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Quasi-Frobenius Modules

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Abstract. Let R be a commutative ring with 1, and let M be a faithful R-module. We say that M is a quasi-Frobenius (in short QF) module if $\mathrm{Hom}_R(P, M)$ is either zero or a simple R-module for each simple R-module P. In this paper we give some charakterizations of QF modules and we study the relation between QF modules and multiplication modules.

Introduction

Let R and S be two rings with identities and let M be an R-S-bimodule. Following G. Azumaya [3], we say that M is a *quasi-Frobenius* (briefly QF) R-S-module if

(1) M is faithful on both R and S,
(2) $\mathrm{Hom}_R(P, M)$ (respectively $\mathrm{Hom}_S(Q, M)$) is either zero or a simple S-module (respectively zero or a simple R-module) for each simple R-module P (respectively S-module Q).

Equivalently, a faithful R-S-bimodule M is QF if and only if $\mathrm{ann}_M(I)$ (respectively $\mathrm{ann}_M(J)$) is either zero or a simple S-module (respectively zero or a simple R-module) for each maximal ideal I of R (respectively J of S).

In this paper we will say that an R-module M is QF if M is a QF R-R-bimodule. On the other hand, a ring R is called a *quasi-Frobenius ring* if R is an Artinian left or right self-injective ring [5].

Let us observe that every QF ring is a QF R-module. However the converse is false, for example $\mathbf{Z}$ as a $\mathbf{Z}$-module is QF but $\mathbf{Z}$ as a ring is not QF.

One of our main concerns in this paper is to generalize some of the basic properties of QF rings to modules in case $R = S$ and R being a commutative ring with identity. We also study QF modules when $S = \mathrm{End}_R(M)$.

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1. Preliminary results

We begin with the following simple remarks.

1.1. Remarks.

(1) *If R is an integral domain then every torsion-free R-module is QF.*

(2) *If the ring R has no non-zero nilpotent elements, then R is a QF R-module.*

Proof. (1) is obvious.
(2) Let P be a simple R-module and assume that $\mathrm{Hom}_R(P, R) \neq 0$, then R contains an ideal say I which is isomorphic to P. Since R has no non-zero nilpotent element, then one can show easily that $I = eR$ for some idempotent e in R. Hence $\mathrm{Hom}_R(P, R) \cong \mathrm{Hom}_R(eR, R) \cong eR$ is a simple R-module. Hence R is a QF R-module. □

A more interesting example of a QF module is given in the following theorem.

1.2. Theorem. *Let R be a Dedekind domain and let K be the field of quotients of R. Let L be an R-submodule of K containing R such that L/R is a faithful R-module. Then L/R is a QF R-module.*

Proof. Let P be a simple R-module, then $P \cong R/I$ for some maximal ideal I of R. Since R is a Dedekind domain, then I is an invertible ideal of R.

We show that $(L \cap I^{-1})/R$ is a simple R-module unless it is zero. Let J/R be an R-submodule of $(L \cap I^{-1})/R$, then $R \subseteq J \subseteq L \cap I^{-1}$ and hence $IR \subseteq IJ \subseteq I(L \cap I^{-1})$ which implies that $I \subseteq IJ \subseteq R$ since $L \cap I^{-1} \subseteq I^{-1}$. Therefore by the maximality of I, either $IJ = I$ or $IJ = R$. Thus either $J = R$ or $J = I^{-1}$ and hence $J = L \cap I^{-1}$.

