We denote by R_2 (respectively, R_3) the β_{i_1}-class containing x_{k_0+1} (respectively, x_{k_1}). Since $x_{k_0} \,\beta\, x_{k_0+1}$, we have $\beta^*_{i_1,j}(R_2) = P_1$. Put $P_2 = \beta^*_{i_1,j}(R_3)$. Clearly, $P_1 \,\alpha_{i_1,j,\beta}\, P_2$. Continuing this process, we obtain β_j-classes $P_3, \ldots, P_r$ such that $P_2 \,\alpha_{i_2,j,\beta}\, P_3 \,\alpha_{i_3,j,\beta} \cdots \alpha_{i_{r-2},j,\beta}\, P_{r-1} \,\alpha_{i_{r-1},j,\beta}\, P_r$ for some $i_2, i_3, \ldots, i_{r-1} \in J$ and the $\beta_{i_{r-1}}$-class $\beta^*_{j,i_{r-1}}(P_r)$ contains y. The last means, in particular, that $p_r \,\beta\, y$ for any $p_r \in P_r$. Put $P = P_0$ and $Q = P_r$. We have $(P, Q) \in \bigvee_{i \in J} \alpha_{i,j,\beta}$. Furthermore, if $p \in P$ and $q \in Q$ then $p \,\beta\, x \,\gamma\, y \,\beta\, q$ whence $p \,\gamma\, q$. This implies that $P \,\gamma_{j,j,\beta}\, Q$. Thus, $(P, Q) \in (\bigvee_{i \in J} \alpha_{i,j,\beta}) \wedge \gamma_{j,j,\beta}$.

According to (1), $(P, Q) \in \bigvee_{i \in J} (\alpha_{i,j,\beta} \wedge \gamma_{j,j,\beta})$. Hence there are β_j-classes $S_0, S_1, \ldots, S_m$ such that $S_0 = P$, $S_m = Q$ and, for any $k = 0, 1, \ldots, m-1$, $(S_k, S_{k+1}) \in \alpha_{i_k,j,\beta} \wedge \gamma_{j,j,\beta}$ for some $i_0, i_1, \ldots, i_{m-1} \in J$. Put $S'_0 = \beta^*_{j,i_0}(S_0)$ and $S'_1 = \beta^*_{j,i_0}(S_1)$. Let $s'_0 \in S'_0$ and $s'_1 \in S'_1$. Clearly, $(s'_0, s'_1) \in \alpha_{i_0}$. Furthermore, by Lemma 2.2, $\gamma_{j,j,\beta} = \gamma_{i_0,j,\beta}$, and therefore, $(s'_0, s'_1) \in \gamma_{i_0}$. Thus, $(s'_0, s'_1) \in \alpha_{i_0} \wedge \gamma_{i_0} \subseteq \alpha \wedge \gamma$. Besides that, if $s_0 \in S_0$ then $x \,\beta\, s_0 \,\beta\, s'_0$ whence $x \,\beta\, s'_0$. Furthermore, put $S''_1 = \beta^*_{j,i_1}(S_1)$ and $S'_2 = \beta^*_{j,i_1}(S_2)$. Using Lemma 2.2 again, we have that if $s''_1 \in S''_1$ and $s'_2 \in S'_2$ then $(s''_1, s'_2) \in \alpha_{i_1} \wedge \gamma_{i_1} \subseteq \alpha \wedge \gamma$. Further, $s'_1 \,\beta\, s''_1$ because $\beta^*_{i_0,j}(S'_1) = S_1 = \beta^*_{i_1,j}(S''_1)$. Continuing this process, we obtain elements $s''_2, s'_3, s''_3, \ldots, s'_{m-1}, s''_{m-1}, s'_m \in A$ such that $s'_2 \,\beta\, s''_2$, $(s''_2, s'_3) \in \alpha_{i_2} \wedge \gamma_{i_2} \subseteq \alpha \wedge \gamma$, $s'_3 \,\beta\, s''_3$, $\ldots$, $s'_{m-1} \,\beta\, s''_{m-1}$ and $(s''_{m-1}, s'_m) \in \alpha_{i_{m-1}} \wedge \gamma_{i_{m-1}} \subseteq \alpha \wedge \gamma$. Furthermore, by the construction, $s'_m \in \beta^*_{j,i_{m-1}}(S_m)$. Since $S_m = Q = P_r$ and $y \in \beta^*_{j,i_{r-1}}(P_r)$, we have that if $s_m \in S_m$ then $s'_m \,\beta\, s_m \,\beta\, y$ whence $s'_m \,\beta\, y$. We find a sequence of elements $x, s'_0, s'_1, s''_1, s'_2, s''_2, \ldots, s'_{m-1}, s''_{m-1}, s'_m, y$ such that $x \,\beta\, s'_0$, $(s'_0, s'_1) \in \alpha \wedge \gamma$, $s'_k \,\beta\, s''_k$ and $(s''_k, s'_{k+1}) \in \alpha \wedge \gamma$ for all $k = 1, 2, \ldots, m-1$ and $s'_m \,\beta\, y$. Thus, $(x, y) \in (\alpha \wedge \gamma) \vee \beta$.

We see that the inclusion (2) always holds. Thus, we have proved the Theorem.

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