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# Equivariant Higher $K$-theory for Compact Lie Group Actions

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## Introduction

The aim of this paper is to construct an equivariant higher $K$-theory for compact Lie group actions in a way analogous to the ones constructed in [2], [3] for finite groups and [6] for profinite groups.

In Section 1, we discuss the category $\mathcal{A}(G)$ of homogeneous spaces on which Mackey functors are defined. In Section 2 we define the higher $K$-groups $K_n^G(G/H, \mathcal{C})$, $n \geq 0$ for any exact category $\mathcal{C}$ and show that $K_n^G(-, \mathcal{C}) : \mathcal{A}(G) \to \mathbb{Z}$-mod is a Mackey functor and that $K_n^G(-, \mathcal{C})$ are $K_0^G(-, \mathcal{C})$-modules. In Section 3, we explore induction techniques in the style of [4] to show that $K_n^G(G/G, \underline{M}(\mathbb{C}))$ (all $n \geq 0$) are hyperelementary computable where $\underline{M}(\mathbb{C})$ is the category of finite dimensional vector spaces over the complex numbers.

In a final section, we briefly discuss possible generalisations of the foregoing to the category of $G$-spaces of $G$-homotopy type of $G$-CW complexes.

## Notations and Conventions

Let $\mathcal{C}$ be an exact category in the sense of Quillen [9]. Then for all $n \geq 0$ we write $K_n(\mathcal{C})$ for the Quillen $K$-groups $\pi_{n+1}(BQ\mathcal{C})$. If $\mathcal{B}$ is any small category, we denote by $[\mathcal{B}, \mathcal{C}]$ the category of covariant functors from $\mathcal{B}$ to $\mathcal{C}$. Unless otherwise stated, $G$ denotes a compact Lie group, and maps between $G$-spaces are $G$-equivariant. If $H$ is a closed subgroup of $G$, we write $N_G(H)$ or just $NH$ for the normaliser of $H$ in $G$. Also we write $R(G)$ for the complex representation ring of $G$.

## 1. Mackey functors on the category $\mathcal{A}(G)$ of homogeneous spaces

**1.1.** Let $G$ be a compact Lie group, $X$ a $G$-space. The component category $\pi_0(G, X)$ is defined as follows: Objects of $\pi_0(G, X)$ are homotopy classes of maps $\alpha : G/H \to X$ where $H$ is a closed subgroup of $G$. A morphism from $[\alpha] : G/H \to X$ to $[\beta] : G/K \to X$ is a $G$-map $\sigma : G/H \to G/K$ such that $\beta\sigma$ is $G$-homotopic to $\alpha$.

Note that since $\mathrm{Hom}(G/H, X) \simeq X^H$ where $\varphi \to \varphi(eH)$, we could consider objects of $\pi_0(G, H)$ as pairs $(H, c)$ where $c \in \pi_0(X^H)$ = the set of path components of $X^H$.

**1.2.** A $G$-ENR (Euclidean Neighbourhood Retract) is a $G$-space that is $G$-homeomorphic to a $G$-retract of some open $G$-subset of some $G$-module $V$. Let $Z$ be a compact $G$-ENR, $f : Z \to X$ a $G$-map. For $\alpha : G/H \to X$ in $\pi_0(G, X)$, we identify $\alpha$ with the path component $X^H_\alpha$ into which $G/H$ is mapped by $\alpha$.

Put $Z(f, \alpha) = Z^H \cap f^{-1}(X^H_\alpha) :=$ subspace of $Z^H$ mapped under $f$ into $X^H_\alpha$. The action of $N_\alpha H/H$ on $Z^H$ induces an action of $N_\alpha H/H$ on $Z(f, \alpha)$ i.e. $Z(f, \alpha)$ is an $\mathrm{Aut}(\alpha)$-space (see [4]). Note that $\mathrm{Aut}(\alpha) = \{\sigma : G/H \to G/H | \alpha\sigma \simeq \alpha\}$ and $N_\alpha(H)_H$, the isotropy group of $\alpha \in \pi_0(X^H)$ is isomorphic to $\mathrm{Aut}(\alpha)$.