The proof will be completed by showing that $\mathrm{Hom}_R(R/I, L/R) \cong (L \cap I^{-1})/R$. So we define a map $\alpha : L \cap I^{-1} \to \mathrm{Hom}_R(R/I, L/R)$ by $\alpha(x) = \alpha_x$ for all $x \in L \cap I^{-1}$, where $\alpha_x : R/I \to L/R$ by $\alpha_x(r + I) = xr + R$ for all $r \in R$. One can check easily that α is a well-defined R-homomorphism. In fact α is an epimorphism. For if $f \in \mathrm{Hom}_R(R/I, L/R)$ then, since R/I is a cyclic R-module generated by $1 + I$, we look at $f(1 + I)$ as an element of L/R, let $f(1 + I) = x + R$ for some $x \in L$. Then

$$I\, f(1 + I) = f(I + I) = f(I) = 0 = I(x + R).$$

Therefore $xI \subseteq R$, that is $x \in I^{-1}$ and hence $x \in L \cap I^{-1}$ and it is clear that $f = \alpha_x$. Thus α is an epimorphism which implies that $(L \cap I^{-1})/\ker\alpha \cong \mathrm{Hom}_R(R/I, L/R)$. Moreover, it is easily checked that $\ker\alpha = R$. Hence $(L \cap I^{-1})/R \cong \mathrm{Hom}_R(R/I, L/R)$. Therefore $\mathrm{Hom}_R(R/I, L/R)$ is a simple R-module which completes the proof. □

The following are special cases of 1.2.

1.3. Corollary. *If R is a Dedekind domain and K is the field of quotients of R, then K/R is a QF R-module.*

1.4. Corollary. *Let p any prime number in $\mathbf{Z}$, and let $Q_p = \left\{ \frac{a}{p^n} \;\middle|\; a, n \in \mathbf{Z} \right\}$. Let $Q_p/\mathbf{Z} = \mathbf{Z}_{p^\infty}$. Then $\mathbf{Z}_{p^\infty}$ is a QF $\mathbf{Z}$-module.*

Proof. It is easily checked that $\mathbf{Z}_{p^\infty}$ is a faithful $\mathbf{Z}$-module. Hence by 1.2, $\mathbf{Z}_{p^\infty}$ is a QF $\mathbf{Z}$-module. □

Note that $\mathbf{Z}_{p^\infty}$ in 1.4 is an Artinian but not Noetherian $\mathbf{Z}$-module.

2. Some characterizations of quasi-Frobenius modules

Consider the following two characterizations for quasi-Frobenius rings.

(1) An Artinian ring R is a QF ring if and only if $\operatorname{ann}_R(\operatorname{ann}_R(I)) = I$ for each simple ideal I of R, [5, Th. 3.4].

(2) An Artinian ring R is a QF ring if and only if every simple R-module is reflexive, [10, Th. 2.1].

We give in this section similar characterizations for QF modules. We start by the following theorem.

2.1. Theorem. *Let M be a faithful R-module. Then M is a QF R-module if and only if $\operatorname{ann}_M(\operatorname{ann}_R(U)) = U$ for each simple R-submodule U of M.*

Proof. Assume that M is a QF R-module and let U be a simple R-submodule of M. Then $U \cong R/I$ for some maximal ideal I of R. Now $\operatorname{ann}_M(\operatorname{ann}_R(U)) = \operatorname{ann}_M(\operatorname{ann}_R(R/I)) = \operatorname{ann}_M(I)$. But M is a QF R-module, then $\operatorname{ann}_M(I)$ is a simple R-module unless it is zero. But $\operatorname{ann}_M(I) = \operatorname{ann}_M(\operatorname{ann}_R(U)) \supseteq U \neq 0$, therefore $\operatorname{ann}_M(I) \neq 0$ is a simple R-module containing U. Hence $\operatorname{ann}_M(I) = U$.

For the converse, let P be a simple R-module such that $\operatorname{Hom}_R(P, M) \neq 0$. Then $P \cong R/I$ for some maximal ideal I of R. Then

$$\operatorname{Hom}_R(P, M) \cong \operatorname{ann}_M(I) \cong \operatorname{ann}_M(\operatorname{ann}_R(P)) = P.$$

Therefore $\operatorname{Hom}_R(P, M)$ is a simple R-module and hence M is a QF R-module. □

Next we recall that an R-module M is said to be *fully stable* if for each R-submodule N of M, $f(N) \subseteq N$ for each R-homomorphism $f : N \to M$, [1].

