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Green Functions and a Conjecture of Geck and Malle[1]

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Abstract. For any special unipotent class C of a split reductive group G over a finite field $\mathbf{F}_q$, a special piece $\widehat{C}$ is defined. It is known that the cardinality of $\widehat{C}(\mathbf{F}_q)$ is a polynomial in q. Geck and Malle proposed a conjectural algorithm for computing these polynomials, and verified it in the case of exceptional groups. In this paper we show, in the case of classical groups, that their conjecture is reduced to another conjecture concerning Springer representations of W. We verify this conjecture in the case where G is of type B_n, C_n or D_n with $n \le 6$.

1. Introduction and the statement of results

1.1. Let W be a finite Coxeter group, and $W^\wedge$ the set of irreducible characters of W. To each $\chi \in W^\wedge$ one can attach a polynomial $D_\chi \in \mathbf{R}[t]$ (where t is an indeterminate), i.e., the generic degree of χ. The a-function $a(\chi)$ is defined as the biggest integer $s \ge 0$ such that t^s divides the polynomial D_χ. Let V be an $\mathbf{R}W$-module affording the reflection character of W. The b-function $b(\chi)$ is defined as the smallest integer $r \ge 0$ such that χ occurs with non-zero multiplicity in $V^{\otimes r}$. We always have $a(\chi) \le b(\chi)$, and χ is called special if the equality holds. The set $W^\wedge$ is partitioned into various two-sided cells of W. Each two-sided cell contains a unique special character of W.

Let S_W be the coinvariant algebra of the symmetric algebra $S(V)$. Then $S_W = \oplus_{i\ge 0} S_W^i$ is a graded regular $\mathbf{R}W$-module. We define, for any character φ of W, a polynomial $R(\varphi) \in \mathbf{Z}[t]$

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by

$$R(\varphi) = \sum_{i\geq 0} \langle \varphi, S_W^i \rangle_W t^i,$$

where $\langle\,,\,\rangle_W$ denotes the inner product of class functions of W, (here the $\mathbf{R}W$-module S_W^i is also regarded as a character of W). For $\chi \in W^\wedge$, $R(\chi)$ is called the fake degree of χ.

1.2. Let G be a connected reductive algebraic group over k, where k is an algebraic closure of a finite field. Let W be the Weyl group of G. By the Springer correspondence there exists a natural injective map from $W^\wedge$ to the set $\mathcal{I}$ of all the pairs $(C, \mathcal{L})$, where C is a unipotent class in G and $\mathcal{L}$ is a G-equivariant irreducible ($\bar{\mathbf{Q}}_l$-) local system on C. We denote by $\mathcal{I}_0$ the subset of $\mathcal{I}$ obtained as the image of this map. If $\mathcal{L}$ is a constant sheaf $\bar{\mathbf{Q}}_l$, any pair $(C, \mathcal{L})$ is contained in $\mathcal{I}_0$. A unipotent class C is called special if $(C, \bar{\mathbf{Q}}_l)$ corresponds to a special character. Then special unipotent classes are in bijective correspondence with special characters of W. This was known by Lusztig [8] for good characteristic case, and was recently verified also for bad characteristic case by Geck and Malle [4]. We define, for a special unipotent class C, a special piece $\widehat{C}$ as a subset of the unipotent variety G_{uni} consisting of all the elements in the closure $\bar{C}$ of C which are not in the closure of any special unipotent class $C' \neq C, C' \subset \bar{C}$. Hence $\widehat{C}$ is an irreducible, locally closed subvariety in G_{uni}. It is known by [14] that special pieces form a partition of G_{uni}. (Note that Th.1.4 and Th.1.5 in Chap. III of [14] substantially contains the proof of the above result. However the notion of special pieces is implicit there).

Let $\mathbf{F}_q$ be a finite field of q elements with $ch\,\mathbf{F}_q = p$. We assume that G has a split $\mathbf{F}_q$-structure with Frobenius map $F : G \to G$. Then all the unipotent classes are F-stable. It is known, once p is fixed, that the cardinality of F-fixed points in a unipotent class C is expressed by a polynomial in q. The classification of unipotent classes however depends on p. Nevertheless, Lusztig has shown in [13] the following fact: Let C be the special unipotent class corresponding to a special character χ of W, and $\widehat{C}$ the special piece associated to C. Then there exists a polynomial $f_\chi \in \mathbf{Z}[t]$ such that

$$|\widehat{C}^F| = f_\chi(q),$$

i.e., the polynomial expressing the cardinality $|\widehat{C}^F|$ is independent of the characteristic of the base field.

1.3. The above result of Lusztig is based on a case by case argument. In [5], Geck and Malle proposed a general algorithm for producing polynomials $f_\chi(q)$ without referring to unipotent classes. Their algorithm is an analogue of the algorithm for computing Green functions (see Section 2), and is given as follows: For $i \in \mathcal{I}_0$, we denote by χ_i the corresponding irreducible character of W. We put $a(i) = a(\chi_i)$. Let us define a matrix $\Omega = (\omega_{ij})_{i,j\in\mathcal{I}_0}$ by

$$\omega_{ij} = t^N R(\chi_i \otimes \chi_j \otimes \varepsilon) \in \mathbf{Z}[t],$$

where N is the number of reflections in W, and ε is the sign character of W. We consider the following system of equations for unknown λ_{ij}, p_{ij} $(i, j \in \mathcal{I}_0)$,

$$
(1.3.1) \qquad \begin{cases} \lambda_{i'j'} = 0 & \text{unless } a(i') = a(j') \\ p_{j'j} = 0 & \text{unless } a(j') > a(j) \text{ or } j = j' \\ p_{jj} = t^{a(j)} & \text{for all } j \in \mathcal{I}_0 \\ \omega_{ij} = \displaystyle\sum_{i', j' \in \mathcal{I}_0} p_{i'i} \lambda_{i'j'} p_{j'j} & \text{for all } i, j \in \mathcal{I}_0 \end{cases}
$$

We choose a total order on $\mathcal{I}_0$ which is compatible with the order reverse to the preorder induced from $a(i)$, and define matrices $\Lambda = (\lambda_{i'j'})_{i',j' \in \mathcal{I}_0}$ and $P = (p_{j'j})_{j',j \in \mathcal{I}_0}$ along this order. Then the last equations in (1.3.1) can be written as

$$
{}^t P \Lambda P = \Omega.
$$

We define an equivalence relation $\sim$ on $\mathcal{I}_0$ by $i \sim j$ if $a(i) = a(j)$, and consider P, Λ as block matrices with respect to this relation. Then P is a block upper triangular matrix, with diagonal block consisting of identity matrix multiplied by $t^{a(i)}$, and Λ is a block diagonal matrix. Moreover, they proved that Λ is non-singular. Hence the system of equations (1.3.1) has a unique solution with $\lambda_{ij}, p_{ij} \in \mathbf{Q}(t)$, rational functions on t. Under this setting, Geck and Malle stated the following conjecture.

Conjecture 1.4. (Geck and Malle [5]) *Let G be as before and W the Weyl group of G. Then*

(i) $\lambda_{ij}, p_{ij} \in \mathbf{Q}[t]$ *for* $i, j \in \mathcal{I}_0$,

(ii) $\sum_i \lambda_{ii} = t^{2N}$, *where the sum is taken over all $i \in \mathcal{I}_0$ such that χ_i is special.*

(iii) *For $i \in \mathcal{I}_0$ such that χ_i is special, λ_{ii} coincides with $f_{\chi_i}(t)$ given in* 1.2, *i.e., if $i = (C, \bar{\mathbf{Q}}_l)$ with C special, we have*

$$
|\widehat{C}^F| = \lambda_{ii}(q).
$$

Remarks 1.5. (i) In the case of type A_n, the equations (1.3.1) are exactly the same as the equations for computing Green functions for G, and the conjecture follows from the properties of Green functions. Geck and Malle verified the conjecture in [5] in the case of all exceptional groups by using the computer.

(ii) The statements (i) and (ii) in the conjecture make sense also for finite Coxeter groups, or even for certain finite complex reflection groups under an appropriate setting. (Note that in the original conjecture, λ_{ij} and p_{ij} are required only in $\mathbf{R}[t]$. But in the case of Weyl groups, $\mathbf{Q}[t]$ would be more appropriate.) Geck and Malle verified that even in those cases, the equations (1.3.1) produce polynomials $\lambda_{ii} = f_\chi \in \mathbf{Z}[t]$ for each special character $\chi \in W^\wedge$, satisfying the equation (ii). Hence those polynomials $f_\chi(t)$ are regarded as those expressing the cardinality of "special pieces" in G^F even if G does not exist.

1.6. Let G be as in 1.2. For a unipotent class C in G, choose $u \in C$ and let $A_G(u)$ be the component group of $Z_G(u)$. Then G-equivariant irreducible local systems on C are in bijective correspondence with irreducible characters of $A_G(u)$. Hence the pair $(C, \mathcal{L}) \in \mathcal{I}$ is also represented by a pair (u, ρ) with $\rho \in A_G(u)^\wedge$. Let $\mathcal{B}_u$ be the variety of Borel subgroups of G containing u. We have an action of $W \times A_G(u)$ on the l-adic cohomology group $H^m(\mathcal{B}_u) =$

$H^m(\mathcal{B}_u, \bar{\mathbf{Q}}_l)$ (the Springer action of W and the natural action of $A_G(u)$). Assume that $(C, \mathcal{L}) \in \mathcal{I}$ is represented by a pair (u, ρ). Then the ρ-isotypic subspace $H^m(\mathcal{B}_u)_\rho$ of $H^m(\mathcal{B}_u)$ is a W-module. Set $d_u = \dim \mathcal{B}_u$. It is known that the pair $(C, \mathcal{L})$ is in $\mathcal{I}_0$ if and only if $H^{2d_u}(\mathcal{B}_u)_\rho \neq 0$, and in that case $W \times A_G(u)$-module $H^{2d_u}(\mathcal{B}_u)_\rho$ is written as $\chi \otimes \rho$ for an irreducible character χ of W. The correspondence $(C, \mathcal{L}) \mapsto \chi$ gives rise to a bijection between $\mathcal{I}_0$ and $W^\wedge$, which is nothing but the Springer correspondence. Note that the condition $(C, \mathcal{L}) \in \mathcal{I}_0$ is also equivalent to the condition $H^i(\mathcal{B}_u)_\rho \neq 0$ for some $i \geq 0$ by [10, 24.4]. Concerning the relationship between Springer representations and the a-function on $W^\wedge$, the followings are known.

1.6.1. ([1]) *Assume that $(u, 1)$ corresponds to $\chi \in W^\wedge$. Then $d_u = b(\chi) \geq a(\chi)$. In particular, if χ is special, then we have $d_u = a(\chi)$.*

1.6.2. ([7], [12]) *More generally, if (u, ρ) corresponds to $\chi \in W^\wedge$, then we have $d_u \geq a(\chi)$.*

We can state the following conjecture, which gives a connection between the a-function and Springer representations occurring in lower cohomology.

Conjecture 1.7. *Let G be as in* 1.2. *Assume that $\chi \in W^\wedge$ corresponds to $(C, \mathcal{L})$, and that $\chi_1 \in W^\wedge$ corresponds to $(C_1, \mathcal{L}_1) = (u_1, \rho_1)$ with $C_1 \subset \bar{C}$.*

(i) *If χ occurs in $H^i(\mathcal{B}_{u_1})_{\rho_1}$ for some i, then $a(\chi_1) \geq a(\chi)$.*

(ii) *In the setting in* (i), *assume further that C is special with $\mathcal{L} = \bar{\mathbf{Q}}_l$ and that $C_1 \not\subset \widehat{C}$. Then $a(\chi_1) > a(\chi)$.*

In the remainder of this section, we assume that p is good for G. Then in the case of exceptional groups the first statement of Conjecture 1.7 is obtained from the following stronger fact which is checked by making use of the tables of Springer correspondence (see, e.g. [3]).

Proposition 1.8. *Let G be of exceptional type, and assume that p is good for G. Assume further that $\chi \in W^\wedge$ corresponds to $(C, \mathcal{L})$, and that χ_1 corresponds to $(C_1, \mathcal{L}_1)$ with $C_1 \subsetneq \bar{C}$. Then we have $a(\chi_1) \geq a(\chi)$.*

In the case of exceptional groups, the second statement of Conjecture 1.7 is also verified by using the table of Green functions. The author is indebted to Frank Lübeck for checking this by using CHEVIE. Note that in contrast to Proposition 1.8, the second statement does not hold if one drops the assumption in (i).

We note also that Proposition 1.8 does not hold in the case of classical groups in general (see 4.13 for counterexamples). In Section 4, we show some properties of Springer representations, and using them we verify Conjecture 1.7 for G of type B_n, C_n or D_n with $n \leq 6$, without computing Green functions. The main purpose of this note is to show the following weaker version of Conjecture 1.4 in the case of classical groups, assuming Conjecture 1.7.

Theorem 1.9. *Let G be of classical type, and assume that p is odd. Assume further that Conjecture* 1.7 *holds for G. Then $\lambda_{ij}, p_{ij} \in \mathbf{Q}[t, t^{-1}]$. Moreover, the statements* (ii), (iii) *in Conjecture* 1.4 *hold for λ_{ii}.*

The idea of the proof is to compare the equations in (1.3.1) with the equations used for computing Green functions, which are obtained by expressing Green functions in terms of certain functions, associated to pairs $(C, \mathcal{L}) \in \mathcal{I}_0$, on the set of unipotent elements. These functions form a basis of the space $\mathcal{V}_0$ of functions generated by Green functions. We construct a new basis of $\mathcal{V}_0$ which has closer relations with the a-function. The theorem is obtained by expressing Green functions in terms of this new basis. It would be interesting to find a geometric interpretation of this basis.