**1.3.** Let $\{Z_i\}$ be a collection of $G$-ENR, $f_i : Z_i \to X$, $G$-maps. Say that $f_i : Z_i \to X$ is equivalent to $f_j : Z_j \to X$ if and only if for each $\alpha : G/H \to X$ in $\pi_0(G, X)$, the Euler characteristic $\chi(Z(f_i, \alpha)/\mathrm{Aut}(\alpha)) = \chi(Z(f_j, \alpha)/\mathrm{Aut}(\alpha))$.

Let $\mathcal{U}(G, X)$ be the set of equivalence classes $[f : Z \to X]$ where addition is given by $[f_0 : Z_0 \to X] + [f_1 : Z_1 \to X] = [f_0 + f_1 : Z_0 + Z_1 \to X]$; the identity element is $\phi \to X$; and the additive inverse of $[f : Z \to A]$ is $[f \circ p : Z \times A \to Z \to X]$, where $A$ is a compact $G$-ENR with trivial $G$-action and $\chi(A) = -1$ (see [4]).

Then $\mathcal{U}(G, X)$ is the free Abelian group generated by $[\alpha]$, $\alpha \in \pi_0(G, X)$ i.e. $[f : Z \to X] = \Sigma n(\alpha)[\alpha]$, where $G/H \times E^n \subset Z$ is an open $n$-cell of $Z$, and the restriction of $f$ to $G/H \times E^n$ defines an element $[\alpha]$ of $\mathcal{U}(G, X)$.

The cell is called an $n$-cell of type $\alpha$. Let $n(\alpha) = \Sigma(-1)^i n(\alpha, i)$ where $n(\alpha, i)$ = number of $i$-cells of type $\alpha$ (see [4]).

If $X$ is a point, write $\mathcal{U}(G)$ for $\mathcal{U}(G, X)$.

**1.4.** For a compact Lie group $G$, the category $\mathcal{A}(G)$ is defined as follows: $ob\mathcal{A}(G) :=$ homogeneous spaces $G/H$. The morphisms in $\mathcal{A}(G)(G/H, G/K)$ are the elements of the Abelian group $\mathcal{U}(G, G/H \times G/K)$ and have the form $\alpha : G/L \to G/H \times G/K$ which can be represented by diagram $\{G/H \xleftarrow{\alpha} G/L \xrightarrow{\beta} G/K\}$, so that $\mathcal{U}(G, G/H \times G/K)$ = free Abelian group on the equivalence classes of diagrams $G/H \xleftarrow{\alpha} G/L \xrightarrow{\beta} G/K$ where two such diagrams are equivalent if there exists an isomorphism $\sigma : G/L \to G/L'$ such that the diagram

$$\begin{tikzcd} & G/L \arrow[dr] \arrow[dd, "\sigma"] & \\ G/H \arrow[ur] \arrow[dr] & & G/K \\ & G/L' \arrow[ur] & \end{tikzcd}$$

commutes up to homotopy.

Composition of morphisms is given by a bilinear map

$$\mathcal{U}(G, G/H_1 \times G/H_2) \times \mathcal{U}(G, G/H_2 \times G/H_3) \to \mathcal{U}(G, G/H_1 \times G/H_3)$$

where the composition of $(\alpha, \beta_1) : A \to G/H_1 \times G/H_2$ and $(\beta_2, \gamma) : B \to G/H_2 \times G/H_3$ yields a $G$-map $(\alpha\bar{\alpha}, \gamma\bar{\gamma}) : C \to G/H_1 \times G/H_3$, where $\bar{\gamma}$, $\bar{\alpha}$ are maps $\bar{\gamma} : C \to B$ and $\bar{\alpha} : C \to A$, respectively.

