Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 41 (2000), No. 1, 233-246.

Möbius Invariants for Pairs of Spheres (S_1^m, S_2^l) in the Möbius Space S^n

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Abstract. In this article we construct a complete system of Möbius-geometric invariants for pairs (S^m, S^l), $l \leq m$, of spheres contained in the Möbius space S^n. It consists of $n - m$ generalized stationary angles. We interpret these invariants geometrically.

1. Introduction

The n-dimensional real Möbius geometry is the n-dimensional sphere S^n considered under the action of the Möbius group G_n, which is the group of all conformal transformations of the Riemannian sphere S^n with the standard metric of constant sectional curvature. By Liouville's theorem the group G_n coincides with the group of those diffeomorphisms of S^n which transform hyperspheres of the Möbius space S^n into hyperspheres, or generally m-spheres into m-spheres, see e. g. W. Blaschke, G. Thomsen [2], or M. A. Akivis, V. V. Goldberg [1]. If one considers S^n as a hyperellipsoid in the $(n+1)$-dimensional real projective space $\boldsymbol{P}^{n+1}$, the m-spheres of S^n are the non-empty intersections of S^n with projective, not tangent $(m+1)$-planes. Therefore, the group G_n consists of all projective transformations preserving the hyperellipsoid S^n. The Möbius group G_n appears as the group of projective transformations generated by pseudo-orthogonal transformations of the underlying vector space provided with a scalar product of index 1 (the $(n+2)$-dimensional Minkowski-space). In B. A. Rosenfeld's book [5] one finds the Möbius-geometric definition of stationary angles for pairs of subspheres having equal dimension $m = l$. We refine his method to handle the general case, and carry out the classification. In spite of the fact that real Möbius geometry is based on pseudo-euclidean linear algebra we obtain the classification only using the elementary theory

0138-4821/93 \$ 2.50

of selfadjoint operators in finite-dimensional euclidean vector spaces. Another approach to the same problem is contained in the paper [3] of A. Montesinos Amilibia, M. C. Romero Fuster and E. Sanabria Codesal. I thank these colleagues very much for the preprint, for interesting discussions, and their kind support of my visit in Valencia.

In Section 2 we describe the mentioned above projective model of the Möbius geometry and fix the basic notations. Section 3 contains the case of hyperspheres, for which a generalized angle is the only conformal invariant. In Section 4 the (generalized) stationary angles between subspheres of arbitrary, in general distinct dimensions are introduced, and the classification of pairs of such subspheres is carried out. Mathematica notebooks containing examples, applications, graphical and algebraical tools especially for $n = 3$ can be found on my homepage under the URL cited at the end of this article.

2. The projective model of the Möbius space

Let $\boldsymbol{V}$ denote the $(n+2)$-dimensional real vector space, and $<\mathfrak{x},\mathfrak{y}>$, $\mathfrak{x},\mathfrak{y} \in \boldsymbol{V}$, a given non-degenerate symmetric bilinear form of index one. We call it the *scalar product.* A basis $(\mathfrak{b}_i)$, $i = 1,\dots,n+2$, is said to be *orthogonal,* if the scalar products fulfil

$$<\mathfrak{b}_i,\mathfrak{b}_j> = \delta_{ij}\epsilon_i, \quad i,j = 1,\dots,n+2, \tag{1}$$

$$\epsilon_i = 1 \text{ for } i = 1,\dots,n+1, \quad \epsilon_{n+2} = -1. \tag{2}$$

In (1) δ_{ij} denotes the Kronecker-symbol. The *projective space* belonging to $\boldsymbol{V}$ is denoted by $\boldsymbol{P}$, as usual it is defined to be the set of all one-dimensional subspaces of $\boldsymbol{V}$. For $\mathfrak{x} \in \boldsymbol{V}, \mathfrak{x} \neq \mathfrak{o}$, we denote by

$$x := [\mathfrak{x}] := \text{ linear hull}\{\mathfrak{x}\} \in \boldsymbol{P} \tag{3}$$

the corresponding point of the projective space $\boldsymbol{P}$. Any subspace $\boldsymbol{W}$ of dimension $\dim \boldsymbol{W} = m+1$ is at the same time a *projective m-plane*, which is the set of points obtained by (3) for $\mathfrak{x} \in W$. The *n-dimensional sphere* S^n is defined to be the projective hyperquadric

$$x \in S^n :\Longleftrightarrow x = [\mathfrak{x}] \in \boldsymbol{P} \text{ and } <\mathfrak{x},\mathfrak{x}> = 0. \tag{4}$$

With respect to any orthogonal basis we get from (4) the equation of S^n in homogeneous coordinates x_i of x in the normal form

$$\sum_{i=1}^{n+1} x_i^2 - x_{n+2}^2 = 0. \tag{5}$$

Obviously, for any orthogonal coordinate system and any $x \in S^n$ one has $x_{n+2} \neq 0$. Norming the coordinates of the points of S^n by $\xi_i := x_i/x_{n+2}$ we obtain the usual equation of the unit hypersphere:

$$\sum_{i=1}^{n+1} \xi_i^2 = 1. \tag{6}$$

The isotropy group G_n of S^n, that is the subgroup of all projective transformations of $\boldsymbol{P}$ preserving S^n, is called the *Möbius group.* It is well-known, and easy to prove, that G_n

is isomorphic to the pseudo-orthogonal group acting on $\boldsymbol{V}$, factored by $\{\mathbf{1}, -\mathbf{1}\}$, where $\mathbf{1}$ denotes the identity transformation of $\boldsymbol{V}$:

$$G_n \cong \mathbf{O}(n+1,1)/\{\mathbf{1}, -\mathbf{1}\} \cong \mathbf{O}(n+1,1)_0 \cup \mathbf{O}(n+1,1)_1. \tag{7}$$

Here $\mathbf{O}(n+1,1)_0$ is the component of the identity of $\mathbf{O}(n+1,1)$, and $\mathbf{O}(n+1,1)_1$ is the coset of the transformations reversing only the orientation of space (or of time, but not of both) in the relativistic interpretation of $[\boldsymbol{V}, <\,,\,>]$. Therefore also G_n has two components, where the component SG_n of the identity preserves the orientation, and the other transformations reverse the orientation of S^n. The sphere S^n considered under the projective action of the group G_n described above, is called the *n-dimensional Möbius space*, and the geometry of this transformation group is the *Möbius geometry.*

Clearly, the points $x \in S^n$ are represented by isotropic vectors $\mathfrak{x} \neq \mathfrak{o}$, $<\mathfrak{x}, \mathfrak{x}> = 0$, $x = [\mathfrak{x}]$. With respect to their position to S^n, there exist three types of projective subspaces:

A. The projective $(m+1)$-planes which *intersect the Möbius space S^n in an m-sphere* (see (16) below); they correspond to the *pseudo-euclidean subspaces* of dimension $m+2$, $m = 0, \ldots, n$, and index 1. A sphere S^0 is a set of two points, and for $m = n$ we get the whole Möbius space.