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2. Green functions

2.1. In this subsection, we shall recall the definition of Green functions, and give an algorithm for computing them. We follow the notation in 1.2. Besides it, we assume that p is good for G, (and $q \equiv 1 \pmod 3$ if G is of type E_8), and that G^F is of split type. Then for each unipotent class C, there exists a good representative $u \in C^F$ as in [16], [2], a "split element" in C, which is unique up to G^F-conjugacy modulo the center of G. Then F acts trivially on the component group $A_G(u)$. The set of G^F-conjugacy classes in C^F is in bijective correspondence with the set $A_G(u)/\sim$ of conjugacy classes in $A_G(u)$. We denote by u_a a representative of the G^F-conjugacy class in C^F corresponding to $a \in A_G(u)/\sim$. For each $w \in W$, the Green function $Q_w : G^F_{\text{uni}} \to \bar{\mathbf{Q}}_l$ is defined by

$$Q_w(g) = \sum_{m=0}^{d_u} \operatorname{Tr}\left((w, a), H^{2m}(\mathcal{B}_u)\right) q^m, \tag{2.1.1}$$

if $g \in G^F_{\text{uni}}$ is G^F-conjugate to u_a. Hence Q_w is a G^F-invariant function on G^F_{uni}. We define, for each $\chi \in W^\wedge$, a function Q_χ on G^F_{uni} by

$$Q_\chi = |W|^{-1} \sum_{w \in W} \chi(w) Q_w.$$

Now assume that χ corresponds to $i = (C, \mathcal{L}) \in \mathcal{I}_0$ under the Springer correspondence. We consider the intersection cohomology $\operatorname{IC}(\bar{C}, \mathcal{L})$ on G. Then it is known, by Borho-MacPherson [1] that the χ-isotypic part $H^{2m}(\mathcal{B}_{u_1})_\chi$ is expressed as

$$H^{2m}(\mathcal{B}_{u_1})_\chi = V_\chi \otimes \mathcal{H}^{2m-2d_u}_{u_1} \operatorname{IC}(\bar{C}, \mathcal{L}), \tag{2.1.2}$$

where V_χ denotes the irreducible W-module affording χ, and $u \in C$. (Here $K = \operatorname{IC}(\bar{C}, \mathcal{L})$ is regarded as a complex on G by extending by 0 outside of $\bar{C}$, and $\mathcal{H}^j_{u_1} K$ denotes the stalk at u_1 of the j-th cohomology sheaf of the complex K.) Hence Q_χ is also expressed, for a split element $g \in G^F_{\text{uni}}$, as

$$Q_\chi(g) = \sum_{m \geq 0} (\dim \mathcal{H}^{2m-2d_u}_g \operatorname{IC}(\bar{C}, \mathcal{L})) q^m. \tag{2.1.3}$$

In what follows, we denote Q_χ by Q_i if χ corresponds to $i = (C, \mathcal{L}) \in \mathcal{I}_0$.

2.2. Let us define, for each $i = (C, \mathcal{L}) \in \mathcal{I}$, a G^F-invariant function ψ_i on G^F_{uni} by

$$\psi_i(g) = \begin{cases} \rho(a) & \text{if } g \text{ is } G^F\text{-conjugate to } u_a, \\ 0 & \text{if } g \notin C^F, \end{cases}$$

where $(C, \mathcal{L})$ is represented by (u, ρ).

Let $\mathcal{V}$ be the $\bar{\mathbf{Q}}_l$-space of G^F-invariant functions on G^F_{uni}. We define an inner product on $\mathcal{V}$ by

$$\langle f, h\rangle = \sum_{g \in G^F_{\text{uni}}} f(g)h(g).$$

Set $\mu_{ij} = \langle \psi_i, \psi_j\rangle$. We note that

2.2.1. *For each $i, j \in \mathcal{I}_o$, there exists a polynomial $\boldsymbol{\mu}_{ij} \in \mathbf{Q}[t]$ such that $\mu_{ij} = \boldsymbol{\mu}_{ij}(q)$. If $i = (C, \bar{\mathbf{Q}}_l)$, then we have $\boldsymbol{\mu}_{ii}(q) = |C^F|$.*

In fact, if (u, ρ) corresponds to $i \in \mathcal{I}_0$, ρ is obtained as the pull back of a character of the component group associated to the adjoint group. But it is known that if G is of adjoint type, $A_G(u)$ is isomorphic to the product of various symmetric groups S_i $(1 \le i \le 5)$. Hence $\rho(a) \in \mathbf{Z}$ for any $a \in A_G(u)$. On the other hand, the classification of unipotent classes and the structure of $A_G(u)$ are independent of p whenever p is good. Furthermore the cardinality of the G^F-conjugacy class containing u_a is expressed as $f_a(q)$ for some polynomial $f_a \in \mathbf{Q}[t]$. The first statement of 2.2.1 follows from these facts. The second statement is clear from the definition.

It is easily verified that the set $\{\psi_i \mid i \in \mathcal{I}\}$ forms a basis of $\mathcal{V}$. In turn it is proved by Lusztig [10] that the set $\{\psi_i \mid i \in \mathcal{I}_0\}$ forms a basis of the subspace $\mathcal{V}_0$ of $\mathcal{V}$ spanned by Green functions Q_i $(i \in \mathcal{I}_0)$. (Here Q_i denotes the function Q_χ if χ corresponds to i). Now one can express Q_i as

$$Q_i = \sum_{j \in \mathcal{I}_0} \pi_{ji}\psi_j \tag{2.2.2}$$

with $\pi_{ji} \in \bar{\mathbf{Q}}_l$. We note that

2.2.3. *There exists a polynomial $\boldsymbol{\pi}_{ij} \in \mathbf{Z}[t]$ such that $\pi_{ij} = \boldsymbol{\pi}_{ij}(q)$.*

In fact, assume that $j = (u_1, \rho_1) \in \mathcal{I}_0$. Then in view of (2.1.1), π_{ji} can be expressed as

$$\pi_{ji} = \sum_{m \ge 0} \langle H^{2m}(\mathcal{B}_{u_1}), \chi \otimes \rho_1\rangle q^m, \tag{2.2.4}$$

where $\langle H^{2m}(\mathcal{B}_{u_1}), \chi \otimes \rho_1\rangle$ denotes the multiplicity of $\chi \otimes \rho_1$ in $H^{2m}(\mathcal{B}_{u_1})$. It is known that the structure of $H^{2m}(\mathcal{B}_{u_1})$ as $W \times A_G(u_1)$-module is independent of p. Hence 2.2.3 holds.

We write $C = C_i$ if $i = (C, \mathcal{L})$. We define a partial order on the set of C_i $(i \in \mathcal{I})$ by $C_i \le C_j$ if C_i is contained in the closure of C_j. Then it is easy to see that $\mu_{ij} = 0$ unless $C_i = C_j$. Moreover, it follows from (2.1.2) and (2.2.4) that we have $\pi_{ij} = 0$ unless $C_j < C_i$ or $i = j$. If $i = j$, we have $\pi_{ii} = q^{d_u}$ for $i = (u, \rho)$.

Now the orthogonality relations for Green functions can be transformed to the following formula for Q_i, (see, e.g. [17]),

$$\langle Q_i, Q_j\rangle = \omega_{ij}(q).$$

The following result gives an algorithm for computing Green functions of G^F.

Proposition 2.3. (Lusztig [10]) *The polynomials $\boldsymbol{\pi}_{ij}, \boldsymbol{\mu}_{ij} \in \mathbf{Q}[t]$ are the unique solution of the following system of equations with unknown $\widetilde{\pi}_{ij}, \widetilde{\mu}_{ij}$,*

$$\tag{2.3.1} \begin{cases} \widetilde{\mu}_{i'j'} = 0 & \textit{unless } C_{i'} = C_{j'}, \\ \widetilde{\pi}_{j'j} = 0 & \textit{unless } C_{j'} < C_j \textit{ or } j = j', \\ \widetilde{\pi}_{jj} = t^{d_u} & \textit{for } j = (u,\rho) \in \mathcal{I}_0, \\ \omega_{ij} = \sum\limits_{i',j' \in \mathcal{I}_0} \widetilde{\pi}_{i'i}\widetilde{\mu}_{i'j'}\widetilde{\pi}_{j'j} & \textit{for all } i,j \in \mathcal{I}_0. \end{cases}$$

In fact, as discussed in 1.3, if we choose a total order on $\mathcal{I}_0$ which is compatible with the preorder for C_i, and define matrices $M = (\widetilde{\mu}_{i'j'})$ and $\varPi = (\widetilde{\pi}_{j'j})$ along this order, the last equation in (2.3.1) can be written as

$${}^t\varPi M \varPi = \varOmega.$$

As in 1.3, we can consider M and $\varPi$ as block matrices with respect to the equivalence relation $\approx$ ($i \approx j$ if $C_i = C_j$). Then M and $\varPi$ have the same shape as $\varLambda$ and P. Moreover M is non-singular since this holds for $t = q$. Hence (2.3.1) has a unique solution $\widetilde{\mu}_{ij}, \widetilde{\pi}_{ij} \in \mathbf{Q}(t)$. But the argument in 2.2 shows that $\widetilde{\mu}_{ij}(t)$ (resp. $\widetilde{\pi}_{ij}(t)$) coincides with $\boldsymbol{\mu}_{ij}(t)$ (resp. $\boldsymbol{\pi}_{ij}(t)$) if t is a power of a good prime. Hence $\widetilde{\mu}_{ij} = \boldsymbol{\mu}_{ij}$ and $\widetilde{\pi}_{ij} = \boldsymbol{\pi}_{ij}$ as asserted.

3. Proof of Theorem 1.9

3.1. Before going into details of the proof of Theorem 1.9, we prepare some combinatorial lemmas. In this section we regard q also as an indeterminate. Let $X = \{1, 2, \dots, k\}$ for some integer $k > 0$. We fix positive integers $d_1, d_2, \dots, d_k$. We denote by $\mathcal{P}_X$ the set of all the subsets of X. For each $I \in \mathcal{P}_X$, let us define $\mathcal{O}_I \in \mathbf{Z}[q]$ by

$$\mathcal{O}_I = \prod_{i \in I}(q^{d_i} - 1) \prod_{i \in X - I}(q^{d_i} + 1),$$

We define, for each $I \in \mathcal{P}_X$, a function $\varPsi_I : \mathcal{P}_X \to \{\pm 1\}$ by

$$\varPsi_I(J) = (-1)^{|I \cap J|}.$$

For abbreviation, we write $\varPsi_i = \varPsi_{\{i\}}$. Then we can write $\varPsi_I = \prod_{i \in I} \varPsi_i$. For each $I \in \mathcal{P}_X$, set $Y_I = \sum_{J \in \mathcal{P}_X} \mathcal{O}_J \varPsi_I(J)$. Then we have

Lemma 3.2. $Y_I = 2^k \prod\limits_{i \in X - I} q^{d_i}.$

Proof. We show the lemma by induction on $k = |X|$. The statement of the lemma is easily verified if $k = 1$. So, we assume that the lemma holds for any proper subset X' of X. First consider the case where I is a proper subset of X. Then there exists $x \in X$ such that $x \notin I$. We decompose $\mathcal{P}_X$ as $\mathcal{P}_X = \mathcal{P}' \cup \mathcal{P}''$, where $\mathcal{P}' = \mathcal{P}_{X'}$ with $X' = X - \{x\}$ and $\mathcal{P}''$ is the set of all the subsets of X containing x. We have

$$Y_I = \sum_{J \in \mathcal{P}'} \mathcal{O}'_J(q^{d_x} + 1)(-1)^{|I \cap J|} + \sum_{J' \in \mathcal{P}''} \mathcal{O}''_{J'}(q^{d_x} - 1)(-1)^{|I \cap J'|},$$

where $\mathcal{O}'_J$ coincides with $\mathcal{O}^{(X')}_J$, a similar object as $\mathcal{O}_J$ defined by replacing X by X', and $\mathcal{O}''_{J'}$ coincides with $\mathcal{O}^{(X')}_{J'-\{x\}}$ under a bijection $\mathcal{P}'' \simeq \mathcal{P}_{X'}$, $(J' \leftrightarrow J' - \{x\})$. Since $|I \cap J'| = |I \cap (J' - \{x\})|$, we have

$$Y_I = 2q^{d_x} Y^{(X')}_I,$$

where $Y^{(X')}_I$ is a similar object as Y_I defined by using $\mathcal{P}_{X'}$. Hence by induction, we obtain the required formula.

