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Simple Counter Examples for the Unsolvability of the Fermat- and Steiner-Weber-Problem by Compass and Ruler

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Abstract. The purpose of this short note is to give counter examples for the unsolvability of the Fermat- and Steiner-Weber-problem by compass and ruler. The used point sets made it possible to obtain for the Fermat-problem polynomials of the degree 3 and 4. Thus, for these counter examples Galois theory and computer algebra is not necessary. In the second part is given a counter example for the construction of the true length of Steiner trees in the three-dimensional space.

1. Notations

Let n points P_i be given. The point M, for wich the sum $\sum_{i=1}^{n} \|P_iM\|$ is minimal is said to be the *Fermat point* of the given point set. The minimal spanning tree between these points is called the *Steiner tree*.

2. The Fermat point in the Euclidean plane for 3 and 4 points

The constructions of the Fermat point for 3 and 4 points are simple. For a convex quadrilateral we obtain the Fermat point by the intersection of the diagonals.

In the case of a triangle Figure 1 describes the construction. The triangles BAP_3, CBP_1, ACP_2 are equilateral. These constructions and facts of the history of these problems are described by Schreiber in [3].

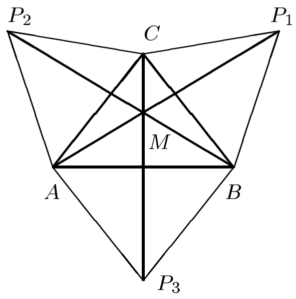

Figure 1

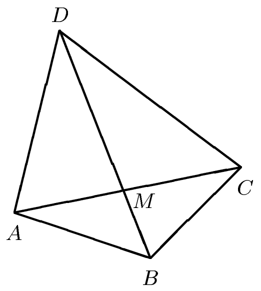

Figure 2

3. The Fermat point in the Euclidean plane for arbitrary finite point sets

Theorem 1. *The construction of the Fermat point, in general, is impossible for* 5 *and more points of the plane.*

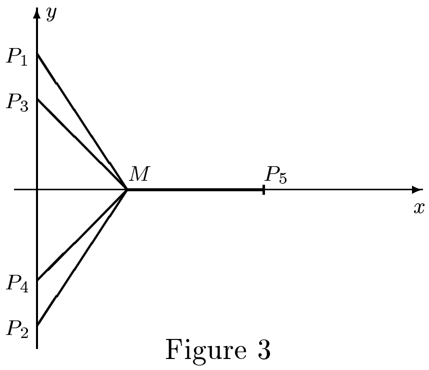

Figure 3

Proof. Let n points $P_1, \ldots, P_n$ (Fig. 3) be given with

odd n	even n
$P_1 = (0, \sqrt{2})$	$P_1 = (0, \sqrt{1.5})$
$P_2 = (0, -\sqrt{2})$	$P_2 = (0, -\sqrt{1.5})$
$P_3 = (0, 1)$	$P_3 = (0, 1)$
$P_4 = (0, -1)$	$P_4 = (0, -1)$
$P_5 = (2, 0)$	$P_5 = (2, 0) = P_6$,
and if $n > 5$ resp. $n > 6$ then	
$P_{2i} = (0, 0)$	$P_{2i-1} = (2, 0)$
$P_{2i+1} = (2, 0)$	$P_{2i} = (0, 0)$
$(3 \le i \le \frac{n-1}{2})$	$(4 \le i \le \frac{n}{2})$.

Let $M = (x, 0)$ be the Fermat point. Then, $f(x) = \sum_{i=1}^{n} \|P_i M\|$ can be calculated by the formulas

$$f(x) = 2\left(\sqrt{x^2+1} + \sqrt{x^2+2}\right) + 2 - x + (n-5) \qquad \text{resp.} \qquad f(x) = 2\left(\sqrt{x^2+1} + \sqrt{x^2+1,5}\,\right) + 2(2-x) + (n-6)$$

A necessary condition for a minimum is $f'(x) = 0$. The derivatives are given by

$$f'(x) = \frac{2x}{\sqrt{x^2+1}} + \frac{2x}{\sqrt{x^2+2}} - 1 \qquad \text{and} \qquad f'(x) = \frac{2x}{\sqrt{x^2+1}} + \frac{2x}{\sqrt{x^2+2}} - 1.$$