As a consequence of 2.1, we have

2.2. Corollary. *Every faithful fully stable R-module is QF.*

Proof. Let M be a faithful fully stable R-module. Then $\operatorname{ann}_M(\operatorname{ann}_R(x)) = (x)$ for each $x \in M$ by [1, Cor. 3.5]. In particular $\operatorname{ann}_M(\operatorname{ann}_R(U)) = U$ for each simple R-submodule U of M. Therefore M is QF by 2.1. □

Note that the converse of 2.2 may not be true in general, for instance $\mathbf{Z}$ as a $\mathbf{Z}$-module is QF but not fully stable since $\operatorname{ann}_{\mathbf{Z}}(\operatorname{ann}_{\mathbf{Z}}(2)) = \mathbf{Z} \neq (2)$.

Recall that a ring R is called a *self-injectice* ring if for each ideal I of R and for each R-homomorphism $f : I \to R$, there exists an element $r \in R$ such that $f(x) = rx$ for each $x \in I$, [2]. Hence we have

2.3. Corollary. *Let R be a self-injective ring. Then R as an R-module is QF.*

Proof. Since R is a self-injective ring, then it can be easily checked that R is a fully stable R-module, and hence the result follows by 2.2. □

Let M and X be two R-modules, X is called *M-reflexive* if the natural map $\phi : X \to X^{**}$ is an R-isomorphism where $X^* = \mathrm{Hom}_R(X, M)$, and ϕ is defined by $(\phi(x))(f) = f(x)$ for all $x \in X$ and $f \in X^*$, see [8]. And recall that an R-module M is called *distinguished* if $\mathrm{ann}_M(I) \neq 0$ for each ideal I of R, see [3].

Using these concepts we extend in the following two theorems the second characterization of QF rings which is mentioned in the introduction.

2.4. Theorem. *Let M be a distinguished QF R-module. Then every simple R-module is M-reflexive.*

Proof. Let P be a simple R-module. Since M is distinguished, then $P^* = \mathrm{Hom}_R(P, M) \neq 0$ by [13]. But M is QF, therefore P^* is a simple R-module. And again since M is distinguished and QF, then P^{**} is a simple R-module. Thus both P and P^{**} are simple R-modules and $\phi : P \to P^{**}$ is a non-zero R-homomorphism, therefore ϕ is an R-isomorphism and hence P is M-reflexive. □

Note that the condition in 2.4 that M is distinguished, cannot be dropped, as it is shown in the following example.

The $\mathbf{Z}$-module $\mathbf{Z}$ is QF. However, if P is any simple $\mathbf{Z}$-module, then $P \cong \mathbf{Z}_p$ for some prime number p, and

$$P^* = \mathrm{Hom}_{\mathbf{Z}}(P, \mathbf{Z}) \cong \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}_p, \mathbf{Z}) = 0$$

which implies that $P^{**} = 0$ and hence $P \ncong P^{**}$, that is, P is not $\mathbf{Z}$-reflexive. Note that $\mathbf{Z}$ is not a distinguished $\mathbf{Z}$-module.

Let A, B and M be any R-modules, a bilinear map $\alpha : A \times B \to M$ is called *regular* if $\alpha(a, b) = 0$ for all $a \in A$ implies $b = 0$ and $\alpha(a, b) = 0$ for all $b \in B$ implies $a = 0$, see [3].

The following theorem gives a converse of 2.4, under a certain condition.

2.5. Theorem. *Let M be a faithful R-module. Assume that for each simple R-module P, $P^{**} \cong P$ and P^* contains a maximal submodule (where $P^* = \mathrm{Hom}_R(P, M)$). Then M is a QF R-module.*

Proof. Let P be a simple R-module. We have to show that P^* is simple. Note that $\mathrm{ann}_P(P^*)$ is an R-submodule of P, it is either 0 or P because of the simplicity of P. If $\mathrm{ann}_P(P^*) = P$, then $f(x) = 0$ for all $x \in P$ and all $f \in P^*$ and this implies that $P^* = 0$ which is a contradiction since $P^{**} \cong P$ by hypothesis. Hence $\mathrm{ann}_P(P^*) = 0$. Therefore it can be easily checked that the pairing $(x, f) \mapsto f(x)$ for all $x \in P$ and all $f \in P^*$ is a regular bilinear map of $P \times P^*$ into M. Now, let U be a maximal submodule of