Next consider the case where $I = X$. Again we decompose $\mathcal{P}_X$ as $\mathcal{P}_X = \mathcal{P}' \cup \mathcal{P}''$, where $\mathcal{P}' = \mathcal{P}_{X'}$ with $X' = \{1, 2, \dots, k-1\}$, and $\mathcal{P}''$ is the set of all the subsets of X containing k. Then we have

$$Y_I = \sum_{J \in \mathcal{P}'} \mathcal{O}'_J(q^{d_k} + 1)(-1)^{|J|} + \sum_{J' \in \mathcal{P}''} \mathcal{O}''_{J'}(q^{d_k} - 1)(-1)^{|J'|}.$$

As before, $\mathcal{O}'_J$ coincides with $\mathcal{O}^{(X')}_J$ and $\mathcal{O}''_{J'}$ coincides with $\mathcal{O}_{J'-\{k\}}$ under a bijection $\mathcal{P}'' \simeq \mathcal{P}_{X'}$. Then the last formula is equal to

$$2 \sum_{J \in \mathcal{P}_{X'}} \mathcal{O}'_J(-1)^{|J|} = 2Y^{(X')}_{X'}.$$

Hence by induction, we have $Y_I = 2^k$ as asserted. This proves the lemma. □

3.3. Let $\mathcal{V}_X$ be the space of $\mathbf{R}(q)$-valued functions on $\mathcal{P}_X$. We define an inner product on $\mathcal{V}_X$ by

$$\langle f, h \rangle_X = \sum_{J \in \mathcal{P}_X} \mathcal{O}_J f(J) h(J)$$

for $f, h \in \mathcal{V}_X$. The inner product $\langle \Psi_I, \Psi_J \rangle$ is computed as follows. Since $\Psi_I = \prod_{i \in I} \Psi_i$, we see that $\Psi_I \Psi_J = \Psi_{I \ominus J}$, where $I \ominus J$ denotes the symmetric difference $I \cup J - I \cap J$. Hence we can write

$$\langle \Psi_I, \Psi_J \rangle_X = Y_{I \ominus J}. \tag{3.3.1}$$

Let us define, for each $J \in \mathcal{P}_X$, a function $\Theta_J \in \mathcal{V}_X$ by

$$\Theta_J = \sum_{J' \subset J} (-1)^{|J| + |J'|} q_{J'} \Psi_{J'}, \tag{3.3.2}$$

where $q_{J'} = \prod_{j \in J'} q^{d_j}$. We show

Lemma 3.4. *Let $I, J \in \mathcal{P}_X$ be such that $|I| < |J|$ or $|I| = |J|$ and $I \neq J$. Then we have $\langle \Psi_I, \Theta_J \rangle_X = 0$. In particular, $\langle \Theta_I, \Theta_J \rangle_X = 0$ for any $I, J \in \mathcal{P}_X$ such that $I \neq J$.*

Proof. It is enough to show the first assertion. We have

$$\begin{aligned}\langle \Psi_I, \Theta_J \rangle_X &= \sum_{J' \subset J} (-1)^{|J|+|J'|} q_{J'} \langle \Psi_I, \Psi_{J'} \rangle \\ &= \sum_{J' \subset J} (-1)^{|J|+|J'|} q_{J'} Y_{I \ominus J'},\end{aligned}$$

by (3.3.1). By our assumption, there exists $x \in J$ such that $x \notin I$. Let $\mathcal{P}_J$ be the set of all the subsets in J. Then $\mathcal{P}_J$ is written as $\mathcal{P}_J = \mathcal{P}'_J \cup \mathcal{P}''_J$, where $\mathcal{P}'_J$ (resp. $\mathcal{P}''_J$) is the subset of $\mathcal{P}_J$ consisting of $J' \subset J$ such that $x \notin J'$ (resp. $x \in J'$). We have $\mathcal{P}'_J \simeq \mathcal{P}''_J$, $J' \leftrightarrow J'' = J' - \{x\}$, and

$$(-1)^{|J|+|J''|} q_{J''} Y_{I \ominus J''} = -(-1)^{|J|+|J'|} q^{d_x} q_{J'} Y_{(I \ominus J') \cup x}. \tag{3.4.1}$$

But, by Lemma 3.2, we have $Y_{(I \ominus J') \cup x} = q^{-d_x} Y_{I \ominus J'}$. Hence the right hand side of (3.4.1) is equal to $-(-1)^{|J|+|J'|} q_{J'} Y_{I \ominus J'}$. This implies that $\langle \Psi_I, \Theta_J \rangle_X = 0$, and we obtain the formula. This proves the lemma. □

3.5. Let W be the Weyl group of type B_n or D_n. A notion of "symbols" was introduced by Lusztig, and irreducible characters of W are parametrized by certain symbols as follows, (see [8, chap.4]). First assume that W is of type B_n. Each irreducible character χ of W is parametrized by an ordered pair $(\alpha; \beta)$ of partitions of n. Here by allowing 0 in the entries of α or β, we may write those partitions as

$$\begin{aligned}\alpha &: 0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_{m+1}, \\ \beta &: 0 \le \beta_1 \le \beta_2 \le \cdots \le \beta_m\end{aligned}$$

for some $m > 0$ such that $\sum \alpha_i + \sum \beta_j = n$. We further assume that $\alpha_1 \neq 0$ or $\beta_1 \neq 0$. Let us define sequences $S = \{\lambda_1, \lambda_2, \dots, \lambda_{m+1}\}$, $T = \{\mu_1, \mu_2, \dots, \mu_m\}$ by $\lambda_i = \alpha_i + (i-1)$, $\mu_j = \beta_j + (j-1)$. Then $\Lambda = \Lambda(\chi) = \binom{S}{T}$ is called a (reduced) symbol of defect 1 and rank n corresponding to χ of W. We denote by $\Phi_{n,1}$ the set of symbols of rank n and defect 1. Thus we have $\Phi_{n,1} \simeq W^\wedge$.

Next assume that W is of type D_n. Each irreducible character χ of W is parametrized by an unordered pair $(\alpha; \beta)$ of partitions of n, (here if $\alpha = \beta$, the pair (α, β) corresponds to two distinct irreducible characters). By allowing 0 in the entries of α or β, we may write them as

$$\begin{aligned}\alpha &: 0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_m, \\ \beta &: 0 \le \beta_1 \le \beta_2 \le \cdots \le \beta_m\end{aligned}$$

for some $m > 0$ such that $\sum \alpha_i + \sum \beta_j = n$. As before, we may assume that $\alpha_1 \neq 0$ or $\beta_1 \neq 0$. Let us define sequences $S = \{\lambda_1, \lambda_2, \dots, \lambda_m\}$, $T = \{\mu_1, \mu_2, \dots, \mu_m\}$ by $\lambda_i = \alpha_i + (i-1)$, $\mu_j = \beta_j + (j-1)$. Then an unordered pair $\Lambda = \Lambda(\chi) = \binom{S}{T}$ is called a (reduced) symbol of defect 0 and rank n corresponding to $\chi \in W^\wedge$. We denote by $\widetilde{\Phi}_{n,0}$ the set of symbols of rank n and defect 0, where the symbols $\binom{S}{T}$ with $S = T$ are counted twice. Thus we have $\widetilde{\Phi}_{n,0} \simeq W^\wedge$.

Using the notion of symbols, the partition of $W^\wedge$ into families is easily described as follows: Take $\chi, \chi' \in W^\wedge$ and let $\Lambda(\chi) = \binom{S}{T}$ and $\Lambda(\chi') = \binom{S'}{T'}$ be associated to χ, χ'. Then χ and χ' are in the same family if and only if $S \cup T = S' \cup T'$ and $S \cap T = S' \cap T'$. Moreover, the condition for χ being special is given as follows: In the case where W is of type B_n, $\chi \in W^\wedge$ is special if and only if $\lambda_1 \le \mu_1 \le \lambda_2 \le \mu_2 \le \cdots$. In the case where W is of type D_n, χ is special if and only if $\lambda_1 \le \mu_1 \le \lambda_2 \le \mu_2 \le \cdots$ or $\mu_1 \le \lambda_1 \le \mu_2 \le \lambda_2 \le \cdots$.

The a-function on $W^\wedge$ is described as follows: For a given $\Lambda = \binom{S}{T}$ as above, we arrange the elements in $S \cup T$ as $\nu_1 \le \nu_2 \le \cdots \le \nu_{2m+1}$ (resp. $\nu_1 \le \nu_2 \le \cdots \le \nu_{2m}$) if W is of type B_n (resp. if W is of type D_n), respectively. We prepare a sequence

$$\nu_1^0 \le \nu_2^0 \le \cdots \le \nu_{2m+1}^0 = 0 \le 0 \le 1 \le 1 \cdots \le m-1 \le m-1 \le m$$

in the case where W is of type B_n, and

$$\nu_1^0 \le \nu_2^0 \le \cdots \le \nu_{2m}^0 = 0 \le 0 \le 1 \le 1 \le \cdots \le m-1 \le m-1$$

in the case where W is of type D_n. Let $a(\Lambda) = a(\chi)$ for $\Lambda = \Lambda(\chi)$. Then we have

$$a(\Lambda) = \sum_{i<j} \inf\{\nu_i, \nu_j\} - \sum_{i<j} \inf\{\nu_i^0, \nu_j^0\}. \tag{3.5.1}$$

In fact, the formula for $a(\Lambda)$ is given in [8, chap.4] in a slightly different form. The formula (3.5.1) for type B_n is found in [11, 4.11]. The similar formula for type D_n is also deduced easily from the one in [8].

3.6. Let $G = Sp_N(k)$ or $SO_N(k)$ with $ch\, k \ne 2$. The unipotent classes and the structure of the centralizer of unipotent elements are described as follows. Assume that $G = Sp_N(k)$. Then the set of unipotent classes of G is in bijection with the set X_N of partitions $\lambda = (1^{r_1}, 2^{r_2}, \dots)$ of N such that r_i is even for odd i. We denote by u_λ a unipotent element in G corresponding to $\lambda \in X_N$. Then we have

$$Z_G(u_\lambda) \simeq \prod_{i:\text{ odd}} Sp_{r_i} \times \prod_{i:\text{ even}} O_{r_i}.$$

Hence the component group $A_G(u_\lambda) = Z_G(u_\lambda)/Z_G^0(u_\lambda)$ is isomorphic to a product $\mathbf{Z}/2\mathbf{Z} \times \cdots \times \mathbf{Z}/2\mathbf{Z}$, where factors are parametrized by the set $\Delta_\lambda = \{i : \text{ even} \mid r_i > 0\}$. We denote by α_i the generator of $\mathbf{Z}/2\mathbf{Z}$ corresponding to $i \in \Delta_\lambda$. It follows that the sets $A_G(u_\lambda)$ and $A_G(u_\lambda)^\wedge$ are parametrized by the set $\mathcal{P}(\Delta_\lambda)$ of all the subsets of Δ_λ. For each $I \subset \Delta_\lambda$, $\rho_I \in A_G(u_\lambda)^\wedge$ is given by

$$\rho_I(\alpha_i) = \begin{cases} 1 & \text{if } i \notin I, \\ -1 & \text{if } i \in I. \end{cases} \tag{3.6.1}$$

Next assume that $G = SO_N(k)$. Then the set of unipotent classes in G is in bijection with the set X'_N of partitions $\lambda = (1^{r_1}, 2^{r_2}, \dots)$ of N such that r_i is even for even i, except that to $\lambda = (2^{r_2}, 4^{r_4}, \dots) \in X'_N$ there correspond two unipotent classes. Again let u_λ stand for a unipotent element corresponding to $\lambda \in X'_N$. Then we have

$$Z_G(u_\lambda) \simeq \prod_{i:\text{ even}} Sp_{r_i} \times \{(x_i) \in \prod_{i:\text{ odd}} O_{r_i} \mid \prod \det x_i = 1\}.$$

Hence $A_G(u_\lambda)$ is isomorphic to an index 2 subgroup of $\prod_{i:\text{odd}} O_{r_i}/SO_{r_i} \simeq \mathbf{Z}/2\mathbf{Z} \times \cdots \times \mathbf{Z}/2\mathbf{Z}$, where factors are parametrized by the set $\varDelta_\lambda = \{i: \text{ odd } \mid r_i > 0\}$ and $A_G(u_\lambda)$ is defined by the condition that $\sum u_i\alpha_i \in A_G(u)$ if and only if $\sum u_i \equiv 0 \pmod 2$. (Here α_i is the generator of $\mathbf{Z}/2\mathbf{Z}$ corresponding to $i \in \varDelta_\lambda$). As in the previous case, the set $A_G(u_\lambda)$ is parametrized by the set $\mathcal{P}_{\text{ev}}(\varDelta_\lambda) = \{I \subset \varDelta_\lambda \mid |I|: \text{ even}\}$, and $A_G(u_\lambda)^\wedge$ is parametrized by the set $\mathcal{P}(\varDelta_\lambda)/\!\sim$, where $\sim$ is the equivalence relation defined by $I \sim \varDelta_\lambda - I$. For each $I \in \mathcal{P}(\varDelta_\lambda)/\!\sim$, ρ_I is defined similarly.

We now consider the $\mathbf{F}_q$-structure on G. We choose a split element u_λ in G^F. Let C_λ be the conjugacy class containing u_λ. Then the set of G^F-conjugacy classes in C_λ^F is parametrized by $A_G(u_\lambda)$, hence by subsets of $\varDelta_\lambda$. We denote by $u_I \in C_\lambda^F$ a representative corresponding to $I \subset \varDelta_\lambda$. We define a subset $\varDelta_\lambda^0$ of $\varDelta_\lambda$ by

$$\varDelta_\lambda^0 = \{i \in \varDelta_\lambda \mid r_i \equiv 2 \pmod 4\}.$$

For each $i \in \varDelta_\lambda^0$, we put $d_i = r_i/2$. For $I \subset \varDelta_\lambda$ let $\mathcal{O}(u_I)$ be the G^F-conjugacy class containing u_I. Then the cardinality $|\mathcal{O}(u_I)|$ is a polynomial in q. Furthermore the following formula is easily verified.

3.6.2. *There exists a rational function $F_\lambda(t)$ such that*

$$|\mathcal{O}(u_I)| = F_\lambda(q) \prod_{i \in I \cap \varDelta_\lambda^0} (q^{d_i} - 1) \prod_{i \in \varDelta_\lambda^0 - I} (q^{d_i} + 1).$$

3.7. We now recall combinatorial objects introduced by Lusztig [9] to describe the generalized Springer correspondence of classical groups. (Here we only need the Springer correspondence). For an even integer $N \geq 2$, let $\Psi_{N,1}$ be the set of pairs $\binom{A}{B}$, called u-symbols, subject to the following conditions: (i) A is a finite subset of $\{0, 1, 2, \dots\}$, B is a finite subset of $\{1, 2, \dots\}$ such that $|A| = |B| + 1$, (ii) A, B contain no consecutive integers, (iii) $\sum_{a \in A} a + \sum_{b \in B} b = \frac{1}{2}N + \frac{1}{2}(|A| + |B|)(|A| + |B| - 1)$. Moreover a u-symbol $\binom{A}{B}$ is called reduced if $0 \notin A$ or $0 \notin B$. Unless otherwise stated, we assume that u-symbols are reduced. We call $\Psi_{N,1}$ the set of (reduced) u-symbols of defect 1 and rank N. Two u-symbols $\binom{A}{B}$, $\binom{A'}{B'}$ are said to be similar if $A \cup B = A' \cup B'$ and $A \cap B = A' \cap B'$. It is known that there exists a natural bijection between the set of similarity classes of $\Psi_{N,1}$ and the set X_N, (see 4.3). Hence we can identify similarity classes and unipotent classes in Sp_N.