**1.5. Remarks.** (i) Each morphism $G/H \xleftarrow{\alpha} G/L \xrightarrow{\beta} G/K$ is the composition of special types of morphisms

$$G/H \xleftarrow{\alpha} G/L \xrightarrow{\text{id}} G/L \quad \text{and} \quad G/L \xleftarrow{\text{id}} G/L \xrightarrow{\beta} G/K\ .$$

(ii) Let $\pi_0$ (or $G$) be the homotopy category of the orbit category or $(G)$, that is, the objects of $\pi_0$ (or $G$) are the homogeneous $G$-spaces $G/H$ and morphisms are homotopy classes $[G/L \to G/K]$ of $G$-maps $G/L \to G/K$. We have a covariant functor $\pi_0$ (or $G$) $\to \mathcal{A}(G)$ given by $[G/L \xrightarrow{\beta} G/K] \to (G/L \xleftarrow{\text{id}} G/L \xrightarrow{\beta} G/K)$ and a contravariant functor $\pi_0$ (or $G$) $\to \mathcal{A}(G)$ given by

$$[G/H \to G/L] \to (G/H \leftarrow G/L \xrightarrow{\text{id}} G/L)$$

(iii) Addition is defined in $\mathcal{A}(G)(G/H, G/K) = \mathcal{U}(G, G/H \times G/K)$ by

$$(G/H \leftarrow G/L \to G/K) + (G/H \leftarrow G/L' \to G/K)$$

$$= (G/H \leftarrow (G/L)\dot{\cup}(G/L') \to G/K)$$

where $(G/L)\dot{\cup}(G/L')$ is the topological sum of $G/L$ and $G/L'$.

**1.6.** Let $R$ be a commutative ring with identity. A Mackey functor $M$ from $\mathcal{A}(G)$ to $\underline{R\text{-mod}}$ is a contravariant additive functor. Note that $M$ is additive if

$$M : \mathcal{A}(G)(G/H, G/K) \to \underline{R\text{-mod}}(M(G/K), M(G/H))$$

is an Abelian group homomorphism.

**1.7. Remarks.** $M$ comprises of two types of induced morphisms

(i) If $\alpha : G/H \to G/K$ is a $G$-map, regarded as an ordinary morphism $\alpha_! : G/H \xleftarrow{\text{id}} G/H \xrightarrow{\alpha} G/K$ of $\mathcal{A}(G)$, we have an induced morphism

$$M(\alpha_!) = M^*(\alpha) =: \alpha^* : M(G/K) \to M(G/H)$$

(ii) If $\alpha$ in (i) is induced from $H \subset K$, i.e. $\alpha(gH) = gK$, call $\alpha^*$ the restriction morphism.

(iii) If we consider $\alpha$ as a transfer morphism $\alpha^! : G/H \leftarrow G/K \to G/K$ in $\mathcal{A}(G)$, then we have

$$M(\alpha^!) =: M_*(\alpha) =: \alpha_* : M(G/H) \to M(G/K)$$

and call $\alpha_*$ the induced homomorphism associated to $\alpha$.

**1.8.** Let $M, N, L$ be Mackey functors on $\mathcal{A}(G)$. A pairing $M \times N \to L$ is a family of bilinear maps $M(S) \times N(S) \to L(S)$: $(x, y) \to x \cdot y$ $(S \in \mathcal{A}(G))$ such that for each $G$-map $f : G/H = S \to T = G/K$ we have

$$\begin{aligned} L^*f(x,y) &= (M^*fx)\cdot(N^*fy) && (x \in M(T),\ y \in N(T));\\ x\cdot(N_*fy) &= L_*f((M^*fx)\cdot y) && (x \in M(T),\ y \in N(S));\\ (M_*fx)\cdot y &= L_*f(x\cdot(N^*fy)) && (x \in N(S),\ y \in N(T)). \end{aligned}$$

A Green functor $V : \mathcal{A}(G) \to \underline{R\text{-mod}}$ is a Mackey functor together with a pairing $V \times V \to V$ such that for each object $S$, the map $V(S) \times V(S) \to V(S)$ turns $V(S)$ into an associative $R$-algebra such that $f^*$ preserves units.

If $V$ is a Green functor, a left $V$-module is a Mackey functor $M$ together with a pairing $V \times M \to M$ such that $M(S)$ is a left $V(S)$-module for every $S \in \mathcal{A}(G)$.