P^*, then P^*/U is a simple R-module. Let $V = \mathrm{ann}_P(U)$, then $V \cong \mathrm{ann}_{P^{**}}(U)$ since $P \cong P^{**}$. But $\mathrm{ann}_{P^{**}}(U) \cong \mathrm{Hom}_R(P^*/U, M)$ by [7, Prop. 23.12, p. 184]. Therefore $V \cong \mathrm{Hom}_R(P^*/U, M) = (P^*/U)^*$. Since V is an R-submodule of P, then either $V = 0$ or $V = P$. If $V = 0$, then $(P^*/U)^* = 0$ and hence $(P^*/U)^{**} = 0$. But by hypothesis $P^*/U \cong (P^*/U)^{**}$ and P^*/U is simple, therefore a contradiction. Hence $V = P$. That is $f(x) = 0$ for all $x \in P$ and all $f \in U$. Therefore $f = 0$ for all $f \in U$ (by the regularity of the pairing $(x, f) \mapsto f(x)$ for all $x \in P$ and all $f \in P^*$). Thus $U = 0$ and hence P^* is a simple R-module. Therefore M is a QF R-module. □

Note that the condition in the previous theorem, that P^* contains a maximal submodule, holds for example in case P^* is a finitely generated [9] or a projective [2] R-module.

3. Quasi-Frobenius modules and multiplication modules

Recall that an R-module M is said to be a *multiplication module* if for each submodule N of M there exists an ideal I of R such that $N = IM$, see [6].

In this section we investigate the relation between multiplication modules and quasi-Frobenius modules. We begin with the following proposition.

3.1. Proposition. *Every faithful multiplication module over a QF ring is a QF module.*

Proof. Let R be a QF ring and let M be a faithful multiplication R-module. Then according to [5], R is an Artinian ring and hence M is a cyclic R-module by [4, Prop. 4]. But M is faithful, therefore $M \cong R$. Since R is a QF ring, then R is a QF R-module. Hence M is a QF R-module. □

We note that if we weaken the condition "R is a QF ring" in 3.1 to R being merely a Noetherian ring and use an extra condition on M we get that M is a QF module as in the following proposition.

3.2. Proposition. *Let R be a Noetherian ring and let M be a faithful multiplication R-module such that for each simple R-module P, $P^{**} \cong P$, where $P^* = \mathrm{Hom}_R(P, M)$. Then M is a QF R-module.*

Proof. M being a faithful multiplication R-module and R a Noetherian ring, then M is a Noetherian R-module, see [6]. Now, if P is any simple R-module, then P is cyclic and hence a finitely generated multiplication module. Moreover EM is a finitely generated submodule of M, where $E = [\mathrm{ann}\, M : \mathrm{ann}\, P]$. Therefore by [12, Th. 3.4], $P^* = \mathrm{Hom}_R(P, M)$ is a finitely generated R-module. Thus P^* contains maximal submodules and hence by 2.5, M is a QF R-module. □

Because of the fact that a faithful multiplication R-module M is Noetherian if and only if R is a Noetherian ring, see [6], the following is an immediate consequence of 3.2.

3.3. Corollary. *If M is a Noetherian faithful multiplication R-module such that for each simple R-module P, $P \cong P^{**}$, then M is a QF R-module.*

For our next result the following remark is needed.

Remark. *IF R is any ring and M is a faithful R-module, then R is isomorphic to a subring of* $\mathrm{End}_R(M)$.

Proof. Is obvious. □

And the following concept is also needed. Given an R-module M, a subring D of $\mathrm{End}_R(M)$ is said to be a *dense* subring of $\mathrm{End}_R(M)$, if given any finite set $\{x_1, x_2, \ldots, x_n\}$ of elements of M and any element f of $\mathrm{End}_R(M)$, there exists an element d of D such that $f(x_i) = d(x_i)$ for all $i = 1, 2, \ldots, n$, see [3].