Next, for any odd integer $N \geq 3$, let $\Psi'_{N,1}$ be the set of pairs $\binom{A}{B}$, also called u-symbols, subject to the following conditions: (i)' A, B are finite subsets of $\{0, 1, 2 \dots\}$ such that $|A| = |B| + 1$, (ii) the same as above, (iii)' $\sum_{a \in A} a + \sum_{b \in B} b = \frac{1}{2}N + \frac{1}{2}((|A| + |B| - 1)^2 - 1)$. Also for an even integer $N \geq 4$, let $\widetilde{\Psi}'_{N,0}$ be the set of unordered pairs $\binom{A}{B}$, where the conditions are the same as (i)', (ii), (iii)' except that $|A| = |B|$, and that $\binom{A}{B}$ is counted twice in the case where $A = B$. A u-symbol $\binom{A}{B}$ is called reduced if $0 \notin A \cap B$. As before any elements in $\Psi'_{N,1}$ or $\widetilde{\Psi}'_{N,0}$ is assumed to be reduced. We define similarity classes for $\Psi'_{N,1}$ or $\widetilde{\Psi}'_{N,0}$ as in the case of $\Psi_{N,1}$. Then it is known that there exists a natural bijection from the set of similarity classes in $\Psi'_{N,1}$ (resp. $\widetilde{\Psi}'_{N,0}$) to the set X'_N for N : odd (resp. N : even). Hence we can identify similarity classes and unipotent classes in these cases also.

3.8. Each similarity class contains a unique u-symbol $\binom{A}{B}$ of the form $A = \{a_1 < a_2 < \cdots\}$, $B = \{b_1 < b_2 < \dots\}$ such that $a_1 \le b_1 \le a_2 \le b_2 \le \cdots$, (or $b_1 \le a_1 \le b_2 \le a_2 \le \cdots$ in the case of $\widetilde{\Psi}'_{N,0}$). Such an element is called a distinguished u-symbol. Let $\xi = \xi_\lambda = \binom{A}{B}$ be a distinguished u-symbol corresponding to $\lambda \in X_N$. We consider the symmetric difference $A \ominus B$. A non-empty subset $\varGamma$ in $A \ominus B$ is said to be an interval if it is of the form $\{i, i+1, \dots, j-1, j\}$ with $i-1 \notin A \ominus B, j+1 \notin A \ominus B$, (and furthermore $i \neq 0$ in the case of $\Psi_{N,1}$). Let $\mathcal{A}_\lambda$ be the set of intervals of $A \ominus B$. Then the cardinality of $\mathcal{A}_\lambda$ is the same as $|\varDelta_\lambda|$, and we can identify $\mathcal{A}_\lambda$ with $\varDelta_\lambda$ in the following way. We arrange the elements in $\mathcal{A}_\lambda$ in an increasing order $\varGamma_1 < \varGamma_2 < \cdots < \varGamma_l$, and also write $\varDelta_\lambda = \{i_1 < i_2 < \cdots < i_l\}$, where $l = |\mathcal{A}_\lambda| = |\varDelta_\lambda|$. We associate i_j to $\varGamma_j$ $(j = 1, \dots, l)$. Then we have $|\varGamma_j| = r_{i_j}$.

In the case of $\Psi_{N,1}$, for each $I \subset \mathcal{A}_\lambda$, a u-symbol $\xi_I = \binom{A_I}{B_I}$ is obtained from ξ by exchanging the entries $A \cap \varGamma$ and $B \cap \varGamma$ for $\varGamma \in I$. (The entries $\{0, 1, 2, \dots, k\}$ which do not belong to any interval remain unchanged). If $\binom{A_I}{B_I} \in \Psi_{N,1}$, this gives an element in the similarity class of ξ. In the case of $\Psi'_{N,1}$ or $\widetilde{\Psi}'_{N,0}$, for each subset $I \subset \mathcal{A}_\lambda$, a u-symbol $\xi_I = \binom{A_I}{B_I}$ or $\binom{B_I}{A_I}$ is obtained from ξ, and it gives an element in the similarity class of ξ if it is in $\Psi'_{N,1}$ or $\widetilde{\Psi}'_{N,0}$. (We regard $\binom{A_I}{B_I}$ as an unordered pair). In both cases, all the elements in the similarity class of ξ are obtained in this way. Hence one can identify elements in the similarity class of ξ with a subset of $A_G(u_\lambda)^\wedge$.

The following result gives a complete description of the Springer correspondence of classical groups.

3.8.1. *Let $\pi : W^\wedge \to \mathcal{I}_0$ be the Springer correspondence. Then under the above identification, the set $\mathcal{I}_0$ coincides with $\Psi_{2n,1}, \Psi'_{2n+1,1}, \widetilde{\Psi}_{2n,0}$ according as $G = Sp_{2n}, SO_{2n+1}$ or SO_{2n}. The correspondence π is given as follows.*

(i) $G = Sp_{2n}$. $\pi : \Phi_{n,1} \hookrightarrow \Psi_{2n,1}$ *is given by*

$$\begin{pmatrix} \lambda_1 < \lambda_2 < \cdots < \lambda_{m+1} \\ \mu_1 < \mu_2 < \cdots < \mu_m \end{pmatrix} \mapsto \begin{pmatrix} \lambda_1 < \lambda_2 + 1 < \cdots < \lambda_{m+1} + m \\ \mu_1 + 1 < \mu_2 + 2 < \cdots < \mu_m + m \end{pmatrix}.$$

(ii) $G = SO_{2n+1}$. $\pi : \Phi_{n,1} \hookrightarrow \Psi'_{2n+1,1}$ *is given by*

$$\begin{pmatrix} \lambda_1 < \lambda_2 < \cdots < \lambda_{m+1} \\ \mu_1 < \mu_2 < \cdots < \mu_m \end{pmatrix} \mapsto \begin{pmatrix} \lambda_1 < \lambda_2 + 1 < \cdots < \lambda_{m+1} + m \\ \mu_1 < \mu_2 + 1 < \cdots < \mu_m + (m-1) \end{pmatrix}.$$

(iii) $G = SO_{2n}$. $\pi : \widetilde{\Phi}_{n,0} \hookrightarrow \widetilde{\Psi}'_{2n,0}$ *is given by*

$$\begin{pmatrix} \lambda_1 < \lambda_2 < \cdots < \lambda_m \\ \mu_1 < \mu_2 < \cdots < \mu_m \end{pmatrix} \mapsto \begin{pmatrix} \lambda_1 < \lambda_2 + 1 < \cdots < \lambda_m + (m-1) \\ \mu_1 < \mu_2 + 1 < \cdots < \mu_m + (m-1) \end{pmatrix}.$$

3.9. Let $\xi = \binom{A}{B}$ be a (not necessarily distinguished) u-symbol belonging to the class of u_λ. For any $i \in \varDelta_\lambda$ such that $|\varGamma_i|$ is even, we can construct a u-symbol ξ_i by replacing $A \cap \varGamma_i$ and $B \cap \varGamma_i$ in the entries of ξ. (If $\xi \in \Psi_{N,1}, \Psi'_{N,1}$ or $\widetilde{\Psi}_{N,0}$, ξ_i is again contained in the same set.) Let us write $A = \{a_1 < a_2 < \cdots\}$ and $B = \{b_1 < b_2 < \cdots\}$. Assume that $|\varGamma_i| = 2d$. Then we can write $A \cap \varGamma_i = \{a_j, a_{j+1}, \cdots, a_{j+d-1}\}$, $B \cap \varGamma_i = \{b_k, b_{k+1}, \cdots, b_{k+d-1}\}$ for some j, k. We have the following lemma.

Lemma 3.10. *Let $\varLambda = \pi^{-1}(\xi)$ and $\varLambda_i = \pi^{-1}(\xi_i)$ be symbols obtained from ξ and ξ_i via the Springer correspondence. Let $\delta = 1$ (resp. $\delta = 0$) in the case of Sp_N (resp. SO_N), respectively.*

(i) *Assume that $j = k + \delta$. Then $a(\varLambda_i) = a(\varLambda)$.*

(ii) *Assume that $j > k + \delta$. Then*

$$\begin{cases} a(\varLambda_i) \geq a(\varLambda) + d & \text{if } a_j < b_k, \\ a(\varLambda_i) \leq a(\varLambda) - d & \text{if } a_j > b_k. \end{cases}$$

(iii) *Assume that $j < k + \delta$. Then*

$$\begin{cases} a(\varLambda_i) \leq a(\varLambda) - d & \text{if } a_j < b_k, \\ a(\varLambda_i) \geq a(\varLambda) + d & \text{if } a_j > b_k. \end{cases}$$

Proof. First consider the case $G = Sp_N$. We assume that $a_j = x < x+1 = b_k$ since the other case is discussed by exchanging $\varLambda$ and $\varLambda_i$. Then we can write

$$\varLambda = \begin{pmatrix} \cdots & x+1-j, & x+2-j, & \cdots, & x+d-j, & \cdots \\ \cdots & x+1-k, & x+2-k, & \cdots, & x+d-k, & \cdots \end{pmatrix},$$

$$\varLambda_i = \begin{pmatrix} \cdots & x+2-j, & x+3-j, & \cdots, & x+d+1-j, & \cdots \\ \cdots & x-k, & x+1-k, & \cdots, & x+d-1-k, & \cdots \end{pmatrix}.$$

If $j = k+1$, we see easily that $a(\varLambda_i) = a(\varLambda)$. Assume that $j \geq k+2$. Then the entries in $\varLambda$ can be written in an increasing order as

$$\cdots \leq x+1-j < x+2-j < \cdots < x+1-k \leq \cdots,$$

which we denote by $\{\nu_i\}$. The a-function $a(\varLambda)$ can be computed by the formula (3.5.1). Let $\{\nu'_i\}$ be the sequence obtained from $\{\nu_i\}$ by replacing $x+1-j$ by $x+2-j$ and $x+1-k$ by $x-k$. Let α be the value defined by a similar formula as (3.5.1) by using $\{\nu'_i\}$ instead of $\{\nu_i\}$. Then we have $\alpha \geq a(\varLambda)+1$. Applying a similar procedure as above for $x+2-j, \cdots, x+d-j$, successively, we see that $a(\varLambda_i) \geq a(\varLambda) + d$.

In the case where $j \leq k$, a similar argument as above shows that $a(\varLambda_i) \leq a(\varLambda) - d$.

Next consider the case where $G = SO_N$. We assume that $a_j = x < x+1 = b_k$. Then we have

$$\varLambda = \begin{pmatrix} \cdots & x+1-j, & x+2-j, & \cdots & x+d-j, & \cdots \\ \cdots & x+2-k, & x+3-k, & \cdots & x+d+1-k, & \cdots \end{pmatrix},$$

$$\varLambda_i = \begin{pmatrix} \cdots & x+2-j, & x+3-j, & \cdots & x+d+1-j, & \cdots \\ \cdots & x+1-k, & x+2-k, & \cdots & x+d-k, & \cdots \end{pmatrix}.$$

Then if $j = k$, we have $a(\Lambda_i) = a(\Lambda)$. The other cases are dealt with in a similar way. This proves the lemma. $\square$

In the special case where ξ is a distinguished u-symbol, we have the following result which is straightforward from Lemma 3.10.

Corollary 3.11. *Assume that ξ is distinguished. Then we have $a(\Lambda_i) \le a(\Lambda) - d$ unless $j = k + \delta$, in which case we have $a(\Lambda_i) = a(\Lambda)$.*

3.12. Let Δ^1_λ be the complement of Δ^0_λ in Δ_λ. Hence $\Delta_\lambda = \Delta^0_\lambda \coprod \Delta^1_\lambda$. For each $I \subset \Delta^1_\lambda$, let Δ'_I be the set of $i \in \Delta^0_\lambda$ such that the interval Γ_i satisfies the relation (i) in Lemma 3.10 with respect to $\xi = \xi_I$. Let $\Delta_I = \Delta^0_\lambda - \Delta'_I$. Then it follows from Lemma 3.10 that there exists $I^* \subset \Delta_I$ satisfying the following property: For each $J \subset \Delta'_I, K \subset \Delta_I$, let ξ_{IJK} be the u-symbol corresponding to $I \cup J \cup (K \ominus I^*) \in \Delta_\lambda$. Set $\Lambda_{IJK} = \pi^{-1}(\xi_{IJK})$. Then $a(\Lambda_{IJ\emptyset})$ is maximal among all $a(\Lambda_{IJK})$ for $K \subset \Delta_I$. Note that, if ξ is distinguished, then in view of Corollary 3.11 we have $I^* = \emptyset$, and so $\Lambda_{\emptyset\emptyset\emptyset}$ corresponds to the u-symbol ξ. More generally as a corollary to Lemma 3.10, we have the following.

Corollary 3.13. *Let the notations be as above. Then*

(i) $a(\Lambda_{IJK}) = a(\Lambda_{IJ'K})$ *for* $J, J' \subset \Delta'_I$.

(ii) $a(\Lambda_{IJK}) < a(\Lambda_{IJK'})$ *if* $K' \subsetneq K \subset \Delta_I$.