**1.9. Remarks.** The Mackey functor as defined in 1.6 is equivalent to the earlier definitions in [2], [3], [7] defined for finite and profinite groups $G$ as functors from the category $\hat{G}$ of $G$-sets to $\underline{R\text{-mod}}$. Observe that if $(M_*, M^*) = M$ is a Mackey functor (bifunctor) $\hat{G} \to \underline{R\text{-mod}}$ (see [7]), we can get $\tilde{M} : \mathcal{A}(G) \to \underline{R\text{-mod}}$ by putting $\tilde{M}(G/H) = M_*(G/H) = M^*(G/H)$ on objects while a morphism $G/H \xleftarrow{\alpha} G/L \xrightarrow{\beta} G/K$ in $\mathcal{A}(G)$ is mapped onto $M(G/H) \underset{M_*(\alpha)}{\longleftarrow} M(G/L) \underset{M^*(\beta)}{\longleftarrow} M(G/K)$ in $\underline{R\text{-mod}}$. Then $M$ is compatible with composition of morphisms.

Conversely, let $\tilde{M} : \mathcal{A}(G) \to \underline{R\text{-mod}}$ be given and for $(\alpha, \beta) \in \mathcal{A}(G)(G/H, G/K)$, let $\tilde{M}^*(\alpha)$ and $\tilde{M}_*(\beta)$ be as defined in 1.7. Then, we can extend $\tilde{M}$ additively to finite $G$-sets to obtain Mackey functors as defined in [2], [3], [7].

**1.10. Universal example of a Green functor.** Define $\bar{V}(G/H) := \mathcal{U}(G, G/H)$, $\mathcal{U}(H, G/G) := \bar{V}(H)$. Now, consider $\mathcal{U}(G, G/H) = \mathcal{U}(G, G/G \times G/H)$, as a morphism set in $\mathcal{A}(G)$. Then, the composition of morphisms

$$\mathcal{U}(G, G/G \times G/K) \times \mathcal{U}(G, G/H \times G/K) \to \mathcal{U}(G, G/G \times G/H)$$

defines an action of $\bar{V}$ on morphisms.

**1.11. Theorem.** [4] *$\bar{V} : \mathcal{A}(G) \to \underline{\mathbb{Z}\text{-mod}}$ is a Green functor, and any Mackey functor $M : \mathcal{A}(G) \to \underline{\mathbb{Z}\text{-mod}}$ is a $\bar{V}$-module.*

## 2. An equivariant higher $K$-theory for $G$-actions

**2.1.** Let $G$ be a compact Lie group, $X$ a $G$-space. We can regard $X$ as a category $\underline{X}$ as follows. The objects of $\underline{X}$ are elements of $X$ and for $x$, $x' \in X$, $\underline{X}(x, x') = \{g \in G | gx = x'\}$.

**2.2.** Let $X$ be a $G$-space, $\mathcal{C}$ an exact category in the sense of Quillen [9]. i.e. $\mathcal{C}$ is an additive category embeddable as a full subcategory of an Abelian category $\mathcal{O}$ such that $\mathcal{C}$ is equipped with a class $\mathcal{E}$ of exact sequences

$$0 \to M' \to M \to M'' \to 0 \tag{I}$$

such that

(i) $\mathcal{E}$ is the class of sequences (I) in the $\mathcal{C}$ that are exact in $\mathcal{O}$.

(ii) $\mathcal{C}$ is closed under extensions in $\mathcal{O}$ that is, if (I) is an exact sequence in $\mathcal{O}$ and $M', M'' \in \mathcal{C}$ then $M \in \mathcal{C}$.

Let $[\underline{X}, \mathcal{C}]$ be the category of functors $\underline{X} \to \mathcal{C}$. Then $[\underline{X}, \mathcal{C}]$ is an exact category where a sequence $0 \to \zeta' \to \zeta \to \zeta'' \to 0$ is exact in $[\underline{X}, \mathcal{C}]$ if and only if

$$0 \to \zeta'(x) \to \zeta(x) \to \zeta''(x) \to 0$$

is exact in $\mathcal{C}$. In particular for $X = G/H$ in $\mathcal{A}(G)$, $[\underline{G/H}, \mathcal{C}]$ is an exact category.