B. The projective m-planes which *are disjoint to the Möbius space S^n;* they correspond to the *euclidean subspaces* of dimension $m+1$, $m = 1, \ldots, n$, on which the restriction of the scalar product is positive definite.

C. The projective m-planes which are *tangent to the Möbius space S^n;* they correspond to the *isotropic subspaces* of dimension $m+1$, $m = 1, \ldots, n$, on which the restriction of the scalar product degenerates and has rank m. Any such m-plane contains exactly one point of the Möbius space S^n, the *contact point.*

The *orthogonality* $\boldsymbol{U} \mapsto \boldsymbol{U}^\perp$ with respect to the pseudo-euclidean scalar product in $\boldsymbol{V}$ defines an involutive bijection of the lattice of all subspaces of $\boldsymbol{V}$, whose projective interpretation is called a *polarity.* It has the following properties:

$$\begin{aligned}
\dim \boldsymbol{U} = k &\iff \dim \boldsymbol{U}^\perp = n+2-k, && (8)\\
\boldsymbol{U} \subset \boldsymbol{W} &\iff \boldsymbol{W}^\perp \subset \boldsymbol{U}^\perp, && (9)\\
\boldsymbol{U} \text{ pseudo-euclidean} &\iff \boldsymbol{U}^\perp \text{ euclidean}, && (10)\\
\boldsymbol{U} \text{ isotropic} &\iff \boldsymbol{U}^\perp \text{ isotropic}, && (11)\\
\boldsymbol{U} \text{ is not isotropic} &\iff \boldsymbol{U} \oplus \boldsymbol{U}^\perp = \boldsymbol{V}, && (12)\\
\boldsymbol{U} \text{ is isotropic} &\iff \dim \boldsymbol{U} \cap \boldsymbol{U}^\perp = 1, && (13)\\
(\boldsymbol{U} + \boldsymbol{W})^\perp &= \boldsymbol{U}^\perp \cap \boldsymbol{W}^\perp, && (14)\\
(\boldsymbol{U} \cap \boldsymbol{W})^\perp &= \boldsymbol{U}^\perp + \boldsymbol{W}^\perp. && (15)
\end{aligned}$$

Of course, the case A of intersecting projective $(m+1)$-planes $\boldsymbol{U}^{m+1}$ (projective dimension!) leads to the natural definition of an m-sphere $S_1^m \subset S^n$ as the intersection

$$S_1^m := \boldsymbol{U}^{m+1} \cap S^n. \tag{16}$$

Since the pseudo-orthogonal group acts transitively on the set of all orthogonal frames one obtains as an immediate consequence of (8) – (15):

Proposition 1. *Definition* (16) *is an equivariant bijection between the set of all m-spheres in S^n and the set X_m of all $(m+2)$-dimensional pseudo-euclidean subspaces of the vector space $\boldsymbol{V}$, $m = 0, 1, \ldots, n$. This set is an open submanifold of the Grassmann manifold $G_{n+2,m+2}$ of all $(m+2)$-dimensional subspaces of $\boldsymbol{V}$; the group $\mathbf{O}(n+1,1)$ acts transitively on X_m. Thus, the Möbius group G_n acts transitively on the set of all m-spheres, which by* (16) *can be identified with X_m. The orthogonality* (10) *yields an equivariant bijection of X_m with the set of all $(n-m)$-dimensional euclidean subspaces of $\boldsymbol{V}$, which is an open submanifold of the Grassmann manifold of all $(n-m)$-dimensional subspaces of $\boldsymbol{V}$, and by* (12) *a bijection of X_m with all decompositions of $\boldsymbol{V}$ into a direct sum of an $(m+2)$-dimensional pseudo-euclidean subspace and its orthogonal complement.* □

In the following we identify X_m with the set of all m-spheres of S^n.

3. Hyperspheres

By Proposition 1 and (8) the set X_{n-1} of all hyperspheres $S^{n-1} \subset S^n$ can be identified with the set of all euclidean one-dimensional subspaces of $\boldsymbol{V}$. Any such subspace contains two spacelike unit vectors $\mathfrak{n}$; therefore the hyper-hyperboloid H^{n+1} defined by the equation

$$<\mathfrak{n}, \mathfrak{n}> = n_1^2 + n_2^2 + \ldots + n_{n+1}^2 - n_{n+2}^2 = 1 \tag{17}$$

is a twofold covering of the manifold X_{n-1} of non-oriented hyperspheres. We denote the hypersphere corresponding to $\mathfrak{n}$ by

$$S(\mathfrak{n}) := S^n \cap [\mathfrak{n}]^{\perp}, \ \mathfrak{n} \in H^{n+1}, \tag{18}$$

and show

Proposition 2. *Let (N, M), $(\hat{N}, \hat{M})$ be two pairs of non-oriented distinct hyperspheres of S^n, $n \geq 1$:*

$$N = S(\mathfrak{n}), \ M = S(\mathfrak{m}), \ \hat{N} = S(\hat{\mathfrak{n}}), \ \hat{M} = S(\hat{\mathfrak{m}}), \quad \mathfrak{n}, \mathfrak{m}, \hat{\mathfrak{n}}, \hat{\mathfrak{m}} \in H^{n+1}.$$

Then (N, M) is G_n-equivalent to $(\hat{N}, \hat{M})$ if and only if the absolute values of the scalar products coincide:

$$|<\mathfrak{n}, \mathfrak{m}>| = |<\hat{\mathfrak{n}}, \hat{\mathfrak{m}}>|. \tag{19}$$

Proof. For the proof we adapt an orthogonal frame $(\mathfrak{b}_i)$ to the pair (N, M), in which the position of N, M depends on $|<\mathfrak{n}, \mathfrak{m}>|$ only; then the transformation defined by

$$g : g\mathfrak{b}_i = \hat{\mathfrak{b}}_i, \quad i = 1, \ldots, n+2,$$

where $(\hat{\mathfrak{b}}_i)$ denotes the adapted frame corresponding to $(\hat{N}, \hat{M})$, is a Möbius transformation with $(gN, gM) = (\hat{N}, \hat{M})$. To this aim we have to distinguish three cases:

Case 1: $|<\mathfrak{n}, \mathfrak{m}>| < 1$, the subspace $[n, m]$ is *euclidean*. Then we choose $\mathfrak{b}_1, \mathfrak{b}_2$ such that

$$\mathfrak{n} = \mathfrak{b}_1, \ \mathfrak{m} = \mathfrak{b}_1 \cos\alpha + \mathfrak{b}_2 \sin\alpha, \ \text{with} \ \cos\alpha = |<\mathfrak{n}, \mathfrak{m}>|, \ 0 < \alpha \leq \pi/2,$$

and complete $\mathfrak{b}_1, \mathfrak{b}_2$ to an orthogonal basis of $\boldsymbol{V}$. Doing the same with $(\hat{N}, \hat{M})$, we obtain

$$g\mathfrak{n} = \hat{\mathfrak{n}}, \quad g\mathfrak{m} = g(\mathfrak{b}_1 \cos\alpha + \mathfrak{b}_2 \sin\alpha) = \hat{\mathfrak{b}}_1 \cos\alpha + \hat{\mathfrak{b}}_2 \sin\alpha = \hat{\mathfrak{m}},$$

what was to show.

Case 2: $|{<}\mathfrak{n}, \mathfrak{m}{>}| > 1$, the subspace $[n, m]$ is *pseudo-euclidean.* Then we choose $\mathfrak{b}_1, \mathfrak{b}_{n+2}$ such that

$$\mathfrak{n} = \mathfrak{b}_1,\ \mathfrak{m} = \mathfrak{b}_1 \cosh\alpha + \mathfrak{b}_{n+2} \sinh\alpha,\ \text{with}\ \cosh\alpha = |{<}\mathfrak{n}, \mathfrak{m}{>}|,\ 0 < \alpha,$$

and complete $\mathfrak{b}_1, \mathfrak{b}_{n+2}$ to an orthogonal basis of $\boldsymbol{V}$.

Case 3: $|{<}\mathfrak{n}, \mathfrak{m}{>}| = 1$, the subspace $[n, m]$ is *isotropic.* Since we consider non-oriented hyperspheres, we may assume ${<}\mathfrak{n}, \mathfrak{m}{>} = 1$. Then we choose $\mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{b}_{n+2}$ such that

$$\mathfrak{n} = \mathfrak{b}_1, \quad \mathfrak{m} = \mathfrak{b}_1 + \mathfrak{b}_2 + \mathfrak{b}_{n+2},$$

and complete $\mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{b}_{n+2}$ to an orthogonal basis of $\boldsymbol{V}$. Since $n \geq 1$, and the hyperspheres are distinct, such a choice is always possible.

Concluding in Cases 2, 3 as in Case 1 one finishes the proof. □

Now we interpret the three cases mentioned in the proof of Proposition 2 geometrically.

Case 1. The hyperspheres N, M intersect. The intersection is a sphere of dimension $n - 2$, and the cosinus of the intersection angle is given by

$$\cos\alpha = |{<}\mathfrak{n}, \mathfrak{m}{>}|, \tag{20}$$

$${<}\mathfrak{n}, \mathfrak{m}{>} = \frac{r^2 + R^2 - d^2}{2rR}, \tag{21}$$

where r, R are the radii of N, M, and d is the distance of the centres of these hyperspheres in the euclidean model of the Möbius space S^n, see below.

Case 2. The hyperspheres N, M are disjoint. Formula (21) remains valid; it expresses the conformal invariant of the two hyperspheres by their euclidean invariants. Clearly, since each conformal invariant is a fortiori an isometric invariant, it is always possible to describe it in terms of a complete isometric invariant system of the given objects. Using the equation $\cosh\alpha = \cos i\alpha$ some authors introduce "imaginary intersection angles", and interpret (20) in this sense, see e.g. B. A. Rosenfeld [5].

Case 3. The hyperspheres N, M have only one point in common, therefore they are tangent to each other at this point.

To prove these statements we denote by $\boldsymbol{W}, \boldsymbol{U}$ the $(n+1)$-dimensional pseudo-euclidean subspaces of $\boldsymbol{V}$ defining the hyperspheres N, M. Since these hyperspheres are distinct, it follows

$$\dim(\boldsymbol{W} + \boldsymbol{U}) = n + 2,\ \text{and}\ \dim(\boldsymbol{W} \cap \boldsymbol{U}) = n.$$

In Case 1 the subspace spanned by $\mathfrak{n}, \mathfrak{m}$

$$[\mathfrak{n}, \mathfrak{m}] = \boldsymbol{W}^\perp + \boldsymbol{U}^\perp = (\boldsymbol{W} \cap \boldsymbol{U})^\perp$$

is euclidean, therefore the intersection $\boldsymbol{W} \cap \boldsymbol{U}$ is pseudo-euclidean and defines an $(n-2)$-dimensional sphere. In Case 2 this intersection is euclidean, and in Case 3 isotropic, from which the qualitative contents of the disjunction follows. Here we applied formulas (10), (11), and (15). It remains to prove (20) and (21). If (21) is proved, then in Case 1 formula (20) is simply the cosinus theorem. Thus it suffices to show (21). To do this we construct Riemannian and euclidean models of the Möbius space S^n and derive a formula which relates the vectors $\mathfrak{n} \in H^{n+1}$ with the centre and the radius of the hypersphere $S(\mathfrak{n})$ in the euclidean model. Let $(\mathfrak{a}_i), i = 1, \ldots, n+2$, be a fixed orthogonal frame in the pseudo-euclidean vector space $\boldsymbol{V}$. The *Riemannian model* is simply the unit n-sphere (6) with centre $\mathfrak{o}$ in the euclidean subspace spanned by $\mathfrak{a}_1, \ldots, \mathfrak{a}_{n+1}$; it can be considered also as the intersection of the isotropic cone with the hyperplane $x_{n+2} = 1$, and therefore one has $\xi_i = x_i, i = 1, \ldots, n+1$. The stereographical projection *sp* of S^n from its north pole $\mathfrak{a}_{n+1}$ onto its equatorial hyperplane E^n spanned by the $\mathfrak{a}_i, i = 1, \ldots, \mathfrak{a}_n$, is given by the formula

$$x = \sum_{i=1}^{n+1} \mathfrak{a}_i \xi_i \in S^n \quad \mapsto \quad sp(x) := \sum_{i=1}^{n} \mathfrak{a}_i \xi_i / (1 - \xi_{n+1}) \in E^n, x \neq \mathfrak{a}_{n+1}, \tag{22}$$

$$sp(\mathfrak{a}_{n+1}) \quad = \quad \infty. \tag{23}$$

Of course, here (6) is supposed; the north pole $\xi_{n+1} = 1$ is mapped onto the infinite point ∞, compactifying the E^n to S^n. Since the stereographical projection (22) is conformal, and transforms k-spheres into k-spheres, we consider the compactified E^n as the *euclidean model* of the Möbius geometry. Hereby one has to take into account that the k-planes, as the images of the k-spheres containing the north pole of S^n under the stereographical projection, are to consider as k-spheres too. From the euclidean point of view the k-planes have infinite radius, and their centres are not defined; both, centre and radius, are concepts of euclidean (or Riemannian), but not of Möbius geometry. Obviously, the models differ by the underlying Riemannian metrics only.