(iii) $d_{u_\lambda} - a(\Lambda_{IJK}) \ge \sum_{j \in K} d_j$.

In fact, (i) and (ii) follows from Lemma 3.10 directly. Moreover, we have

$$a(\Lambda_{IJ\emptyset}) - a(\Lambda_{IJK}) \ge \sum_{j \in K} d_j$$

by a repeated use of Lemma 3.10. Since $d_{u_\lambda} - a(\Lambda_{IJ\emptyset}) \ge 0$ by 1.6.2, we obtain (iii). (The property (iii) in the corollary is not used later).

3.14. As described in 3.6, the set $A_G(u_\lambda)^\wedge$ is parametrized by $\mathcal{P}(\Delta_\lambda)$ or $\mathcal{P}(\Delta_\lambda)/\sim$. Let $\rho_I \in A_G(u_\lambda)^\wedge$ be as in (3.6.1). Then for $i = (u_\lambda, \rho_I) \in \mathcal{I}_0$, we obtain a G^F-invariant function ψ_i as defined in 2.2, which we denote by ψ_I. Now $I \subset \Delta_\lambda$ can be written uniquely in the form $I = I_1 \cup I_2 \cup (I_3 \ominus I_1^*)$, where $I_1 \subset \Delta^1_\lambda, I_2 \subset \Delta'_{I_1}, I_3 \subset \Delta_{I_1}$ as in 3.12. We define a G^F-invariant function θ_I on G^F_{uni} by $\theta_I = \psi_{I_1}\psi_{I_2}\theta'_{I_3}$, where

$$\theta'_{I_3} = \sum_{J' \subset I_3} (-1)^{|I_3| + |J'|} q_{J'} \psi_{J' \ominus I_1^*} = \psi_{I_1^*} \sum_{J' \subset I_3} (-1)^{|I_3| + |J'|} q_{J'} \psi_{J'}$$

with $q_{J'} = \prod_{j \in J'} q^{d_j}$. The following property is easily verified by using Corollary 3.13 (ii).

3.14.1. *θ_I is written in the form $\theta_I = q_{I_3}\psi_I + \alpha$, where α is a linear combination of ψ_J, with coefficients in $\mathbf{Z}[q]$, such that $J \subsetneq I$ and that $a(J) > a(I)$.*

When $i = (u_\lambda, \rho_I)$ varies over all the elements in $\mathcal{I}_0$, the function θ_I gives rise to a basis of $\mathcal{V}_0$ (cf. 2.2). For each $i = (u_\lambda, \rho_I) \in \mathcal{I}_0$, we define $a(I)$ as $a(I) = a(i)$. Then we have the following lemma.

Lemma 3.15. *Let $I, J \subset \varDelta_\lambda$. Then $\langle \theta_I, \theta_J\rangle = 0$ unless $a(I) = a(J)$.*

Proof. To simplify the notation, we write $\rho_I(u_K), \mathcal{O}(u_K)$ for a unipotent element u_K as $\rho_I(K), \mathcal{O}(K)$. For $I, J \subset \varDelta_\lambda$, we can write

$$\langle \psi_I, \psi_J\rangle = \sum_K |\mathcal{O}(K)|\rho_I(K)\rho_J(K),$$

where K runs over all the elements in $\mathcal{P}(\varDelta_\lambda)$ (resp. $\mathcal{P}_{\mathrm{ev}}(\varDelta_\lambda)$) in the case where $G = Sp_N$ (resp. $G = SO_N$), respectively. We decompose I and J as $I = I_1 \cup I_2 \cup (I_3 \ominus I_1^*)$, $J = J_1 \cup J_2 \cup (J_3 \ominus J_1^*)$, where $I_1, J_1 \in \varDelta^1_\lambda$, $I_2 \in \varDelta'_{I_1}, I_3 \in \varDelta_{I_1}, J_2 \in \varDelta'_{J_1}, J_3 \in \varDelta_{J_1}$. We also decompose K as $K = K_1 \cup K_2'$ with $K_1 \subset \varDelta^1_\lambda$ and $K_2' \in \varDelta^0_\lambda$. By 3.6.2, $|\mathcal{O}(K)|$ only depends on the K_2' parts. Hence, the last formula is equal to (up to non-zero scalar)

$$\sum_{K_1 \in \varDelta^1_\lambda} \rho_{I_1}(K_1)\rho_{J_1}(K_1) \cdot \sum_{K_2' \in \varDelta^0_\lambda} |\mathcal{O}(K_2')|\rho_{I'}(K_2')\rho_{J'}(K_2'), \tag{3.15.1}$$

where $I' = I_2 \cup (I_3 \ominus I_1^*)$, $J' = J_2 \cup (J_3 \ominus J_1^*)$. Moreover, the first part $\sum \rho_{I_1}(K_1)\rho_{J_1}(K_1)$ coincides with the inner product for characters ρ_{I_1} and ρ_{J_1} of the elementary abelian 2-group generated by $\varDelta^1_\lambda$. Hence it is 0 unless $I_1 = J_1$.

We assume that $I_1 = J_1$. We decompose $K_2' \in \varDelta^0_\lambda$ as $K_2' = K_2 \cup K_3$ with $K_2 \in \varDelta'_{I_1} = \varDelta'_{J_1}$, $K_3 \in \varDelta_{I_1} = \varDelta_{J_1}$. Then

$$\begin{aligned}
&\sum_{K_2' \in \varDelta^0_\lambda} |\mathcal{O}(K_2')|\rho_{I'}(K_2')\rho_{J'}(K_2') \\
&= c \sum_{K_2 \in \varDelta'_{I_1}} |\mathcal{O}(K_2)|\rho_{I_2}(K_2)\rho_{J_2}(K_2) \sum_{K_3 \in \varDelta_{I_1}} |\mathcal{O}(K_3)|\rho_{I_3 \ominus I_1^*}(K_3)\rho_{J_3 \ominus I_1^*}(K_3),
\end{aligned}$$

since $\rho_{I_3 \ominus I_1^*}(K_3) = \rho_{I_3}(K_3)\rho_{I_1^*}(K_3)$. Here c is a non-zero constant depending only on I_1.

We now compute the inner product $\langle \psi_I, \theta_J\rangle$ under the assumption $|I| \le |J|$. The previous computation shows that

$$\langle \psi_I, \theta_J\rangle = c \sum_{K_2 \in \varDelta_{I_1}} |\mathcal{O}(K_2)|\rho_{I_2}(K_2)\rho_{J_2}(K_2) \sum_{K_3 \in \varDelta_{I_1}} |\mathcal{O}(K_3)|\rho_{I_3}(K_3)\theta'_{J_3}(K_3).$$

The last sum of the right hand side coincides with $\langle \Psi_{I_3}, \varTheta_{J_3}\rangle_X$ in the notation of 3.3, applied for $X = \varDelta_{I_1}$ under the identification $\Psi_{I_3} \leftrightarrow \psi_{I_3}$, $\varTheta_{J_3} \leftrightarrow \theta'_{J_3}$. Hence by Lemma 3.4, we know that $\langle \Psi_{I_3}, \varTheta_{J_3}\rangle_X = 0$, and so $\langle \psi_I, \theta_J\rangle = 0$ unless $I_3 = J_3$. It follows that $\langle \theta_I, \theta_J\rangle = 0$ unless $I_1 = J_1$ and $I_3 = J_3$. Now assume that $\langle \theta_I, \theta_J\rangle \ne 0$. Then I and J differs only by I_2 and J_2 parts. But then by Corollary 3.13 (i), we see that $a(I) = a(J)$. This proves the lemma. □

3.16. For $i = (u, \rho_I) \in \mathcal{I}_0$, we write θ_I as θ_i. Also in this case we write q_i for $q_{I_3} = \prod_{j \in I_3} q^{d_j}$. We consider the expression $Q_i = \sum_j \pi_{ji}\psi_j$ as in (2.2.2). For each $i \in \mathcal{I}_0$, let us define a G^F-invariant function $\widetilde{\theta}_i$ on G^F_{uni} by

$$\widetilde{\theta}_i = \sum_j \pi_{ji} q^{-a(i)} q_j^{-1} \theta_j, \tag{3.16.1}$$

where the sum is taken over all j such that $a(j) = a(i)$. If $i \neq j$, then $\pi_{ji} = 0$ unless $C_j < C_i$. This implies that the $\widetilde{\theta}_i$ $(i \in \mathcal{I}_0)$ form a basis of $\mathcal{V}_0$. In view of 3.14.1, $\widetilde{\theta}_i$ is a linear combination of ψ_j with coefficients in $\mathbf{Z}[q, q^{-1}]$. Hence, if we set $\lambda_{ij} = \langle \widetilde{\theta}_i, \widetilde{\theta}_j \rangle$, 2.2.1 implies that $\lambda_{ij} \in \mathbf{Q}[q, q^{-1}]$. Moreover, it follows from Lemma 3.15 that $\lambda_{ij} = 0$ unless $a(i) = a(j)$. We can express Q_i in terms of $\widetilde{\theta}_j$ as $Q_i = \sum_{j \in \mathcal{I}_0} p_{ji} \widetilde{\theta}_j$. Now Conjecture 1.7 (i) implies that $a(j) \geq a(i)$ unless $\pi_{ji} = 0$. Hence in view of 3.14.1, we see that $p_{ji} = 0$ unless $a(j) > a(i)$ or $i = j$, and that $p_{ii} = q^{a(i)}$. Moreover, we have $p_{ji} \in \mathbf{Q}[q, q^{-1}]$. Finally, the fourth formula in (1.3.1) is obtained from the relation $\langle Q_i, Q_j \rangle = \omega_{ij}(q)$. Summing up the above argument, we have the following.

Proposition 3.17. *Let us write $\lambda_{ij} = \langle \widetilde{\theta}_i, \widetilde{\theta}_j \rangle$, and $Q_i = \sum_{j \in \mathcal{I}_0} p_{ji} \widetilde{\theta}_j$. Assume that Conjecture 1.7 holds for G. Then λ_{ij}, p_{ij} satisfy the relations in (1.3.1). Moreover, we have $\lambda_{ij}, p_{ij} \in \mathbf{Q}[q, q^{-1}]$.*

Now in order to prove Theorem 1.9, it is enough to show the following lemma.

Lemma 3.18. *Let λ_{ii} be as above. Then*

(i) $\sum_i \lambda_{ii} = q^{2N}$*, where the sum is taken over all $i \in \mathcal{I}_0$ such that χ_i is special.*

(ii) *For $i = (C, \bar{\mathbf{Q}}_l) \in \mathcal{I}_0$ with C special, λ_{ii} coincides with $|\widehat{C}^F|$.*

Proof. First we argue on the group $G_{\mathbf{C}}$, the algebraic group defined over $\mathbf{C}$. Let $C = C_i$ be the corresponding special unipotent class in $G_{\mathbf{C}}$ with $u \in C$. Then by [6], $\widehat{C}$ is a rational homology manifold, and so the restriction $\mathrm{IC}(\bar{C}, \mathbf{C})|_{\widehat{C}}$ of $\mathrm{IC}(\bar{C}, \mathbf{C})$ to $\widehat{C}$ is a constant sheaf $\mathbf{C}$. It follows from (2.1.2) that $H^{2m}(\mathcal{B}_{u_1})_{\chi \otimes \rho_1}$ is non-zero only when $m = d_u$ and $\rho_1 = 1$. Moreover in this case $\langle H^{2m}(\mathcal{B}_{u_1}), \chi \otimes \rho_1 \rangle = 1$. The corresponding fact also holds for G. Hence we see that $\pi_{ji} = q^{d_u}$ for $j = (C_j, \bar{\mathbf{Q}}_l)$ such that $C_j \subset \widehat{C}_i$. On the other hand, by Conjecture 1.7 (ii), if $C_j \subset \bar{C}_i$ and $C_j \not\subset \widehat{C}_i$, and if $a(j) = a(i)$, we must have $\pi_{ji} = 0$. Moreover by [11], if $j = (C_j, \bar{\mathbf{Q}}_l)$ with $C_j \subset \widehat{C}_i$, then $a(i) = a(j)$. Hence in (3.16.1) the sum is taken exactly over $j = (C_j, \bar{\mathbf{Q}}_l) \in \mathcal{I}_0$ such that $C_j \subset \widehat{C}_i$. Furthermore by a remark in 3.12, we see that $q_j = 1$ and $\theta_j = 1_{C_j}$ in this case. Hence $\widetilde{\theta}_i = \sum_j \theta_j$ coincides with the characteristic function $1_{\widehat{C}_i}$ on $\widehat{C}_i^F$. This implies that $\lambda_{ii} = \langle \widetilde{\theta}_i, \widetilde{\theta}_i \rangle = |\widehat{C}_i^F|$. Thus we have $\sum_i \lambda_{ii} = \sum_i |\widehat{C}_i^F| = q^{2N}$. This proves the lemma, and so the theorem follows. □

3.19. It is likely that a similar construction of a basis $\widetilde{\theta}_i$ will work also for the case of exceptional groups. As an example, we consider the case of F_4. So, assume that G is of type F_4, and let C be a unipotent class in G. If the a-function is constant for all $\chi_i \in W^\wedge$ belonging to C, we may choose $\theta_i = \psi_i$. In the case of F_4, there exist two unipotent classes where the a-function is non-constant. They are given as follows; (here we follow the notation in [3]).

$$
\begin{aligned}
C = F_4(a_2): &\qquad a(\phi_{9,2}) = 2,\ a(\phi''_{2,4}) = 1, \\
C = A_2: &\qquad a(\phi''_{8,9}) = 9,\ a(\phi''_{1,2}) = 4.
\end{aligned}
$$

In the case where $C = F_4(a_2)$, we have $A_G(u) \simeq \mathbf{Z}/2\mathbf{Z}$, and $|Z_G(u)^F| = |Z_G(u')^F| = 2q^8$ for two representatives u, u' of G^F-classes in C^F. Therefore $\langle \psi_i, \psi_{i'} \rangle = 0$ for $i, i' \in \mathcal{I}_0$ belonging to C, and so we may choose $\theta_i, \theta_{i'}$ so that they coincide with $\psi_i, \psi_{i'}$. In the case where $C = A_2$, we have $A_G(u) \simeq \mathbf{Z}/2\mathbf{Z}$, and

$$|Z_G(u)^F| = 2q^{17}(q^2-1)(q^3-1), \qquad |Z_G(u')^F| = 2q^{17}(q^2-1)(q^3+1),$$

for two representatives u, u' in G^F-classes in C^F. Let $i = (u, 1), i' = (u, -1) \in \mathcal{I}_0$, where -1 denotes the non-trivial character of $A_G(u)$. We put $\theta_i = q^3\psi_i - \psi_{i'}$, $\theta_{i'} = \psi_{i'}$. Then $\langle \theta_i, \theta_{i'} \rangle = 0$, and one can define $\widetilde{\theta}_i$ $(i \in \mathcal{I}_0)$ in a similar way as (3.16.1). In this way, we can verify the same statement as in Theorem 1.9 for $G = F_4$.