To solve the equations we put $\cos\alpha = \frac{-2x}{\sqrt{x^2+1}}$ and $\cos\beta = \frac{-2x}{\sqrt{x^2+2}}$. Thus we get

$$-2\cos\alpha - 2\cos\beta - 1 = 0 \qquad \text{and} \qquad -2\cos\alpha - 2\cos\beta - 2 = 0,$$

$$\cos\alpha = -\frac{1+2\cos\beta}{2} \qquad \text{and} \qquad \cos\alpha = -(1+\cos\beta).$$

Hence $\tan\alpha = -\frac{1}{x}$ yields

$$\tan\beta = -\frac{\sqrt{2}}{x} = \sqrt{2}\frac{1}{-x} = \sqrt{2}\tan\alpha \quad \text{and} \quad \tan\beta = -\frac{\sqrt{1.5}}{x} = \sqrt{1.5}\frac{1}{-x} = \sqrt{1.5}\tan\alpha.$$

Squaring equations and using

$$\tan^2 x = \frac{1-\cos^2 x}{\cos^2 x} \qquad \tan x = \frac{\sin x}{\cos x} \qquad \sin^2 x + \cos^2 x = 1$$

we now obtain

$$\frac{1-\cos^2\beta}{\cos^2\beta} = 2\left(\frac{1-(\frac{1+2cos\beta}{2})^2}{(\frac{1+2cos\beta}{2})^2}\right) \quad \text{and} \quad \frac{1-cos^2\beta}{cos^2\beta} = 1.5\left(\frac{1-(1+cos\beta)^2}{(1+cos\beta)^2}\right).$$

The substitution $\cos\beta = x$ and some elementary operations give the needed equations noted in the variable x.

	5 points in the plane	6 points in the plane
quartic	$4x^4 + 4x^3 - 3x^2 + 4x + 1 = 0$	$0.5x^4 + x^3 + 2x + 1 = 0$
monic quartic	$x^4 + 4x^3 - 12x^2 + 64x + 64 = 0$	$x^4 + 2x^3 + 4x + 2 = 0$
reduced quartic	$x^4 - 18x^2 + 96x - 15 = 0$	$x^4 - 6x^2 + 40x - 3 = 0$
resolvent cubic	$x^3 - 48x - 8064 = 0$	$x^3 - 1536 = 0$

By Eisenstein's theorem it is easy to see, that the quartics and the resolvent cubics are irreducible over $\mathbb{Q}$. Consequently, the quartics do not have constructible solutions. Therefore, the abscissa of the minimal point M and finally the point M is not constructible. □

4. The Fermat point in the Euclidean three-dimensional space

Theorem 2. *In general it is impossible to construct the minimal point for 4 and more points of the Euclidean three-dimensional space by strict usage of compass and ruler.*

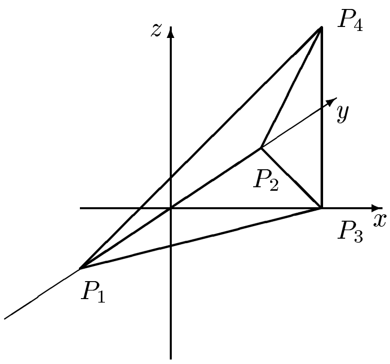

Figure 4

Let 4 points corresponding to Figure 4 with the coordinates

$$P_1 = (0, -1, 0); P_2 = (0, 1, 0); P_3 = (1, 0, 0); P_4 = (1, 0, 1)$$

be given.

The following properties of the Fermat point M we will use by the computation of its coordinates.

1. The Fermat point M belongs to the x-z-plane.
2. The bisecting lines of the angles P_2MP_1 and P_3MP_4 belong to the same straight line.
3. $\angle P_2MP_1 = \angle P_3MP_4$.