Lemma 3. *The hypersphere $S(z, r) \subset E^n$ with centre z and radius r is the image under the stereographical projection* (22) *of the hypersphere $S(\mathfrak{n})$ defined by* (18), *where $\mathfrak{n}$ denotes the spacelike unit vector*

$$\mathfrak{n}(z, r) = (2z + \mathfrak{a}_{n+1}(-1 - r^2 + |z|^2) + \mathfrak{a}_{n+2}(1 - r^2 + |z|^2))/2r, \quad |z|^2 = \sum_{j=1}^{n} z_j^2. \tag{24}$$

For the proof we first define:

Definition 1. *A set or sequence of $k+2$ points $p_j \in S^n$, $j = 1, \ldots, k+2$, $k \leq n$, is said to be in general position, if they do not belong to any l-sphere of lower dimension $l < k$.*

Now we take $n+1$ points of the hypersphere $S(z, r)$ in general position. Their images under the *inverse stereographical projection*

$$y \in E^n \mapsto sp^{-1}(y) = \frac{2y + \mathfrak{a}_{n+1}(<y, y> - 1)}{<y, y> + 1} \tag{25}$$

are $n+1$ points p_i in general position, spanning the hypersphere $sp^{-1}(S(z, r)) \subset S^n$. The corresponding isotropic vectors are linearly independent, and their normed generalized cross product gives $\mathfrak{n}(z,r)$. The elementary calculation following this way leads to formula (24). Finally, calculating the scalar product of two vectors of the shape (24) we obtain (21). □

Remark 1. Orientation. Changing the order in the sequence

$$(p_1, \dots, p_{n+1})$$

changes the sign of $\mathfrak{n}$ according to the signature of the permutation. Therefore H^{n+1} corresponds bijectively to the manifold of all oriented hyperspheres of the Möbius space S^n.

Remark 2. Hyperplanes. Formulas (21) and (24) are related to the euclidean model; they make no sense for hyperplanes. If the hyperplane of E^n is defined by the equation $<\mathfrak{b}, \mathfrak{x}> = p$, instead of (24) we get a corresponding vector $\mathfrak{n}(\mathfrak{b}, p) \in H^{n+1}$ by

$$\mathfrak{n}(\mathfrak{b}, p) = \frac{\mathfrak{b} + (\mathfrak{a}_{n+1} + \mathfrak{a}_{n+2})p}{|\mathfrak{b}|}, \tag{26}$$

where $|\mathfrak{b}|$ denotes the norm of $\mathfrak{b}$. If another hyperplane is defined by $\hat{\mathfrak{b}}, \hat{p}$, we obtain

$$<\mathfrak{n}(\mathfrak{b}, p), \mathfrak{n}(\hat{\mathfrak{b}}, \hat{p})> = <\mathfrak{b}, \hat{\mathfrak{b}}>.$$

Here for simplicity $\mathfrak{b}, \hat{\mathfrak{b}}$ are assumed to be unit vectors. Therefore the conformal invariant is the cosinus of the angle between the normal vectors of the intersecting or parallel hyperplanes; parallel means tangent to each other at infinity. Of course, this invariant never can be larger than one. Finally, in case of the mutual position of a hyperplane and a hypersphere, we have for its conformal invariant in euclidean terms

$$<\mathfrak{n}(\mathfrak{b}, p), \mathfrak{n}(z, r)> = \frac{<\mathfrak{b}, z> - p}{r|\mathfrak{b}|}, \tag{27}$$

and this coincides with the cosinus of the intersection angle if the intersection is not empty.

4. Pairs of subspheres

In this section we classify pairs (S_1^m, S_2^l) of subspheres of S^n under the action of the Möbius group G_n. We assume

A. The dimensions fulfil $n > m \geq l \geq 0$.
B. The pair is in *general position,* i.e. there does not exist a hypersphere S^{n-1} with $S_1^m \cup S_2^l \subset S^{n-1}$.

If assumption B is not fulfilled one considers the smallest subsphere containing $S_1^m \cup S_2^l$ and applies the methods described below. We often omit the dimension superscripts and fix the notations as follows: The pseudo-euclidean subspace of dimension $m+2$ defining S_1 is $\boldsymbol{W} \in X_m$, and the $(l+2)$-dimensional pseudo-euclidean subspace defining S_2 is $\boldsymbol{U} \in X_l$ (see Proposition 1). Assumption B is equivalent to each of the following equations:

$$\boldsymbol{W} + \boldsymbol{U} = \boldsymbol{V}, \ \boldsymbol{W}^{\perp} \cap \boldsymbol{U}^{\perp} = \mathfrak{o}. \tag{28}$$

Therefore the sum $\boldsymbol{W}^\perp + \boldsymbol{U}^\perp$ is direct. From (15) and (8) we get

$$\dim \boldsymbol{U} \cap \boldsymbol{W} = m + l - n + 2 \geq 0, \tag{29}$$

$$\dim \boldsymbol{U}^\perp + \boldsymbol{W}^\perp = 2n - m - l. \tag{30}$$

Relating qualitative different positions of the subspheres and algebraic properties of $\boldsymbol{U} \cap \boldsymbol{W}$, we shall distinguish four cases:

Case 1. $\boldsymbol{U} \cap \boldsymbol{W}$ is euclidean, or the null vector $\mathfrak{o}$. Then $\boldsymbol{U}^\perp + \boldsymbol{W}^\perp$ is pseudo-euclidean. The subspheres S_1, S_2 are disjoint, and there exist hyperspheres separating them:

$$\Sigma_1 = S(\mathfrak{n}) \supset S_1, \Sigma_2 = S(\mathfrak{m}) \supset S_2 \text{ with } \Sigma_1 \cap \Sigma_2 = \emptyset. \tag{31}$$

Indeed, by (30) we have $\dim \boldsymbol{U}^\perp + \boldsymbol{W}^\perp \geq 2$. Since $\boldsymbol{U}^\perp + \boldsymbol{W}^\perp$ is pseudo-euclidean, one can find a timelike vector $\mathfrak{x} = \mathfrak{n} + \mathfrak{m} \in \boldsymbol{U}^\perp + \boldsymbol{W}^\perp$, where $\mathfrak{m} \in \boldsymbol{U}^\perp$, $\mathfrak{n} \in \boldsymbol{W}^\perp$ must be spacelike. Since their span $[\mathfrak{m}, \mathfrak{n}]$ is pseudo-euclidean, the corresponding hyperspheres are disjoint (Section 3, Case 2).