4. Springer representations

4.1. In this section we shall prove some properties of Springer representations of W related to the a-function in the case of classical groups. This makes it possible to check Conjecture 1.7 in the case of low rank classical groups without computing Green functions. In what follows we assume that $G = Sp_N$ or SO_N with rank $G = n$. First we recall some results from [16]. Let P be a maximal parabolic subgroup of G with a Levi subgroup L of the same type as G. We consider the variety $\mathcal{B}_u$ as before for a fixed unipotent element $u = u_\lambda$ with $\lambda \in X_N$ or X'_N. Let $\mathcal{P}_u$ be the variety of parabolic subgroups of G containing u and conjugate to P. We have a natural surjective map $\pi : \mathcal{B}_u \to \mathcal{P}_u$. According to [16], the map π has the following locally trivial filtration: There exists a filtration of $\mathcal{P}_u$,

$$\mathcal{P}_u = Y_0 \supset Y_1 \supset Y_2 \supset \cdots$$

such that for each $x \in Y_i - Y_{i+1}$, the fibre $\pi^{-1}(x)$ is isomorphic to $\mathcal{B}^L_{u'}$, the variety of Borel subgroups of L containing a unipotent element $u' = u_{\lambda'}$ in L. Here $\lambda' \in X_{N-2}$ (resp. $\lambda' \in X'_{N-2}$) is obtained from the Young diagram of λ by (under the notation in [16])

Case I-(a_1), (a_2) replacing two rows of length k by two rows of length $k-1$, for $k \in \varDelta_\lambda$,

Case I-(b_1), (b_2) replacing one row of length k by a row of length $k-2$, for $k \in \varDelta_\lambda$,

Case II replacing two rows of length k by two rows of length $k-1$, for $k \notin \varDelta_\lambda$.

Moreover, $Y_i - Y_{i+1}$ has the following form: It is isomorphic to an affine space $\mathbf{A}^r$ for some r in Case II, Case I-(a_1) or Case I-(b_2), to $\mathbf{A}^r \coprod \mathbf{A}^r$ in Case I-(a_2), and to $\mathbf{A}^r - \mathbf{A}^{r-1}$ in Case I-(b_1). The Case I-(a_2) occurs only when r_k is even for $k \in \varDelta_\lambda$, and the Case I-(b_2) occurs only when r_k is odd for $k \in \varDelta_\lambda$. Set $X_i = \pi^{-1}(Y_i - Y_{i+1})$. Then in each case, we have $X_i \simeq (Y_i - Y_{i+1}) \times \mathcal{B}^L_{u'}$, or its étale covering version holds. Let W' be the Weyl group of L, regarded as a Weyl subgroup of W of type C_{n-1} or D_{n-1}. The cohomology group $H^m_c(X_i)$ has a natural structure of $W' \times A_G(u)$-module, which is compatible with the action of $W' \times A_G(u)$ on $H^m(\mathcal{B}_u)$ obtained by restriction from $W \times A_G(u)$. Then we have

Proposition 4.2. ([16, Prop. 2.4]) *Let $X = X_i$ be as above for some i. Then the following formulas hold for any $m \geq 0$, where d is a positive integer depending only on X.*

(i) *Case II* $H_c^m(X) \simeq H^{m-d}(\mathcal{B}_{u'}^L)$,

(ii) *Case I-*(a_1) $H_c^m(X) \simeq H^{m-d}(\mathcal{B}_{u'}^L)$,

(iii) *Case I-*(a_2) $H_c^m(X) \simeq H^{m-d}(\mathcal{B}_{u'}^L) \oplus H^{m-d}(\mathcal{B}_{u'}^L)$,

(iv) *Case I-*(b_1) $H_c^m(X) \simeq H^{m-d}(\mathcal{B}_{u'}^L)^\tau \oplus H^{m-d+1}(\mathcal{B}_{u'}^L)^\tau$,

(v) *Case I-*(b_2) $H_c^m(X) \simeq H^{m-d}(\mathcal{B}_{u'}^L)$.

The isomorphisms are those of W'-modules. τ is the element $\alpha'_k \cdot \alpha'_{k-2} \in A_L(u')$, where α'_k, α'_{k-2} are generators of $A_L(u')$ corresponding to $k, k-2 \in \Delta_{\lambda'}$ as in 3.6, and the right hand side stands for the τ-invariant subspace. The action of $A_G(u)$ on $H_c^m(X)$ is described by the action of $A_L(u')$ as follows: In the cases (i) *and* (ii), *the actions on both sides are compatible with the natural map $A_G(u) \to A_L(u')$ given by $\alpha_i \mapsto \alpha'_i$. In* (iii), *the same is true for α_i $(i \neq k)$, and α_k permutes two components of the right hand side. In* (iv), *the actions are compatible with the map $A_G(u) \to A_L(u')/\langle\tau\rangle$. In* (v), *the same as* (i), (ii) *is true for α_i $(i \neq k)$, while α_k acts as α'_{k-2} on the right hand side.*

4.3. For a locally closed subvariety X' of $\mathcal{B}_u$, let $H_c^*(X') = \sum(-1)^m H_c^m(X')$. Note that we have $H^*(\mathcal{B}_u) = \bigoplus_{m\geq 0} H^{2m}(\mathcal{B}_u)$. Then for X as in 4.1, $H_c^*(X)$ is a virtual $W' \times A_G(u)$-module. We denote by $H_c^*(X)_\rho$ its ρ-isotypic subspace for each $\rho \in A_G(u)^\wedge$. It follows from Proposition 4.2 that we have

$$H_c^*(X)_\rho \simeq \begin{cases} H^*(\mathcal{B}_{u'}^L)_{\rho'} & \text{case (i) or (ii)}, \\ H^*(\mathcal{B}_{u'}^L)_{\rho'} \oplus H^*(\mathcal{B}_{u'}^L)_{\rho''} & \text{case (iii)}, \\ H^0(\mathcal{B}_{u'}^L)_{\rho'} & \text{case (iv)}, \\ H^*(\mathcal{B}_{u'}^L)_{\rho'} & \text{case (v)}, \end{cases} \tag{4.3.1}$$

where in the case (i), $\rho' \in A_L(u')^\wedge$ is obtained from ρ by the isomorphism $A_G(u) \simeq A_L(u')$ if $r_{k-1} \neq 0$, and by setting $\rho'(\alpha'_{k-1}) = 1$ if $r_{k-1} = 0$, and $\rho'(\alpha'_i) = \rho(\alpha_i)$ for $i \neq k$. While in the case (ii), ρ' is obtained by $\rho'(\alpha'_k) = \rho(\alpha_k)$ if $r_k \geq 3$, and similarly for $i \neq k$. In the case (iii), $\rho', \rho'' \in A_L(u')$ are obtained by $\rho'(\alpha'_k) = 1, \rho'(\alpha'_k) = -1$ if $\rho_k \geq 4$, and similarly for $i \neq k$. In the case (iv), $H^0(\mathcal{B}_{u'}^L) \simeq \mathbf{C}$ is a trivial W'-module, and so $H_c^*(X)_\rho = 0$ except when $\rho(\alpha_k) = \rho(\alpha_{k-2})$ and $\rho(\alpha_i) = 1$ for $i \neq k, k-2$. In the case (v), $H_c^*(X) \neq 0$ only when $\rho(\alpha_k) = \rho(\alpha_{k-2})$, and then we have $\rho'(\alpha'_k) = \rho(\alpha_k)$.

Summing up the above arguments, we have an isomorphism of W'-modules,

$$H^*(\mathcal{B}_u)_\rho \simeq \bigoplus_{(u',\rho')} m_{u',\rho'} H^*(\mathcal{B}_{u'}^L)_{\rho'} \oplus U, \tag{4.3.2}$$

where U is a trivial W'-module and $m_{u',\rho'}$ denotes the multiplicity of the W'-modules. The sum is taken over all the pairs (u', ρ') as in (i), (ii), (iii) and (v) in (4.3.1).

4.4. We consider a pair (u, ρ) such that $H^*(\mathcal{B}_u)_\rho \neq 0$. Then by 1.6, $(u, \rho) \in \mathcal{I}_0$. Let $\chi \in W^\wedge$ be the character corresponding to (u, ρ) via the Springer correspondence, and for (u', ρ') appearing in the right hand side of (4.3.2) let $\chi' \in (W')^\wedge$ be the corresponding character. We shall investigate the relationship between $a(\chi)$ and $a(\chi')$. In order to avoid too much complication, we pose the following assumption on (u, ρ). (In fact, a weaker assumption is enough for the discussion below. See Remark 4.12.)

4.4.1. *$\rho \in A_G(u)^\wedge$ is the character satisfying the condition that $\rho(\alpha_i) = 1$ for any $i \in \varDelta_\lambda$ such that r_i is odd.*

In the following, we denote by $D(u) = D(\lambda)$ the Young diagram associated to $u = u_\lambda$. Let $u = u_\lambda$ with $\lambda = (1^{r_1}, 2^{r_2}, \dots, t^{r_t})$. So, t is the length of the longest rows in $D(\lambda)$. We define an integer $t(u)$, for $G = Sp_{2n}$ or SO_{2n}, by

$$t(u) = \begin{cases} r_t - 2 & \text{if } t \in \varDelta_\lambda \text{ and } r_t \geq 2, \\ r_t - 1 & \text{otherwise,} \end{cases}$$

and for $G = SO_{2n+1}$ by

$$t(u) = \begin{cases} r_t - 2 & \text{if } r_t \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$

We have the following lemma.

Lemma 4.5. *Assume that (u, ρ) satisfies the condition* 4.4.1.

(i) *For any (u', ρ') occurring in the expression in* (4.3.2)*, we have $a(\chi) \geq a(\chi') + t(u)$.*

(ii) *Assume further that u' is obtained from u by replacing the rows, which are not largest among the rows in $D(u)$. Then we have $a(\chi) > a(\chi') + t(u)$.*

Proof. We give the proof only in the case where $G = Sp_{2n}$ or SO_{2n}. The case where $G = SO_{2n+1}$ is dealt with by a trivial modification. (The difference of these two cases arises from the maps in 3.8.1.) Let ξ be the u-symbol associated to (u, ρ). Then by our assumption 4.4.1, any interval of odd length is in the same position as in the distinguished u-symbol associated to u, (see 3.9). We consider each (u', ρ') separately for the cases (i), (ii), (iii) and (v) in Proposition 4.2. Let ξ' be the u-symbol associated to (u', ρ').

First assume that (u', ρ') is as in (i). Then ξ is written as

$$\xi = \begin{pmatrix} \dots & a, & a+2, & \dots & a+2r-2, & \dots \\ \dots & a, & a+2, & \dots & a+2r-2, & \dots \end{pmatrix},$$

where $2r$ is the number of rows of length k in $D(\lambda)$. Then ξ' is obtained from ξ by replacing a by $a-1$. (The choice of a is taken subject to the condition that it produces a u-symbol in $\Psi_{N-2,1}$ or $\widetilde{\Psi}'_{N-2,0}$). Hence the corresponding symbol $\varLambda' = \varLambda(\chi')$ is also obtained from $\varLambda = \varLambda(\chi)$ by replacing the element b corresponding to a by $b-1$. In the case where $a+2r-2$ is the largest number in the entries of ξ, we have $t(u) = 2r-1$. In this case the order of elements $a \leq a \leq a+2 \leq \dots \leq a+2r-2$ remains unchanged by the transformation from ξ to ξ'. Hence we have $a(\chi') = a(\chi) - 2r + 1 = a(\chi) - t(u)$. If $a+2r-2$ is not the largest number, it may happen that $a \leq a$ goes to $b_1 < b_2$ by the transformation $\xi \to \xi'$ and $\varLambda'$ is obtained from $\varLambda$ by replacing b_2 by $b_2 - 1$. Since the order of the other parts remains unchanged by 4.4.1, we have

$$\begin{aligned} a(\chi') &= a(\chi) - \sharp\{x \in \xi \mid x \geq a+2\} \\ &< a(\chi) - t(u). \end{aligned}$$

Next assume that (u', ρ') is as in (ii). Then ξ contains an interval $\varGamma = \{a, a+1, \dots, a+r_k-1\}$ corresponding to $k \in \varDelta_\lambda$, and ξ' is obtained from ξ by replacing $a+1$ by a. Hence if $\varGamma$ lies in the top part of ξ, we see that $a(\chi') = a(\chi) - r_k + 2$. The general case is also dealt with easily, and we have $a(\chi') < a(\chi) - t(u)$.