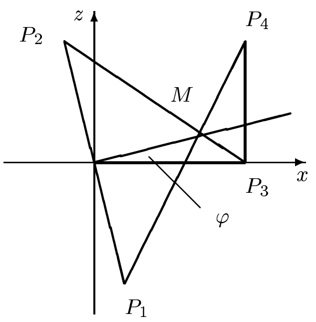

Figure 5

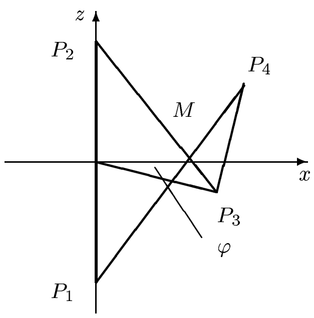

Figure 6

Let $M(x, 0, z)$ be the minimal point. At first we turn $\triangle P_1MP_2$ with the angle $\frac{\pi}{2}$ in the z-x-plane corresponding to Figure 5. Now we are able to construct the minimal point if and only if the angle φ is constructible. To simplify the calculation we turn the figure with the angle φ so that $\overline{P_1P_2}$ belongs to the z-axis (Fig. 6). The equations for the diagonales $\overline{P_2P_3}$, $\overline{P_1P_4}$ are given by

$$z = -\frac{1 - \sin\varphi}{\cos\varphi} \cdot x + 1 \quad \text{and} \quad z = -\frac{\cos\varphi + \sin\varphi + 1}{\cos\varphi - \sin\varphi} \cdot x - 1.$$

Additional $z = 0$ holds. The transformation of the system of equations gives

$$\frac{1-\sin\varphi}{\cos\varphi} = \frac{\cos\varphi+\sin\varphi+1}{\cos\varphi-\sin\varphi}$$

coinciding with property 3. By substituting

$$\sin\varphi = x \text{ and } \cos\varphi = \sqrt{1-\sin^2\varphi} = \sqrt{1-x^2}$$

we have the quartic

$$8x^4 - 4x^3 - 7x^2 + 2x + 1 = (x-1)(8x^3 + 4x^2 - 3x - 1) = 0.$$

The root $x = 1$ does not solve the construction. Therefore the solution needed is a root of the cubic polynomial

$$8x^3 + 4x^2 - 3x - 1 = 0.$$

This polynomial has the approximate roots

$$\begin{array}{ll} x_1 \approx -0,776 & (8x_1 \approx -6,208) \\ x_2 \approx -0,287 & (8x_2 \approx -2,296) \\ x_3 \approx 0,562 & (8x_3 \approx 4,496) \end{array}$$

Since the values $8x_1$;$8x_2$;$8x_3$ are not integer the roots are not rational and consequently not constructible. By the condition $0 = -\frac{1-\sin\phi}{\cos\phi}x + 1$ we get the minimal point with

$$M = (\frac{\cos\varphi\cdot\cos\varphi}{1-\sin\varphi}; 0; \frac{\cos\varphi\cdot\sin\varphi}{1-\sin\varphi}).$$

Since $\sin\varphi$ is not constructible, the minimal point for the given point set is also not constructible with compass and ruler. For more than 4 points we use the same point set as in the plain. If we rotate $\overline{P_3P_4}$ around the x-axis with the angle $\frac{\pi}{2}$ we get points out of the z-x-plane. The proof of Theorem 1 analogously holds. □

5. The Steiner tree in the Euclidean three-dimensional space

Theorem 3. *Generally the construction of the true length of all parts of the Steiner tree is impossible for* 4 *or more points of the Euclidean three-dimensional space.*

Let the tetrahedron $P_1P_2P_3P_4$ with the points

$$P_1 = (0, \frac{-1}{4}, 0); P_2 = (0, \frac{1}{4}, 0); P_3 = (1, \frac{1}{4}, 0); P_4 = (1, \frac{1}{4}, \frac{2\sqrt{3}}{3})$$

be given.

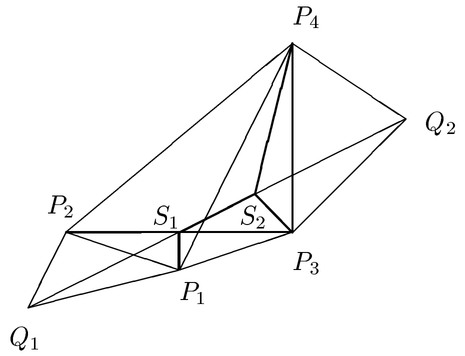

Figure 7

The triangles $P_1P_2Q_1$ and $P_4P_3Q_2$ are equilateral. We will consider the path between $\overline{P_1P_2}$ and $\overline{P_3P_4}$. The Steiner points S_1, S_2 are connected with exactly 3 other points of the Steiner tree. Figure 8 describes the construction in the plane.