Case 2. $\boldsymbol{U} \cap \boldsymbol{W}$ is pseudo-euclidean, and we have $k = \dim \boldsymbol{U} \cap \boldsymbol{W} \geq 2$. Then this intersection contains isotropic vectors, and defines a subsphere of dimension $k - 2$.

Case 3. $\boldsymbol{U} \cap \boldsymbol{W}$ is pseudo-euclidean, and we have $k = \dim \boldsymbol{U} \cap \boldsymbol{W} = 1$. Then this one-dimensional, timelike intersection does not contain isotropic vectors, and the subspheres do not intersect: $S_1 \cap S_2 = \emptyset$. In difference to Case 1 there do not exist hyperspheres Σ_1, Σ_2 which separate the subspheres, i.e. have properties (31); the subspheres S_1, S_2 are *interlaced.* Indeed, for any two linearly independent vectors $\mathfrak{m}, \mathfrak{n}$ of the euclidean subspace $(\boldsymbol{U} \cap \boldsymbol{W})^\perp = \boldsymbol{U}^\perp + \boldsymbol{W}^\perp$ the span $[\mathfrak{m}, \mathfrak{n}]$ is euclidean, and by Section 3, Case 1, the corresponding hyperspheres intersect. As examples we mention a hypersphere S_1, and a 0-sphere S_2, the two points of which lie on different sides of S_1, or two interlaced circles in S^3. By (29), in Case 3 we always have $m + l = n - 1$.

Case 4. $\boldsymbol{U} \cap \boldsymbol{W}$ is isotropic. Since in any isotropic subspace of the pseudo-euclidean space $\boldsymbol{V}$ there exists a uniquely defined isotropic subspace of dimension one, the intersection of the subspheres is a uniquely defined point: $S_1 \cap S_2 = \{x_0\}$. Examples are a hypersphere S_1 and any tangent l-sphere S_2, or two circles in S^3 which intersect in exactly one point; circles in S^3 intersecting in two points, or having a common tangent, are not in general position.

To obtain a complete system of invariants for pairs of subspheres we apply the method of stationary angles, see e.g. B. A. Rosenfeld [5], §3.3, §11.3, H. Reichardt [4], §5.3. We consider the function $<\mathfrak{n}, \mathfrak{m}>$ under the conditions

$$\mathfrak{n} \in \boldsymbol{W}^\perp, \quad \mathfrak{m} \in \boldsymbol{U}^\perp, \quad <\mathfrak{n}, \mathfrak{n}> = 1, \quad <\mathfrak{m}, \mathfrak{m}> = 1. \tag{32}$$

Since $\boldsymbol{W}^\perp, \boldsymbol{U}^\perp$ are euclidean vector spaces, the vector pair $(\mathfrak{n}, \mathfrak{m})$ varies on a compact set, and therefore exist maxima and minima of $<\mathfrak{n}, \mathfrak{m}>$. Introducing the Lagrange multipliers λ, μ we consider the function

$$f(\mathfrak{n}, \mathfrak{m}, \lambda, \mu) = <\mathfrak{n}, \mathfrak{m}> - \lambda(<\mathfrak{n}, \mathfrak{n}> - 1) - \mu(<\mathfrak{m}, \mathfrak{m}> - 1). \tag{33}$$

At an extremum the differentials of f with respect to $\mathfrak{n}, \mathfrak{m}, \lambda, \mu$ must vanish. Deriving f with respect to $\mathfrak{n}, \mathfrak{m}$ we obtain the conditions

$$<d\mathfrak{n}, \mathfrak{m}> - 2\lambda<d\mathfrak{n}, \mathfrak{n}> = <d\mathfrak{n}, \mathfrak{m} - 2\lambda\mathfrak{n}> = 0, \tag{34}$$

$$<d\mathfrak{m}, \mathfrak{n}> - 2\mu<d\mathfrak{m}, \mathfrak{m}> = <d\mathfrak{m}, \mathfrak{n} - 2\mu\mathfrak{m}> = 0. \tag{35}$$

The derivations with respect to λ, μ only reproduce the conditions (32). The decompositions

$$\boldsymbol{V} = \boldsymbol{W} \oplus \boldsymbol{W}^\perp = \boldsymbol{U} \oplus \boldsymbol{U}^\perp \tag{36}$$

allow to introduce the linear maps

$$p_{1,2} : \mathfrak{m} \in \boldsymbol{U}^\perp \mapsto pr_{\boldsymbol{W}^\perp}(\mathfrak{m}) \in \boldsymbol{W}^\perp, \quad p_{2,1} : \mathfrak{n} \in \boldsymbol{W}^\perp \mapsto pr_{\boldsymbol{U}^\perp}(\mathfrak{n}) \in \boldsymbol{U}^\perp. \tag{37}$$

Keeping in mind the conditions (32) we get from the second equations (34), (35) the equivalent conditions

$$2\lambda\mathfrak{n} = p_{1,2}(\mathfrak{m}), \quad 2\mu\mathfrak{m} = p_{2,1}(\mathfrak{n}). \tag{38}$$

Now we have to find real numbers λ, μ such that this equations admit non-trivial solutions $\mathfrak{n}, \mathfrak{m}$. We consider the linear endomorphism

$$A = p_{1,2} \circ p_{2,1} \in \mathrm{End}(\boldsymbol{W}^\perp). \tag{39}$$

If $\mathfrak{n}, \mathfrak{m}$ is a solution of (38) then it follows

$$A(\mathfrak{n}) = 4\lambda\mu\mathfrak{n}.$$

Therefore we shall look for eigenvalues of A. The linear maps $p_{1,2}$, $p_{2,1}$, A operate between euclidean vector spaces. We show

Lemma 4. *The operator $A \in \mathrm{End}(\boldsymbol{V})$ is selfadjoint, and the formulas*

$$p'_{2,1} = p_{1,2}, \quad p'_{1,2} = p_{2,1}, \quad A' = A = p'_{2,1} \circ p_{2,1} \tag{40}$$

are valid. The eigenvalues of A are nonnegative. The eigenspace of A to the eigenvalue zero is the kernel of $p_{2,1}$.