Next assume that (u', ρ') is as in (iii). As in the case (ii), ξ contains an interval $\varGamma = \{a, a+1, \dots, a+r_k-1\}$. Let ξ', ξ'' be the u-symbols associated to $(u', \rho'), (u', \rho'')$, respectively. Then ξ' is obtained from ξ by replacing $a+1$ by a, and ξ'' is obtained from ξ' by exchanging lower entries and upper entries of $\varGamma' = \{a+2, a+3, \dots, a+r_k-2\}$ contained in ξ'. (Note that $|\varGamma'|$ is even.) Hence a similar argument as in the case (ii) can be applied.

Finally assume that (u', ρ') is as in (v). Let $\varGamma$ be an interval $\{a, a+1, \dots, a+r_k-1\}$ contained in ξ and corresponding to $k \in \varDelta_\lambda$. In the case where $r_{k-1} = 0$, by using the fact that $\rho(\alpha_k) = \rho(\alpha_{k-2})$ if $k-2 \in \varDelta_\lambda$, we see that ξ' is obtained from ξ by replacing a by $a-1$. Hence a similar argument as in the case (i) shows the required inequality. However, a somewhat different phenomenon occurs in the case where $r_{k-1} \neq 0$. In this case r_{k-1} is even since $k-1 \notin \varDelta_\lambda$, and ξ can be expressed as

$$\xi = \begin{pmatrix} \dots & a-2r', & \dots & a-4, & a-2, & a+1, & \dots \\ \dots & a-2r', & \dots & a-4, & a-2, & a, & \dots \end{pmatrix},$$

or in the form obtained by exchanging a and $a+1$ in the above form. Here $2r' = r_{k-1}$. (We assumed here that $r_k \geq 3$. But the argument below works as well for the case $r_k = 1$.) We may assume that ξ is as above. Then ξ' is given as

$$\xi' = \begin{pmatrix} \cdots & a-2r'+1, & \cdots & a-3, & a-1, & a+1, & \cdots \\ \cdots & a-2r'-1, & \cdots & a-5, & a-3, & a-1, & \cdots \end{pmatrix},$$

i.e., the last element a in $\varGamma$ is replaced by $a-1$, and each pair $(a-2i, a-2i)$ is changed to a pair $(a-2i-1, a-2i+1)$. Note that again we used the property that $\rho(\alpha_k) = \rho(\alpha_{k-2})$ if $k-2 \in \varDelta_\lambda$. We express ξ as $\xi = \binom{A}{B}$ with $A = \{\dots \leq a_2 \leq a_1\}$ and $B = \{\dots \leq b_2 \leq b_1\}$. Then by our assumption 4.4.1, $a+1$ and a can be written as $a+1 = a_j$, $a = b_{j+e}$ for some j with $e \in \{1, 0, -1\}$. Moreover in this case $\varLambda$ and $\varLambda'$ can be expressed as

$$\varLambda = \begin{pmatrix} \cdots & b-r', & \cdots & b-2, & b-1, & b+1, & \cdots \\ \cdots & b-r'+e, & \cdots & b-2+e, & b-1+e, & b+e, & \cdots \end{pmatrix},$$

$$\varLambda' = \begin{pmatrix} \cdots & b-r'+1, & \cdots & b-1, & b, & b+1, & \cdots \\ \cdots & b-r'-1+e, & \cdots & b-3+e, & b-2+e, & b-1+e, & \cdots \end{pmatrix}$$

for some b (b is an element corresponding to a under the map π in 3.8.1). Let $Z = \{z_1 \leq z_2 \leq \cdots\}$ be the sequence consisting of all the entries in $\varLambda$ arranged in an increasing order, and Z' the similar sequence for $\varLambda'$. Then Z' is obtained from Z by replacing $b+1$ by b if $e = 1$, by replacing $b-r'$ by $b-r'-1$ if $e = 0$, and by replacing $b-r'-1, b-r', b-1$ by $b-r'-2, b-r'-1, b$, respectively if $e = -1$. It follows that

$$a(\chi') = \begin{cases} a(\chi) - 2j & \text{if } e = 1, \\ a(\chi) - 2r' - 2j + 1 & \text{if } e = 0, \\ a(\chi) - 4r' - 2j + 3 & \text{if } e = -1. \end{cases}$$

In any case we have $a(\chi') < a(\chi) - t(u)$. The lemma is now proved. $\square$

4.6. We consider the setting in Conjecture 1.7. So $u_1 \in C_1, u \in C$ with $C_1 < C$, and let $\chi, \chi_1 \in W^\wedge$ be characters corresponding to $(u,\rho), (u_1,\rho_1)$ respectively. Let $\Lambda(\chi) = \binom{A}{B}$ be a symbol associated to χ, and let c be the largest element in the entries in $\Lambda(\chi)$ such that the replacement $c \to c-1$ produces a symbol Λ' of rank $n-1$. Set

$$s(\chi) = \sharp\{x \in A \cup B \mid x \geq c\} - 1.$$

Then if we denote by χ' the character of W' corresponding to Λ', we have

$$a(\chi) = a(\chi') + s(\chi). \tag{4.6.1}$$

Moreover, χ' appears in the restriction of χ to W'. We have the following proposition.

Proposition 4.7. *Let the notations be as above. Suppose that Conjecture* 1.7 *holds for* W'. *Let* $\rho_1 \in A_G(u_1)^\wedge$ *be the character satisfying the condition* 4.4.1. *Assume that* χ *occurs in* $H^*(\mathcal{B}_{u_1})_{\rho_1}$. *Then we have* $a(\chi_1) \geq a(\chi) + t(u_1) - s(\chi)$.

Proof. Assume that χ occurs in $H^i(\mathcal{B}_{u_1})_{\rho_1}$ for some $i \geq 0$. Let $\chi' \in (W')^\wedge$ be as in 4.6. We may assume that χ' is not the trivial character. In fact, if $\chi' = 1_{W'}$, then $\chi = 1_W$ and the proposition is easily verified for χ. Hence by (4.3.2), there exists a pair (u_1', ρ_1') such that χ' occurs in $H^*(\mathcal{B}_{u_1'})_{\rho_1'}$. Let χ_1' be the character corresponding to (u_1', ρ_1'). Then by our assumption, we have $a(\chi_1') \geq a(\chi')$. On the other hand, by applying Lemma 4.5 to χ_1 and χ_1', we have

$$a(\chi_1) \geq a(\chi_1') + t(u_1).$$

It follows that

$$a(\chi_1) \geq a(\chi') + t(u_1) = a(\chi) + t(u_1) - s(\chi).$$

by (4.6.1). Hence the proposition holds. □

The following special case seems worth mentioning.

Proposition 4.8. *Under the same assumption as in Proposition* 4.7, *let* $u_1 = u_{\lambda_1}$ *with* $\lambda_1 = (1^{r_1}, 2^{r_2}, \dots, t^{r_t})$ *be such that* $t, t-2 \in \Delta_{\lambda_1}, r_t = 1$ *and that* $\rho_1(\alpha_t) \neq \rho_1(\alpha_{t-2})$, *(if* $r_{t-2} = 0$, *we assume that* $\rho_1(\alpha_t) \neq 1$*). Further assume that* $s(\chi) = 0$. *Then we have* $a(\chi_1) > a(\chi)$.

Proof. Let (u_1', ρ_1') be as in the proof of Proposition 4.7. We note that u_1' satisfies the assumption in Lemma 4.5 (ii), since $r_t = 1$ and $\rho_1(\alpha_t) \neq \rho_1(\alpha_{t-2})$, (see 4.3). In this case $t(u_1) = 0$ and we have $a(\chi_1) > a(\chi_1')$ by Lemma 4.5 (ii). The proposition then follows from a similar argument as in the previous proposition. □

4.9. By making use of the following formula of Spaltenstein on Springer representations, we can exploit another type of criterion for the above inequality. Let G be of type B_n or C_n. For each $\chi \in W^\wedge$, we define the parity $p(\chi)$ by the condition that $p(\chi) = 0$ (resp. $p(\chi) = 1$) if $\chi(-1) = \chi(1)$ (resp. $\chi(-1) = -\chi(1)$). For a pair $(u,\rho) \in \mathcal{I}_0$, let $\chi_{u,\rho}$ be the corresponding character of W. Then by Spaltenstein [15], the following formula holds.

4.9.1. *Assume that $\chi \in W^\wedge$ occurs in $H^{2i}(\mathcal{B}_u)_\rho$. Then we have*

$$i \equiv d_u + p(\chi) + p(\chi_{u,\rho}) \pmod 2.$$

We now consider the split $\mathbf{F}_q$-structure on G with Frobenius map F. We choose $u \in G^F_{\text{uni}}$ as in 2.1. For an F-stable locally closed subvariety X' of $\mathcal{B}_u$, let F^* be the map on $H^i_c(X')$ induced from F. We denote by $H^i_c(X')^0$ (resp. $H^i_c(X')^1$) the sum of generalized eigenspaces of F^* corresponding to the eigenvalues q^{2j} (resp. q^{2j+1}) for $j \geq 0$. Note that the eigenvalue of F^* on $H^{2i}(\mathcal{B}_u)$ is q^i. Hence 4.9.1 implies that χ occurs in either $H^*(\mathcal{B}_u)^0_\rho$ or $H^*(\mathcal{B}_u)^1_\rho$, and does not occur simultaneously.

Let $X = X_i \subset \mathcal{B}_u$ be as in 4.1. It is possible to choose a filtration so that all the X are F-stable. Since taking 0- or 1-part is an exact functor, we have, for $\varepsilon \in \{0,1\}$,

$$H^*(\mathcal{B}_u)^\varepsilon = \bigoplus H^*_c(X)^\varepsilon,$$

where the sum is taken over all the pieces X appearing in the filtration in 4.1. The isomorphisms in Proposition 4.2 are all F-stable, where u' can be taken to be a split element except in the case (iv). In each case (i) $\sim$ (v) (in the last case (v) we need to assume that $r_{k-1} = 0$), there exists a maximal choice of X so that $d = 2d_u - 2d_{u'}$. We have the following lemma.

Lemma 4.10. *Let G be of type B_n or C_n. Assume that $\chi \in W^\wedge$ occurs in $H^*(\mathcal{B}_{u_1})^\varepsilon_{\rho_1}$ for $\varepsilon \in \{0,1\}$. Let X be a piece satisfying the condition $d = 2d_{u_1} - 2d_{u'_1}$. Let $(u'_1, \rho'_1) \in (W')^\wedge$ be the pair corresponding to X in* (4.3.1)*. Set $\chi_1 = \chi_{u_1,\rho_1}$ as before, and $\chi'_1 = \chi_{u'_1,\rho'_1}$. Let $\chi' \in (W')^\wedge$ be a character occurring in the restriction of χ to W', satisfying the condition that*

$$p(\chi) + p(\chi_1) \not\equiv p(\chi') + p(\chi'_1) \pmod 2. \tag{4.10.1}$$

Then χ' does not occur in $H^*_c(X)^\varepsilon_{\rho_1}$.

Proof. Suppose that χ' occurs in $H^*_c(X)^\varepsilon_{\rho_1}$. Then χ' occurs in $H^*(\mathcal{B}^L_{u'})^{\varepsilon'}_{\rho'_1}$ with $\varepsilon' \equiv \varepsilon + d_{u_1} - d_{u'_1}$. By 4.9.1 we have

$$\varepsilon' \equiv d_{u'_1} + p(\chi') + p(\chi'_1).$$

On the other hand, since χ occurs in $H^*(\mathcal{B}_{u_1})^\varepsilon_{\rho_1}$, we have

$$\varepsilon \equiv d_{u_1} + p(\chi) + p(\chi_1).$$

This contradicts (4.10.1). □

Proposition 4.11. *Let G be of type B_n or C_n. Under the same assumption as in Proposition 4.7, let $u_1 = u_{\lambda_1}$ with $\lambda_1 = (1^{r_1}, 2^{r_2}, \dots, t^{r_t})$ be such that* (a) *$t \in \Delta_{\lambda_1}$ with $r_t = 2$, or* (b) *$t \in \Delta_{\lambda_1}$ with $r_t = 1, r_{t-1} = 0$. Let $\chi', \chi'_1 \in (W')^\wedge$ be as in 4.6 for χ, χ_1, respectively. Assume further that $\chi, \chi_1, \chi', \chi'_1$ satisfy the condition* (4.10.1)*. Then we have $a(\chi_1) > a(\chi) + t(u_1) - s(\chi)$.*

Proof. Let (u'_1, ρ'_1) be the pair corresponding to χ'_1. Then u'_1 is obtained from u_1 by removing two boxes from the top and second top rows of $D(u_1)$ in the case (a), and removing two boxes from the top row of $D(u_1)$ in the case (b), respectively. In view of (4.3.1), there exists a piece X corresponding to (u'_1, ρ'_1). In our case, such X is unique (it is the last one in the filtration of $\mathcal{B}_{u_1}$), and it satisfies the assumption of Lemma 4.10. Hence we see that χ' does not occur in $H_c^*(X)^{\varepsilon}_{\rho_1}$. It follows that χ' occurs in another $H_c^*(X')^{\varepsilon}_{\rho_1}$. If we denote by (u_1'', ρ_1'') the pair corresponding to X', (u_1'', ρ_1'') satisfies the assumption in Lemma 4.5 (ii). Hence we have $a(\chi_1) > a(\chi_1'') + t(u_1)$, where $\chi_1'' \in (W')^\wedge$ corresponds to (u_1'', ρ_1''). Now the assertion follows by a similar argument as in Proposition 4.8. □

Remark 4.12. Until now, we have assumed the condition 4.4.1 for (u_1, ρ_1). But this condition is not essential. What is necessary in the proof of Lemma 4.5 is actually that (in the notation there) in the u-symbol ξ, any interval of odd length has the same shape as in the case of distinguished u-symbol. So, Proposition 4.8 and Proposition 4.11 can be applied, for example, to the case of (u_1, ρ_1) with $u_1 = (1^3 35)$, and $\rho_1(\alpha_3) = \rho_1(\alpha_5) = -1$.