Now we have the following proposition.

3.4. Proposition. *If M is a QF R-module such that R is dense in* $\mathrm{End}_R(M)$*, then M is a QF* $\mathrm{End}_R(M)$*-module.*

Proof. Put $S = \mathrm{End}_R(M)$. Let U be a simple S-module such that $\mathrm{Hom}_S(U, M) \neq 0$. Then U can be considered as an S-submodule of M. And by the last remark U is an R-submodule of M. Since R is dense in S, then it can be easily seen that U is a simple R-module. Hence $\mathrm{Hom}_R(U, M)$ is a simple R-module because M is a QF R-module. Moreover if $f \in \mathrm{Hom}_S(U, M)$, $x \in M$ and $r \in R$, then

$$f(r, x) = f(\lambda_r(x)) = \lambda_r(f(x)) = r(f(x))$$

(where $\lambda_r : M \to M$ is such that $\lambda_r(x) = rx$ for all $r \in R$ and $x \in M$). Hence $f \in \mathrm{Hom}_R(U, M)$ and thus $\mathrm{Hom}_S(U, M) \subseteq \mathrm{Hom}_R(U, M)$. The simplicity of $\mathrm{Hom}_R(U, M)$ implies that $\mathrm{Hom}_R(U, M) = \mathrm{Hom}_S(U, M)$. Therefore $\mathrm{Hom}_S(U, M)$ is a simple R-module and hence a simple S-module. Thus M is a QF S-module. □

As an immediate consequence of 3.4 we have the following

3.5. Corollary. *Let M be a multiplication QF R-module. Then M is a QF S-module where $S = \mathrm{End}_R(M)$.*

Proof. Since M is a faithful multiplication R-module, then by [11, Prop. 1.5], R is dense in S and hence the result follows by 3.4. □

Now, we need the following lemma.

3.6. Lemma. *Let M be a finitely generated faithful multiplication R-module and let N be an R-submodule of M. Then N is a simple R-module if and only if there exists a simple ideal I of R such that $N = IM$.*

Proof. Since M is a multiplication R-module, then there exists an ideal I of R such that $N = IM$, see [6].

Assume that N is simple. Let J be a non-zero ideal of R such that $J \subseteq I$, then $JM \subseteq IM$. But IM is a simple R-module and $JM \neq 0$, hence $JM = IM$, then by [14, Cor. of Th. 9] $J = I$. Thus I is a simple ideal.

Conversely: Suppose that I is a simple ideal and let K be a non-zero R-submodule of N, then there exists a non-zero ideal J of R such that $K = JM$. Then $JM \subseteq IM$ and according to [14, Cor. of Th. 9], this implies that $J \subseteq I$. Thus $J = I$ and hence $K = N$ which completes the proof. □

We are now ready to give the following result.

3.7. Theorem. *Let M be a finitely generated faithful multiplication R-module. Then the following statements are equivalent:*

(1) $\operatorname{ann}_M(\operatorname{ann}_R(N)) = N$ *for each simple R-submodule N of M,*
(2) $\operatorname{ann}_R(\operatorname{ann}_R(I)) = I$ *for each simple ideal I of R.*

Proof. Let I be a simple ideal of R, then $N = IM$ is a simple R-module by 3.6, and

$$\begin{aligned}\operatorname{ann}_M(\operatorname{ann}_R(N)) &= \operatorname{ann}_M(\operatorname{ann}_R(IM))\\ &= \operatorname{ann}_M(\operatorname{ann}_R(I)) \qquad \text{(since } M \text{ is faithful)}\\ &\supseteq \operatorname{ann}_R(\operatorname{ann}_R(I))M \qquad \text{(since for any ideal } J,\ \operatorname{ann}_M(J) \supseteq \operatorname{ann}_R(J)M\text{)}.\end{aligned}$$