Proof. For arbitrary vectors $\mathfrak{x} \in \boldsymbol{U}^\perp, \mathfrak{y} \in \boldsymbol{W}^\perp$ we denote the decompositions with respect to the orthogonal direct sums (36) by

$$\mathfrak{x} = \mathfrak{x}_{\boldsymbol{W}} + \mathfrak{x}_{\boldsymbol{W}^\perp}, \quad \mathfrak{y} = \mathfrak{y}_{\boldsymbol{U}} + \mathfrak{y}_{\boldsymbol{U}^\perp}.$$

By the definitions we have

$$<p'_{2,1}(\mathfrak{x}), \mathfrak{y}> = <\mathfrak{x}, p_{2,1}(\mathfrak{y})> = <\mathfrak{x}, \mathfrak{y}_{\boldsymbol{U}^\perp}> = <\mathfrak{x}, \mathfrak{y}>.$$

The last equation follows because $\mathfrak{x} \in \boldsymbol{U}^\perp$ is orthogonal to $\mathfrak{y}_{\boldsymbol{U}}$. Since $\mathfrak{y} \in \boldsymbol{W}^\perp$ we continue

$$<\mathfrak{x}, \mathfrak{y}> = <\mathfrak{x}_{\boldsymbol{W}^\perp}, \mathfrak{y}> = <p_{1,2}(\mathfrak{x}), \mathfrak{y}>.$$

Comparing the first and the last term of these equations we obtain the first statement of (40), from which the other both follow easily. Now let $\mathfrak{a}$ be an eigenvector of A to the eigenvalue α. By (40) it follows

$$<\mathfrak{a}, A(\mathfrak{a})> = \alpha<\mathfrak{a}, \mathfrak{a}> = <p_{2,1}(\mathfrak{a}), p_{2,1}(\mathfrak{a})>.$$

Since $\boldsymbol{W}^{\perp}, \boldsymbol{U}^{\perp}$ are euclidean vector spaces, and $\mathfrak{a} \neq \mathfrak{o}$ we obtain $\alpha \geq 0$, and the last statement. □

Because A is a selfadjoint endomorphism, it has $n-m$ real eigenvalues

$$\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_{n-m} \geq 0. \tag{41}$$

Let $(\mathfrak{a}_i)$, $i = 1, \ldots, n-m$, be an orthonormal basis of eigenvectors with $A(\mathfrak{a}_i) = \alpha_i \mathfrak{a}_i$. Applying (40) we obtain

$$<p_{2,1}(\mathfrak{a}_i), p_{2,1}(\mathfrak{a}_j)> = <\mathfrak{a}_i, A(\mathfrak{a}_j)> = <\mathfrak{a}_i, \mathfrak{a}_j>\alpha_j = \delta_{i\,j}\alpha_j. \tag{42}$$

Let r_A denote the rank of A. We define

$$\mathfrak{b}_j := p_{2,1}(\mathfrak{a}_j)/\sqrt{\alpha_j}, \quad j = 1, \ldots, r_A. \tag{43}$$

Applying (40) one easily calculates

$$<\mathfrak{b}_i, \mathfrak{b}_j> = \delta_{i\,j}, \quad i, j = 1, \ldots, r_A \tag{44}$$

$$<\mathfrak{b}_i, \mathfrak{a}_k> = \delta_{i\,k}\sqrt{\alpha_k}, \quad k = 1, \ldots, n-m. \tag{45}$$

The extremal properties of the eigenvalues show that the angles defined by

$$\cos\beta_k = \sqrt{\alpha_k}, \quad \text{if } \alpha_k \leq 1,\ 0 \leq \beta_k \leq \pi/2, \tag{46}$$

can be considered as the stationary angles between the hyperspheres fulfilling (31). Of course, this is true only if the hyperspheres intersect, as we discussed in Section 3. Generally one can speak only about the stationary values of $<\mathfrak{n}, \mathfrak{m}>$ under the conditions (31). Now we complete the orthonormal sequence $(\mathfrak{b}_i), i = 1, \ldots, r_A$, to an orthonormal basis of $\boldsymbol{U}^{\perp}$. As a consequence of Lemma 4 we have

$$<p_{1,2}(\mathfrak{b}_i), \mathfrak{a}_j> = <\mathfrak{b}_i, p_{2,1}(\mathfrak{a}_j)> = <\mathfrak{b}_i, \mathfrak{b}_j>\sqrt{\alpha_j} = \delta_{i\,j}\sqrt{\alpha_j}$$

for $i = 1, \ldots, n-l,\ j = 1, \ldots, n-m$. If $j \leq r_A$ this follows from definition (43), and for $j > r_A$ we remember that by Lemma 4 we have

$$\ker p_{2,1} = \boldsymbol{W}^{\perp} \cap \boldsymbol{U} = [\mathfrak{a}_{r_A+1}, \ldots, \mathfrak{a}_{n-m}]. \tag{47}$$

It follows

$$p_{1,2}(\mathfrak{b}_j) = \mathfrak{a}_j\sqrt{\alpha_j} \text{ for } j \leq r_A, \quad p_{1,2}(\mathfrak{b}_k) = \mathfrak{o} \text{ for } k > r_A. \tag{48}$$

Thus, in analogy with (47) we conclude

$$\ker p_{1,2} = \boldsymbol{U}^{\perp} \cap \boldsymbol{W} = [\mathfrak{b}_{r_A+1}, \ldots, \mathfrak{b}_{n-l}]. \tag{49}$$

For the following the maximum α_1 of the eigenvalues is deciding. We shall relate its value to the four cases discussed at the beginning of this section. We define

Definition 2. *The orthogonal bases* $(\mathfrak{a}_i)$, $(\mathfrak{b}_j)$ *of* $\boldsymbol{V}$ *fulfilling* (1) *are said to be adapted to the subspheres* S_1, S_2, *if*
1. $(\mathfrak{a}_1, \dots, \mathfrak{a}_{n-m})$ *is a basis of eigenvectors of* A *such that* (41) *and* $A(\mathfrak{a}_i) = \mathfrak{a}_i \alpha_i$ *are valid;*
2. $(\mathfrak{b}_1, \dots, \mathfrak{b}_{n-l})$ *is a basis of* $\boldsymbol{U}^{\perp}$ *such that* (45) *is satisfied for* $i = 1, \dots, n-l$.