4.13. By applying the previous results, one can verify Conjecture 1.7 for G of type B_n, C_n or D_n with $n \le 6$. First consider the statement (i) in the conjecture. The following gives a list of the cases where Proposition 1.8 can not be applied, i.e, $C_{u_1} < C_u$, but $a(\chi_1) < a(\chi)$. In the following, we give $\Lambda_1 = \Lambda(\chi_1)$ and $\Lambda = \Lambda(\chi)$. Note that for $B_n, (n \le 4), C_n, (n \le 4)$ and $D_n, (n \le 6)$ there exist no such χ and χ_1.

$$C_5: \quad u_1 = (1^2 2^2 4), \qquad u = (2^3 4).$$
$$\Lambda_1 = \begin{pmatrix} 1\ 3\ 4 \\ 0\ 1 \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} 0\ 2\ 4 \\ 1\ 2 \end{pmatrix}, \begin{pmatrix} 0\ 1\ 2 \\ 2\ 4 \end{pmatrix},$$
$$a(\Lambda_1) = 5, \qquad a(\Lambda) = 6.$$

$$C_6: \quad u_1 = (1^4 2^2 4), \qquad u = (1^2 2^3 4).$$
$$\Lambda_1 = \begin{pmatrix} 1\ 2\ 4\ 5 \\ 0\ 1\ 2 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0\ 1\ 3\ 5 \\ 1\ 2\ 3 \end{pmatrix}, \begin{pmatrix} 0\ 1\ 2\ 3 \\ 1\ 3\ 5 \end{pmatrix},$$
$$a(\Lambda_1) = 10, \qquad a(\Lambda) = 11.$$

$$B_5: \quad u_1 = (12^2 3^2), \qquad u = (1^2 3^3).$$
$$\Lambda_1 = \begin{pmatrix} 0\ 1\ 2 \\ 2\ 4 \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} 0\ 2\ 3 \\ 1\ 3 \end{pmatrix}, \begin{pmatrix} 1\ 2\ 3 \\ 0\ 3 \end{pmatrix},$$
$$a(\Lambda_1) = 6, \qquad a(\Lambda) = 7.$$

$$B_6: \quad u_1 = (12^2 35), \qquad u = (1^2 3^2 5).$$
$$\Lambda_1 = \begin{pmatrix} 0\ 1\ 2 \\ 2\ 5 \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} 0\ 2\ 4 \\ 1\ 3 \end{pmatrix}, \begin{pmatrix} 1\ 2\ 4 \\ 0\ 3 \end{pmatrix}, \begin{pmatrix} 0\ 3\ 4 \\ 1\ 2 \end{pmatrix}, \begin{pmatrix} 1\ 3\ 4 \\ 0\ 2 \end{pmatrix},$$
$$a(\Lambda_1) = 6, \qquad a(\Lambda) = 7.$$

In each of the above cases, it is enough to show that χ does not occur in $H^*(\mathcal{B}_{u_1})_{\rho_1}$. In the case of type C_5 or C_6, we have $s(\chi) = t(u_1) = 0$. In these cases, Proposition 4.8 (or Proposition 4.7) can be applied to show that χ does not occur in $H^*(\mathcal{B}_{u_1})_{\rho_1}$. Note that if $\chi \in W^\wedge$ is given by a pair (α, β) of partitions of n, it is easy to see that $p(\chi) = (-1)^{|\beta|}$. In the case of type B_6, we have $t(u_1) = 1$, and $s(\chi) = 0$ for the first two χ, and $s(\chi) = 1$ for the other two χ. For each of these 4 cases, we can check that (4.10.1) holds. Hence by applying Proposition 4.11, one knows that $a(\chi_1) \geq a(\chi)$ if χ occurs in $H^*(\mathcal{B}_{u_1})_{\rho_1}$. This contradicts $a(\chi_1) < a(\chi)$. Finally consider the case of type B_4. In this case, neither Proposition 4.8 nor Proposition 4.11 can be applied. However, in this case, we consider $\Lambda' = \Lambda(\chi')$ of the following type,

$$\Lambda' = \begin{pmatrix} 0\ 1\ 3 \\ 1\ 3 \end{pmatrix} \quad \text{for} \quad \Lambda = \begin{pmatrix} 0\ 2\ 3 \\ 1\ 3 \end{pmatrix},$$

and

$$\Lambda' = \begin{pmatrix} 0\ 2\ 3 \\ 0\ 3 \end{pmatrix} = \begin{pmatrix} 1\ 2 \\ 2 \end{pmatrix} \quad \text{for} \quad \Lambda = \begin{pmatrix} 1\ 2\ 3 \\ 0\ 3 \end{pmatrix},$$

respectively. According to each step of the filtration of $\mathcal{B}_{u_1}$, we have two kinds of (u_1', ρ_1'). The corresponding $\Lambda_1' = \Lambda(\chi_1')$ are

$$\Lambda_1' = \begin{pmatrix} 0\ 1\ 2 \\ 2\ 3 \end{pmatrix} \quad \text{or} \quad \Lambda_1' = \begin{pmatrix} 0\ 1\ 2 \\ 1\ 4 \end{pmatrix}.$$

The pieces X corresponding to the pairs (u_1', ρ_1') always satisfy the condition in Lemma 4.10. Moreover, for each χ' and χ_1' as above, we see that (4.10.1) holds. Hence by Lemma 4.10, χ' does not occur in any of $H_c^*(X)_{\rho_1}^{\varepsilon}$, where X runs over all the pieces in the filtration of $\mathcal{B}_{u_1}$. This means that χ does not occur in $H^*(\mathcal{B}_{u_1})_{\rho_1}$ as asserted. Now the statement (i) of the conjecture is verified for these cases.

4.14. Next consider the statement (ii) of the conjecture. So assume that u is special and χ is the special character corresponding to $(u, 1)$. Let χ_1 correspond to (u_1, ρ_1) with $C_1 < C$ such that $C_1 \not\subset \widehat{C}$. The following is the list of pairs χ and χ_1 such that $a(\chi_1) \leq a(\chi)$. There are no such pairs for $C_n, (n \leq 3), B_n, (n \leq 3)$ and $D_n, (n \leq 5)$. We note that in the list below, Λ_1 and Λ are always in the same family and so we have $a(\Lambda_1) = a(\Lambda)$.

$$\begin{array}{llll}
C_4: & u_1 = (2^2 4), \quad \Lambda_1 = \begin{pmatrix} 2\ 3 \\ 0 \end{pmatrix}, & u = (4^2), & \Lambda = \begin{pmatrix} 0\ 3 \\ 2 \end{pmatrix}, \\[2ex]
C_5: & u_1 = (2^2 6), \quad \Lambda_1 = \begin{pmatrix} 2\ 4 \\ 0 \end{pmatrix}, & u = (46), & \Lambda = \begin{pmatrix} 0\ 4 \\ 2 \end{pmatrix}, \\[2ex]
 & u_1 = (1^2 2^2 4), \ \Lambda_1 = \begin{pmatrix} 1\ 3\ 4 \\ 0\ 1 \end{pmatrix}, & u = (1^2 4^2), & \Lambda = \begin{pmatrix} 0\ 1\ 4 \\ 1\ 3 \end{pmatrix}, \\[2ex]
C_6: & u_1 = (2^2 8), \quad \Lambda = \begin{pmatrix} 2\ 5 \\ 0 \end{pmatrix}, & u = (48), & \Lambda = \begin{pmatrix} 0\ 5 \\ 2 \end{pmatrix}, \\[2ex]
 & u_1 = (246), \quad \Lambda_1 = \begin{pmatrix} 3\ 4 \\ 0 \end{pmatrix}, & u = (6^2), & \Lambda = \begin{pmatrix} 0\ 4 \\ 3 \end{pmatrix}, \\[2ex]
 & u_1 = (1^2 2^2 6), \ \Lambda_1 = \begin{pmatrix} 1\ 3\ 5 \\ 0\ 1 \end{pmatrix}, & u = (1^2 46), & \Lambda = \begin{pmatrix} 0\ 1\ 5 \\ 1\ 3 \end{pmatrix},
\end{array}$$

$$u_1 = (1^4 2^2 4),\ \Lambda_1 = \begin{pmatrix} 1\ 2\ 4\ 5 \\ 0\ 1\ 2 \end{pmatrix}, \qquad u = (1^4 4^2),\ \ \Lambda_1 = \begin{pmatrix} 0\ 1\ 2\ 5 \\ 1\ 2\ 4 \end{pmatrix},$$

$$B_4: \quad u_1 = (135),\ \ \Lambda_1 = \begin{pmatrix} 0\ 1 \\ 4 \end{pmatrix}, \qquad u = (1^2 7),\ \ \Lambda = \begin{pmatrix} 0\ 4 \\ 1 \end{pmatrix},$$

$$u_1 = (1^3 3^2),\ \ \Lambda_1 = \begin{pmatrix} 0\ 1\ 2 \\ 1\ 4 \end{pmatrix}, \qquad u = (1^4 5),\ \ \Lambda = \begin{pmatrix} 0\ 1\ 4 \\ 1\ 2 \end{pmatrix},$$

$$B_5: \quad u_1 = (137),\ \ \Lambda_1 = \begin{pmatrix} 0\ 1 \\ 5 \end{pmatrix}, \qquad u = (1^2 9),\ \ \Lambda = \begin{pmatrix} 0\ 5 \\ 1 \end{pmatrix},$$

$$u_1 = (15^2),\ \ \Lambda_1 = \begin{pmatrix} 0\ 2 \\ 4 \end{pmatrix}, \qquad u = (137),\ \ \Lambda = \begin{pmatrix} 0\ 4 \\ 2 \end{pmatrix},$$

$$u_1 = (1^3 35),\ \ \Lambda_1 = \begin{pmatrix} 0\ 1\ 2 \\ 1\ 5 \end{pmatrix}, \qquad u = (1^4 7),\ \ \Lambda = \begin{pmatrix} 0\ 1\ 5 \\ 1\ 2 \end{pmatrix},$$

$$u_1 = (12^2 3^2),\ \Lambda_1 = \begin{pmatrix} 0\ 1\ 2 \\ 2\ 4 \end{pmatrix}, \qquad u = (1^2 2^2 5),\ \ \Lambda = \begin{pmatrix} 0\ 2\ 4 \\ 1\ 2 \end{pmatrix},$$

$$u_1 = (1^5 3^2),\ \ \Lambda_1 = \begin{pmatrix} 0\ 1\ 2\ 3 \\ 1\ 2\ 5 \end{pmatrix}, \qquad u = (1^6 5),\ \ \Lambda = \begin{pmatrix} 0\ 1\ 2\ 5 \\ 1\ 2\ 3 \end{pmatrix},$$

$$B_6: \quad u_1 = (139),\ \ \Lambda_1 = \begin{pmatrix} 0\ 1 \\ 6 \end{pmatrix}, \qquad u = (1^2, 11),\ \ \Lambda = \begin{pmatrix} 0\ 6 \\ 1 \end{pmatrix},$$

$$u_1 = (157),\ \ \Lambda_1 = \begin{pmatrix} 0\ 2 \\ 5 \end{pmatrix}, \qquad u = (139),\ \ \Lambda = \begin{pmatrix} 0\ 5 \\ 2 \end{pmatrix},$$

$$u_1 = (1^3 37),\ \ \Lambda_1 = \begin{pmatrix} 0\ 1\ 2 \\ 1\ 6 \end{pmatrix}, \qquad u = (1^4 9),\ \ \Lambda = \begin{pmatrix} 0\ 1\ 6 \\ 1\ 2 \end{pmatrix},$$

$$u_1 = (12^2 35),\ \Lambda_1 = \begin{pmatrix} 0\ 1\ 2 \\ 2\ 5 \end{pmatrix}, \qquad u = (1^2 2^2 7),\ \ \Lambda = \begin{pmatrix} 0\ 2\ 5 \\ 1\ 2 \end{pmatrix},$$

$$u_1 = (13^4),\ \ \Lambda_1 = \begin{pmatrix} 0\ 1\ 2 \\ 3\ 4 \end{pmatrix}, \qquad u = (1^2 3^2 5),\ \ \Lambda = \begin{pmatrix} 0\ 2\ 4 \\ 1\ 3 \end{pmatrix},$$

$$u_1 = (1^7 3^2),\ \ \Lambda_1 = \begin{pmatrix} 0\ 1\ 2\ 3\ 4 \\ 1\ 2\ 3\ 6 \end{pmatrix}, \qquad u = (1^8 5),\ \ \Lambda = \begin{pmatrix} 0\ 1\ 2\ 3\ 6 \\ 1\ 2\ 3\ 4 \end{pmatrix},$$

$$D_6: \quad u_1 = (13^2 5),\ \ \Lambda_1 = \begin{pmatrix} 0\ 1 \\ 3\ 4 \end{pmatrix}, \qquad u = (1^2 5^2),\ \ \Lambda = \begin{pmatrix} 0\ 3 \\ 1\ 4 \end{pmatrix}.$$

In order to verify the statement (ii) in the conjecture, it is enough to show that χ does not occur in $H^*(\mathcal{B}_{u_1})_{\rho_1}$ for each case listed above. In all the cases in $C_4 \sim C_6$ and D_6, Proposition 4.8 can be applied to show that χ does not occur in $H^*(\mathcal{B}_{u_1})_{\rho_1}$. On the other hand, in all the cases in $B_4 \sim B_6$, Proposition 4.11 can be applied to show the required property. Note in these cases, the situation described in Remark 4.12 appears, and we have to apply the proposition in this modified form. In this way, the statement (ii) in the conjecture is verified.

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