Now, assume (1), then $\operatorname{ann}_M(\operatorname{ann}_R(N)) = N$ and $N = IM \supseteq \operatorname{ann}_R(\operatorname{ann}_R(I))M$. Hence $I \supseteq \operatorname{ann}_R(\operatorname{ann}_R(I))$ by [14, Cor. of Th. 9]. But $\operatorname{ann}_R(\operatorname{ann}_R(I)) \supseteq I$, and thus (2) follows. Conversely: Assume (2), and let N be a simple R-submodule of M, then by 3.6, there exists a simple ideal I of R such that $N = IM$. By (2), $\operatorname{ann}_R(\operatorname{ann}_R(I)) = I$. Then

$$\begin{aligned}\operatorname{ann}_M(\operatorname{ann}_R(N)) &= \operatorname{ann}_M(\operatorname{ann}_R(IM))\\ &= \operatorname{ann}_M(\operatorname{ann}_R(I)) \qquad \text{(since } M \text{ is faithful)}\\ &\supseteq \operatorname{ann}_R(\operatorname{ann}_R(I))M = IM = N.\end{aligned}$$

But $N \subseteq \operatorname{ann}_R(\operatorname{ann}_R(N))$, therefore (1) follows. □

The following are some consequences of 3.7.

3.8. Corollary. *A finitely generated faithful multiplication R-module is QF if and only if* $\operatorname{ann}_R(\operatorname{ann}_R(I)) = I$ *for each simple ideal I of R.*

Proof. Is obvious by 2.1 and 3.7. □

3.9. Corollary. *If R is a self-injective ring and M is a finitely generated faithful multiplication R-module, then M is a QF R-module.*

Proof. Since R is a self-injective ring, then by [1, Prop. 3.4], $\operatorname{ann}_R(\operatorname{ann}_R(I)) = I$ for each cyclic ideal I of R, in particular for each simple ideal I of R. Therefore M is a QF R-module by 3.8. □

We end this section by the following example.

3.10. Example. Let X be an infinite set and let $R = P^X$ be the power set of X. For all $A, B \in R$, define $A + B = A \cup B / A \cap B$ and $AB = A \cap B$. Then R is a Boolean ring.

Let I be an ideal of R generated by all singletons in X. In fact I is the set of all finite subsets of X (I is not finitely generated). I is a pure ideal and hence I is a multiplication ideal, that is a multiplication R-module. Note that the simple R-modules are generated by singletons and each simple R-module contains only two elements. Let P be a simple R-module. Clearly I contains a copy of P and hence $\mathrm{Hom}_R(P, I) \neq 0$. Moreover $\mathrm{Hom}_R(P, I)$ is simple. In fact $\mathrm{Hom}_R(P, I) \cong P$ since I contains a unique copy of P. Hence I is a QF R-module.

4. Quasi-Frobenius bimodules

Let R and S be two commutative rings with identities. We consider in this section QF R-S-modules.

Because of the fact that the endomorphism ring of a multiplication module is a commutative ring, see [11], we start with the following proposition.

4.1. Proposition. *Let M be a multiplication QF R-module and let $S = \mathrm{End}_R(M)$. Then M is a QF R-S-module.*

Proof. Since M is a multiplication QF R-module, then by 3.5, M is a QF S-module. Let I be a maximal ideal of R such that $\mathrm{ann}_M(I) \neq 0$, then $\mathrm{ann}_M(I)$ is a simple R-module. Note that $\mathrm{ann}_M(I)$ is an R-submodule of M, therefore it is an S-submodule of M (since R is dense in S because M is a faithful multiplication R-module, see [11, Th. 1.5]). In fact $\mathrm{ann}_M(I)$ is a simple S-submodule of M, for if U is an S-submodule of $\mathrm{ann}_M(I)$, then U is an R-submodule of $\mathrm{ann}_M(I)$ (since R is dense in S), and hence either $U = 0$ or $U = \mathrm{ann}_M(I)$. Thus $\mathrm{ann}_M(I)$ is a simple S-module.