Lemma 5. *Under the assumptions* A, B *the following disjunction is valid:*

$$\begin{aligned} \alpha_1 > 1 &\iff \boldsymbol{U} \cap \boldsymbol{W} \ \textit{euclidean,} \\ \alpha_1 = 1 &\iff \boldsymbol{U} \cap \boldsymbol{W} \ \textit{isotropic,} \\ \alpha_1 < 1 &\iff \boldsymbol{U} \cap \boldsymbol{W} \ \textit{pseudo-euclidean.} \end{aligned}$$

Proof. We consider $\boldsymbol{U}^{\perp} + \boldsymbol{W}^{\perp} = (\boldsymbol{U} \cap \boldsymbol{W})^{\perp}$, and adapt the bases. If $\alpha_1 > 1$, we conclude from (45) that the span $[\mathfrak{a}_1, \mathfrak{b}_1]$ is a pseudo-euclidean subspace of $\boldsymbol{U}^{\perp} + \boldsymbol{W}^{\perp}$, which therefore must be pseudo-euclidean too. Conversely, if this is the case, we find a timelike vector $\mathfrak{x} = \mathfrak{u} + \mathfrak{w}$, $\mathfrak{u} \in \boldsymbol{U}^{\perp}, \mathfrak{w} \in \boldsymbol{W}^{\perp}$. Let $\mathfrak{u}_0, \mathfrak{w}_0$ be the corresponding normed vectors. Since the span $[\mathfrak{u}, \mathfrak{w}] = [\mathfrak{u}_0, \mathfrak{w}_0]$ is pseudo-euclidean, the determinant of the scalar products $1 - {<}\mathfrak{u}_0, \mathfrak{w}_0{>}^2$ must be negative, and by the maximum property of the eigenvalue α_1 we get

$$1 < \ {<}\mathfrak{u}_0, \mathfrak{w}_0{>}^2 \ \leq \alpha_1.$$

Now let $\alpha_1 = 1$. By (45) we have ${<}\mathfrak{b}_1, \mathfrak{a}_1{>} = 1$. Therefore the vector $\mathfrak{z} = \mathfrak{a}_1 - \mathfrak{b}_1$ satisfies ${<}\mathfrak{z}, \mathfrak{z}{>} = 0$. By (28) we have $\mathfrak{z} \neq \mathfrak{o}$, and therefore $\mathfrak{z}$ is isotropic. From (45) it follows ${<}\mathfrak{z}, \mathfrak{a}_k{>} = 0$ for $k = 1, \dots, n-m$, and ${<}\mathfrak{z}, \mathfrak{b}_i{>} = 0$ for $i = 1, \dots, r_A$. Finally, from (49) we obtain ${<}\mathfrak{z}, \mathfrak{b}_j{>} = 0$ for $j = r_A + 1, \dots, n-l$. Therefore $\boldsymbol{U}^{\perp} + \boldsymbol{W}^{\perp}$ is isotropic, and the orthogonal space $\boldsymbol{U} \cap \boldsymbol{W}$ too. In the converse, assume that $\boldsymbol{U}^{\perp} + \boldsymbol{W}^{\perp}$ is isotropic. Then we find a vector $\mathfrak{y} = \mathfrak{u} + \mathfrak{w} \neq \mathfrak{o}$ being orthogonal to $\boldsymbol{U}^{\perp} + \boldsymbol{W}^{\perp}$, with $\mathfrak{u} \in \boldsymbol{U}^{\perp}, \mathfrak{w} \in \boldsymbol{W}^{\perp}$. Especially we have

$${<}\mathfrak{y}, \mathfrak{u}{>} = {<}\mathfrak{u}, \mathfrak{u}{>} + {<}\mathfrak{w}, \mathfrak{u}{>} = 0, \quad {<}\mathfrak{y}, \mathfrak{w}{>} = {<}\mathfrak{u}, \mathfrak{w}{>} + {<}\mathfrak{w}, \mathfrak{w}{>} = 0,$$

which yields

$$\frac{{<}\mathfrak{w}, \mathfrak{u}{>}^2}{{<}\mathfrak{u}, \mathfrak{u}{>}{<}\mathfrak{w}, \mathfrak{w}{>}} = 1.$$

By the maximum property of α_1 we conclude $\alpha_1 \geq 1$, but $\alpha_1 > 1$ cannot take place since then $\boldsymbol{U}^{\perp} + \boldsymbol{W}^{\perp}$ would be pseudo-euclidean, as already shown. The third equivalence is an immediate consequence of the others and (41). □

By (43) and (47) we know the components of the eigenvectors of A in $\boldsymbol{U}^{\perp}$. Our aim is now to calculate the components of these vectors in $\boldsymbol{U}$. Let $q : \boldsymbol{W}^{\perp} \to \boldsymbol{U}$ denote the projection defined by the orthogonal decompositions (36). Using $q(\mathfrak{x}) = \mathfrak{x} - p_{2,1}(\mathfrak{x})$ and applying (43) – (45) one calculates

$${<}q(\mathfrak{a}_i), q(\mathfrak{a}_j){>} = \delta_{ij}(1 - \alpha_j). \tag{50}$$

Case 1. $\alpha_1 > 1$. *Then the space* $\boldsymbol{U}^{\perp} + \boldsymbol{W}^{\perp}$ *is pseudo-euclidean, the spheres are disjoint:* $S_1 \cap S_2 = \emptyset$, *and can be separated by hyperspheres. The eigenvalue* α_1 *has multiplicity* 1, *and the eigenvalues* $\alpha_i, i \geq 2$, *satisfy* $\alpha_i < 1$. *We remember* (46), *and set*

$$\sqrt{\alpha_1} = \cosh \beta_1, \ \beta_1 > 0.$$

There exist adapted bases such that

$$\mathfrak{a}_1 = \mathfrak{b}_1 \cosh \beta_1 + \mathfrak{b}_{n+2} \sinh \beta_1 \tag{51}$$

$$\mathfrak{a}_i = \mathfrak{b}_i \cos \beta_i + \mathfrak{b}_{n-l+i-1} \sin \beta_i, \ (i = 2, \dots, n-m). \tag{52}$$