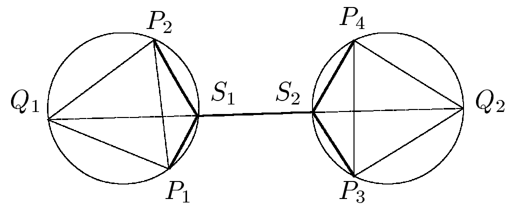

Figure 8

We obtain the tree in the space analogously if we rotate the triangles in such a way that $\overline{Q_1Q_2}$ belongs to the plane of $\triangle P_4P_3Q_2$ and of $\triangle P_1P_2Q_1$. That means, we have to determine the angle α for the rotation of $\triangle P_1P_2Q_1$ around the axis $\overline{P_1P_2}$ and the angle β for the rotation of $\triangle P_4P_3Q_2$ around the axis $\overline{P_3P_4}$. Thus we get the formulas

$$\frac{\frac{1}{\sqrt{3}}}{1+\cos\beta} = \tan\alpha \qquad\qquad \frac{\frac{1}{4}}{1+\frac{\sqrt{3}}{4}\cos\alpha} = \tan\beta$$

for the x-y-plane and the x-z-plane. Simple transformations give

$$\cos\beta = \frac{1}{\sqrt{3}}\cot\alpha - 1 \qquad\qquad \frac{1}{4+\sqrt{3}\cos\alpha} = \tan\beta.$$

With $\tan\beta = \frac{\sqrt{1-\cos^2\beta}}{\cos\beta}$ we get

$$\frac{1}{4+\sqrt{3}\cos\alpha} = \frac{\sqrt{1-(\frac{1}{\sqrt{3}}\cot\alpha - 1)^2}}{\frac{1}{\sqrt{3}}\cot\alpha - 1}$$

and squaring the equation gives

$$(\frac{2}{\sqrt{3}}\cot\alpha - \frac{1}{3}\cot^2\alpha)(3\cos^2\alpha + 8\sqrt{3}\cos\alpha + 17) = 1.$$

Now we substitute $\sqrt{3}\cos\alpha = x$ and $\cot\alpha = \sqrt{\frac{x^2}{3-x^2}}$. Thus we obtain

$$(\frac{2}{\sqrt{3}}\cdot\sqrt{\frac{x^2}{3-x^2}} - \frac{1}{3}\cdot\frac{x^2}{3-x^2})(x^2+8x+17) = 1.$$

Through simple transformations the octic polynomial is given as

$$13x^8 + 208x^7 + 1232x^6 + 2912x^5 + 154x^4 - 9648x^3 - 10152x^2 + 81 = 0.$$

Replacing x by $\frac{X}{13}$ and multiplying by 13^7, we now obtain the monic polynomial

$$X^8 + 208X^7 + 16016X^6 + 492128X^5 + 338338X^4 - 275556528X^3 - \\ 3769366536X^2 + 5082629877.$$

Since the polynomial is seperable we can use the fact that the Galois group of $f(x)$ in $\mathbb{Z}_p$ is a subgroup of the Galois group in $\mathbb{Z}$. If $f(x)$ has a factorisation

$$f(x) = (x^n + ... + a_n)\cdot ...\cdot(x^m + ... + b_m)$$

then the Galois group contains a permutation consistung of m and n cycles. The factorisations in $\mathbb{Z}_p$ give:

(1)	$p = 7$	$1x^8 + 5x^7 + 0x^6 + 0x^5 + 5x^3 + 2x^2 + 0x + 3$ $(1x+2)(1x^2+6x+6)(1x^5+4x^4+6x^3+1x^2+4x+2)$
(2)	$p = 23$	$1x^8 + 1x^7 + 8x^6 + 20x^5 + 8x^4 + 9x^3 + 10x^2 + 0x + 16$ $(x^1+14)(1x^7+10x^6+6x^5+5x^4+7x^3+3x^2+14x+11)$
(3)	$p = 79$	$1x^8 + 50x^7 + 58x^6 + 37x^5 + 60x^4 + 54x^3 + 43x^2 + 0x + 4$ is irreducible over $\mathbb{Z}_{79}$.

Thus the Galois group of $f(x)$ contains an 8-cycle, a 7-cycle and a transposition. These permutations generate the symmetric group $\mathbf{S_8}$. Consequently the polynomial is not solvable by radicals and so the roots are not constructible. Numerical calculations demonstrate the fact that the considerd tree is the shortest of the three possible spanning trees. □

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