Now, let L be a maximal ideal of S such that $\mathrm{ann}_M(L) \neq 0$, $\mathrm{ann}_M(L)$ is a simple S-submodule. Let V be an R-submodule of $\mathrm{ann}_M(L)$, then $rx \in V$ for all $r \in R$ and $x \in V$. Let $f \in S$, then there exists $t \in R$ such that $f(x) = tx$ (since R is dense in S). Therefore $f(x) \in V$ for all $f \in S$ and $x \in V$. Hence V is an S-submodule of $\mathrm{ann}_M(L)$, which implies that either $V = 0$ or $V = \mathrm{ann}_M(L)$. Hence $\mathrm{ann}_M(L)$ is a simple R-module and this completes the proof. □

Now, we consider the following concept:

Let M be an R-S-module. A left R-module A and a right S-module B are said to form an *orthogonal pair with respect to M*, if there exists a regular bilinear map of $A \times B$ into M.

Next we give the following two lemmas:

4.2. Lemma. *Let M be a QF R-S-module, let X be an R-module and Y be an S-module which form an orthogonal pair with respect to M. Then:*

(1) *If X is simple, then $X^* \cong Y$.*
(2) *If Y is simple, then $X \cong Y^*$.*

Proof. (1) Let $\alpha : X \times Y \to M$ be a regular bilinear map. For each $y \in Y$, define $\alpha_y : X \to M$ by $\alpha_y(x) = \alpha(x, y)$ for all $x \in X$. It can be easily seen that α_y is a well-defined R-homomorphism, and hence $\alpha_y \in X^*$. Define $\lambda : Y \to X^*$ such that $\lambda(y) = \alpha_y$ for all $y \in Y$. Clearly λ is an S-homomorphism. Moreover, if $\lambda(y) = 0$ for some $y \in Y$, then $\alpha_y = 0$ and therefore $\alpha(x, y) = 0$ for all $x \in X$ which implies that $y = 0$ by the regularity of α. Therefore Y is isomorphic to an S-submodule of X^*. But M is a QF R-S-module and $X^* \neq 0$ (since $\alpha_y \in X^*$), hence X^* is a simple S-module, therefore $X^* \cong Y$. Similarly for (2). $\square$

4.3. Lemma. *Let M be a QF R-S-module. Let U be a simple R-submodule of M and V be a simple S-submodule of M. Then:*

(1) *$S/\operatorname{ann}_S(V)$ is a simple S-module.*
(2) *$R/\operatorname{ann}_R(V)$ is a simple R-module.*

Proof. (1) Define $\lambda : \big(S/\operatorname{ann}_S(U)\big) \times U \to M$ by $\lambda(a + \operatorname{ann}_S(U), x) = xa$ for all $a \in S$ and all $x \in U$. It can easily be checked that λ is a regular bilinear map. Therefore $S/\operatorname{ann}_S(U)$ and U form an orthogonal pair with respect to M. Hence 4.2 implies that $U^* \cong S/\operatorname{ann}_S(U)$ and thus $U^* \neq 0$. But M is a QF R-S-module, then U^* is a simple S-module. Therefore $S/\operatorname{ann}_S(U)$ is a simple S-module. Similarly for (2). $\square$

Finally, we have the following proposition.

4.4. Proposition. *Let M be a QF R-S-module. Then $\operatorname{ann}_M(\operatorname{ann}_S(U)) = U$ for each simple R-submodule U of M and $\operatorname{ann}_M(\operatorname{ann}_R(V)) = V$ for each simple S-submodule V of M.*

Proof. Let U be a simple R-submodule of M. Then $\operatorname{ann}_M(\operatorname{ann}_S(U)) \cong \operatorname{Hom}_S(S/\operatorname{ann}_S(U), M)$ by [1]. But by 4.3, $S/\operatorname{ann}_S(U)$ is a simple S-module and since M is a QF R-S-module, then $\operatorname{Hom}_S(S/\operatorname{ann}_S(U), M)$ is a simple R-module and since it contains U, it is equal to U. The second part is proved similarly. $\square$

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