Proof. By Lemma 5 the space $\boldsymbol{U}^\perp + \boldsymbol{W}^\perp$ is pseudo-euclidean. By norming $q(\mathfrak{a}_1)$ we define $\mathfrak{b}_{n+2}$ and obtain (51). There cannot be any other eigenvector with an eigenvalue $\alpha > 1$, since then, by (50), we would have two orthogonal timelike vectors, what is impossible in a pseudo-euclidean space of index 1. In particular, α_1 has multiplicity 1. Assume $\alpha_2 = 1$. Then from (50), for $i = j = 2$, would follow $q(\mathfrak{a}_2) = \mathfrak{o}$, or $q(\mathfrak{a}_2)$ isotropic. The latter cannot take place, since this vector is orthogonal to $\mathfrak{b}_{n+2}$ and therefore belongs to an euclidean subspace. On the other hand, $q(\mathfrak{a}_2) = \mathfrak{o}$ implies $\mathfrak{a}_2 \in \boldsymbol{W}^\perp \cap \boldsymbol{U}^\perp$, what contradicts (28). Thus, $0 \leq \alpha_i < 1$ holds true for $i = 2, \dots, n-m$. Norming the vectors $q(\mathfrak{a}_i)$ we find the orthonormal sequence $(\mathfrak{b}_{n-l+i-1})_i$ such that (52) holds. Completing the already defined vectors to orthonormal bases of $\boldsymbol{U}$ and $\boldsymbol{U}^\perp$, respectively, we get the adapted bases with properties (51), (52). □

Case 2. $\alpha_1 < 1$, *and* $\dim \boldsymbol{U} \cap \boldsymbol{W} > 1$. *Then, by* (29) *and Lemma* 5, $\boldsymbol{U} \cap \boldsymbol{W}$ *is pseudo-euclidean, and the intersection* $S_1 \cap S_2$ *is a subsphere of dimension* $m + l - n$. *There exist adapted bases such that*

$$\mathfrak{a}_i = \mathfrak{b}_i \cos \beta_i + \mathfrak{b}_{n-l+i} \sin \beta_i, \ (i = 1, \dots, n-m). \tag{53}$$

Proof. Equation (46) is a correct definition for $k = 1, \dots, n-m$, giving stationary angles of the intersecting subspheres. By (50) we obtain as projections in $\boldsymbol{U}$ an orthogonal sequence, which we can norm and take as part of the adapted basis. Analogously to (52) we obtain (53). □

Case 3. $\alpha_1 < 1$, *and* $\dim \boldsymbol{U} \cap \boldsymbol{W} = 1$. *Then the subspheres are interlaced and one has* $m + l - n = -1$. *There exist adapted bases such that* (53) *holds.*

Proof. The first statement is already proved, see Case 3 at the beginning of this section and (29). The rest can be proved as in Case 2. □

Case 4. $\alpha_1 = 1$. *Then the multiplicity of this eigenvalue is* 1. *The intersection* $\boldsymbol{U} \cap \boldsymbol{W}$ *is isotropic, and the subspheres intersect in a single point. There exist* $n - m - 1$ *stationary angles defined by* (46) *for* $k = 2, \dots, n-m$ *and adapted bases such that* (52) *and the following equation hold:*

$$\mathfrak{a}_1 = \mathfrak{b}_1 + \mathfrak{b}_{n+1} + \mathfrak{b}_{n+2}. \tag{54}$$

Proof. By Lemma 5 the second statement is true. Since $q(\mathfrak{a}_1)$ is an isotropic vector in $\boldsymbol{U}$ which by (50) is orthogonal to $q(\boldsymbol{W}^\perp)$ we find an orthogonal basis $(\mathfrak{b}_j)$, $j = n-l+1, \dots, n+2$ of $\boldsymbol{U}$, such that (54) holds. If the multiplicity of $\alpha_1 = 1$ would be greater than 1, then an analogous representation would exist for $\mathfrak{a}_2$, where the isotropic parts were proportional, since the one-dimensional isotropic subspace in $q(\boldsymbol{W}^{\perp})$ is uniquely defined. Thus, together with (54) we would have

$$\mathfrak{a}_2 = \mathfrak{b}_2 + (\mathfrak{b}_{n+1} + \mathfrak{b}_{n+2})\kappa$$

for a certain constant $\kappa \neq 0$. Multiplying (54) by κ, and subtracting, we would get a non-vanishing vector

$$\mathfrak{a}_1\kappa - \mathfrak{a}_2 = \mathfrak{b}_1\kappa - \mathfrak{b}_2 \in \boldsymbol{U}^{\perp} \cap \boldsymbol{W}^{\perp},$$

in contradiction to (28). Now, for $k = 2, \ldots, n-m$, we may apply (46); norming and numbering the vectors $q(\mathfrak{a}_k)$ in an appropriate way, we obtain (52). Since $q(\mathfrak{a}_1)$ is orthogonal to $q(\boldsymbol{W}^{\perp})$, the orthogonal complement of the span $[q(\mathfrak{a}_2), \ldots, q(\mathfrak{a}_{n-m}]$ in $\boldsymbol{U}$ is an at least two-dimensional pseudo-euclidean subspace containing $q(\mathfrak{a}_1)$, in which we may realize the equation (54); completing to orthogonal bases of $\boldsymbol{U}, \boldsymbol{U}^{\perp}$ we get the required adapted bases.□

We summarize the results and finish the article with the following

Theorem 6. *Under the assumptions A, B the eigenvalues* (41) *of the selfadjoint operator A, defined by* (39), *are a complete system of invariants under the Möbius group for pairs of subspheres* (S_1^m, S_2^l) *of the Möbius space* S^n. *The cases* 1 – 4 *of the mutual position of the spheres correspond to the cases* 1 – 4 *characterized by the maximal eigenvalue.*

Proof. Obviously, by their definition, the eigenvalues α_i are invariant under Möbius transformations. In each of the cases we proved the existence of adapted bases with respect to which the eigenbasis $(\mathfrak{a}_i)$ of the space $\boldsymbol{W}^{\perp}$ is expressed with respected to the basis $(\mathfrak{b}_i)$, $i = 1, \ldots, n+2$, corresponding to the subsphere S_2^l, with coefficients uniquely defined by the eigenvalues α_i. Therefore, if another pair $(\hat{S}_1^m, \hat{S}_2^l)$ possesses the same eigenvalues, and $(\hat{\mathfrak{b}}_i)$ is a corresponding adapted basis, then the Möbius transform g defined by $g(\mathfrak{b}_i) = (\hat{\mathfrak{b}}_i)$ transforms the pairs into each other:

$$(gS_1,\, gS_2) = (\hat{S}_1,\, \hat{S}_2).$$

We remark that Case 2 and Case 3 differ by the dimensions: In Case 3 we have $m+l-n = -1$, and in Case 2 $m+l-n \geq 0$ is the dimension of the subsphere $S_1 \cap S_2$. □

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Received February 22, 1999