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Affine Super-Pythagorean Geometry

Victor Pambuccian

Department of Integrative Studies, ASU West
P. O. Box 37100, Phoenix AZ 85069-7100, USA
e-mail: pamb@math.west.asu.edu

Super-Pythagorean fields can be thought of as arising from a weakening of the requirement of Euclideanity, while preserving the requirement of Pythagoreanity. To be precise: F is Pythagorean if and only if, given $C \in \dot{F}/\dot{F}^2$ distinct from $-\dot{F}^2$, there is an ordering of F in which all members of C are positive. F is said to be super-Pythagorean if, given any finite subgroup C of $\dot{F}/\dot{F}^2$ not containing $-\dot{F}^2$, there is an ordering of F in which all members of C are positive (cf. [2] or [3]).[1] Euclidean fields can be ordered in only one way. Given that Pythagorean and Euclidean fields arise naturally as coordinate fields of planes with free mobility, and of those satisfying the circle axiom, it is natural to ask whether super-Pythagorean fields have a natural geometric meaning.

In [5] we have provided axiom systems for Euclidean planes with super-Pythagorean coordinate fields with or without order. Two axioms of those axiom systems, A7 and A8, are artificial, in the sense that they do not reflect simple geometric intentions. This shortcoming will be remedied in this paper, where we shall display two axioms with clear geometric content to replace the above two. Moreover, in their new formulations these axioms will be affine statements.

We start with a simple example of a super-Pythagorean, non-Euclidean field. Let E_1 and E_2 be the Euclidean closures (inside the quadratic closure of $\mathbb{Q}$; cf. [1]) of the fields $\langle \mathbb{Q}(\sqrt{2}), <_1 \rangle$ and $\langle \mathbb{Q}(\sqrt{2}), <_2 \rangle$, where $<_1$ and $<_2$ stand for the two orders ($\sqrt{2} >_1 0$, $\sqrt{2} <_2 0$) that $\mathbb{Q}(\sqrt{2})$ admits.[2] Then $S := E_1 \cap E_2$ is a super-Pythagorean (since it can be ordered in exactly two ways), clearly non-Euclidean field.

We now show that super-Pythagorean fields may be defined by a simpler set of conditions than

[1] $\dot{F}$ stands for $F \setminus \{0\}$.
[2] One could take E_1 and E_2 to be the real closures of the two fields inside the same algebraic closure of $\mathbb{Q}$.

(1) $-1 \notin F^2$, (2) $(\forall a)\, 1+a^2 \in F^2$, (3) $(\forall x)\, x \in F^2 \vee -x \in F^2 \vee 1+x \in F^2 \vee x(1+x) \in F^2$, by which they were defined in [5, p. 256].

Lemma. *Super-Pythagorean fields are commutative fields that satisfy*

$$(1) \quad -1 \notin F^2, \qquad (3') \quad (\forall x) \; -x \in F^2 \vee 1+x \in F^2 \vee x(1+x) \in F^2.$$

Proof. To show that fields satisfying (1) and (3′) satisfy (2) and (3) as well, let $x = a^2$ in (3′) to get (2) (since, by (1), for $a \neq 0$, $-a^2$ cannot be in F^2), and notice that (3′)⇒(3) is logically valid. To show that fields satisfying (2) and (3) satisfy (3′) as well, notice that if for some x none of the disjuncts in (3′) holds, then x must be a square by (3). But then $1+x$ must be a square by (2), so the second disjunct in (3′) must hold. □

This implies that A7 may be replaced by the axiom $ab \equiv cc \to a = b$, and A8 simplified by removing the line $x = 3a$ from its statement. But we can do better.

Let Δ be an axiom system (for example, the one in [7]) in the language $L_{\|}$ (containing a single quaternary relation $\|$, with $ab \parallel cd$ to be interpreted as 'ab and cd are either parallel or collinear', as primitive notion) for Fanoian Pappian affine planes. Let $\lambda(abc) :\leftrightarrow ab \parallel ac \wedge a \neq b \wedge b \neq c \wedge c \neq a$ stand for 'a, b, c are different and collinear', and let $L(abc) :\leftrightarrow ab \parallel ac$. Following [4, p. 338] (cf. also [6, Proposition 1]) we define, for any three points e, o, a with $\lambda(eoa)$

$$Q(eoa) :\leftrightarrow (\exists bcd)\, \lambda(oab) \wedge \neg L(oac) \wedge \lambda(ocd) \wedge ec \parallel bd \wedge bc \parallel ad$$

to be read as 'o does not lie between e and a'. Notice that if the points o, a, b, e are assigned the coordinates $(0,0), (1,0), (u,0), (v,0)$ respectively, then $Q(eoa)$ states that $v = u^2$. When added to Δ, the axiom

C1 $\quad \lambda(abc) \to Q(cab) \vee Q(abc) \vee Q(bca)$

implies precisely that the coordinate field satisfies (3′), as may be seen by coordinatizing the plane with $a = (0,0)$, $b = (1,0)$, $c = (-x, 0)$. There is nothing more natural than asking for C1 to hold, for it says that among three collinear points at least one is not between the other two. When added to Δ, the axiom

C2 $\quad \lambda(abc) \wedge \neg L(abd) \wedge ad \parallel be \wedge bd \parallel ce \wedge de \parallel ab \to \neg Q(abc)$

implies that the coordinate field satisfies (1), for C2 states that 'if b is the midpoint of ac, then it is not true that b does not lie between a and c (in the sense of Q)'.

We can now state, for $\Delta_1 = \Delta \cup \{\mathrm{C1}, \mathrm{C2}\}$ and $\Delta_2 = \Delta_1 \cup \{\mathrm{A9}, \mathrm{A10}, \mathrm{A11'}, \mathrm{A12}\}$, A9, A10, A12 being the (strict) betweenness axioms from [5, p. 257] and A11′ having the same antecedent as axiom C2 with $Z(abc)$ as consequent (i. e. the natural way to state that the midpoint lies between the endpoints in affine geometry), the following

Theorem. *$\mathfrak{M}$ is a model of Δ_1 (or Δ_2) iff $\mathfrak{M}$ is isomorphic to an affine plane over a super-Pythagorean (ordered) field.*

References

[1] Becker, E.: *Euklidische Körper und euklidische Hüllen von Körpern.* J. Reine Angew. Math. **268/269** (1974), 41–52.

[2] Elman, R.; Lam, T. Y.: *Quadratic forms over formally real fields and pythagorean fields.* Amer. J. Math. **94** (1972), 1155–1194.

[3] Lam, T. Y.: *Orderings, Valuations and Quadratic Forms.* CBMS 52, American Mathematical Society, Providence 1983.

[4] Lenz, H.: *Fastgeordnete Körper.* Arch. Math. **11** (1960), 333–338.

[5] Pambuccian, V.: *Euclidean superpythagorean geometry.* Beitr. Algebra Geom. **39** (2), (1998), 255–258.

[6] Royden,H. L.: *Remarks on primitive notions for elementary Euclidean and non-Euclidean geometry.* In: The Axiomatic Method (eds. L. Henkin, P. Suppes, A. Tarski), North-Holland, Amsterdam 1959, 86–96.

[7] Szmielew, W.: *From Affine to Euclidean Geometry.* PWN Warsaw, D. Reidel Dordrecht 1983.

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