**2.3. Example.** The most important example of $[\underline{G/H}, \mathcal{C}]$ is when $\mathcal{C}$ is the category $\underline{M}(\mathbb{C})$ of finite dimensional vector spaces over the field $\mathbb{C}$ of complex numbers. Here, the category $[\underline{G/H}, \underline{M}(\mathbb{C})]$ can be identified with the category of $G$-vector bundles on the compact $G$-space $G/H$ where for any $\zeta \in [G/H, \underline{M}(\mathbb{C})]$. $x \in G/H$, $\zeta(x) \in \underline{M}(\mathbb{C})$ is the fibre $\hat{\zeta}_x$ of the vector bundle $\hat{\zeta}$ associated with $\zeta$. Indeed, $\hat{\zeta}$ is completely determined by $\zeta_{\bar{e}}$ where $\bar{e} = eH$ (see [10]).

**2.4. Definition.** *For $X = G/H$ and all $n \geq 0$, define $K_n^G(X, \mathcal{C})$ as the $n^{th}$ algebraic $K$-group of the exact category $[\underline{X}, \mathcal{C}]$ with respect to fibre-wise exact sequence introduced in* 2.2.

**2.5. Theorem.** (i) *For all $n \geq 0$, $K_n^G(-, \mathcal{C}) : \mathcal{A}(G) \to \mathbb{Z}$-mod is a Mackey functor,*
(ii) $K_0^{\mathcal{C}}(-, \mathcal{C}) : \mathcal{A}(G) \to \underline{\mathbb{Z}\text{-mod}}$ *is a Green functor and $K_n^G(-, \mathcal{C})$ is a $K_0^G(-, \mathcal{C})$-module for all $n \geq 0$.*

Before proving 2.5, we first briefly discuss pairings and module structures on higher $K$-theory of exact categories.

**2.6.** Let $\mathcal{E}, \mathcal{E}_1, \mathcal{E}_2$ be three exact categories and $\mathcal{E}_1 \times \mathcal{E}_2$ the product category. An exact pairing $\mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E} : (M_1, M_2) \to M_1 \circ M_2$ is a covariant functor from $\mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}$ such that $\mathcal{E}_1 \times \mathcal{E}_2((M_1, M_2), (M_1', M_2')) = \mathcal{E}_1(M_1, M_1') \times \mathcal{E}_2(M_2, M_2') \to \mathcal{E}(M_1 \circ M_2,\ M_1' \circ M_2')$ is bi-additive and bi-exact, that is, for a fixed $M_2$, the functor $\mathcal{E}_1 \to \mathcal{E}$ given by $M_1 \to M_1 \circ M_2$ is additive and exact and for fixed $M_1$, the functor $\mathcal{E}_2 \to \mathcal{E} : M_2 \to M_1 \circ M_2$ is additive and exact. It follows from [12] that such a pairing gives rise to a $K$-theoretic product $K_i(\mathcal{E}_1) \times K_j(\mathcal{E}_2) \to K_{i+j}(\mathcal{E})$ and in particular to natural pairing $K_0(\mathcal{E}_1) \circ K_n(\mathcal{E}_2) \to K_n(\mathcal{E})$ which could be defined as follows.

Any object $M_1 \in \mathcal{E}$ induces an exact functor $M_1 : \mathcal{E}_2 \to \mathcal{E} : M_2 \to M_1 \circ M_2$ and hence a map $K_n(M_1) : K_n(\mathcal{E}_2) \to K_n(\mathcal{E})$. If $M_1' \to M_1 \to M_1''$ is an exact sequence in $\mathcal{E}$, then we have an exact sequence of exact functors $M_1^{'*} \to M_1^* \to M_1^{''*}$ from $\mathcal{E}_2$ to $\mathcal{E}$ such that for each object $M_2 \in \mathcal{E}_2$ the sequence $M_1'(M_2) \to M_1^*(M_2) \to M_1^{''*}(M_2)$ is exact in $\mathcal{E}$ and hence by a result of Quillen [9] induces a relation $K_n(M_1^{'*}) + K_n(M_1^{''*}) = K_n(M_1^*)$. So the map $M_1 \to K_n(M_1) \in \mathrm{Hom}(K_n(\mathcal{E}_2),\ K_n(\mathcal{E}))$ induces a homomorphism $K_0(\mathcal{E}_1) \to \mathrm{Hom}(K_n(\mathcal{E}), K_n(\mathcal{E}))$ and hence a pairing $K_0(\mathcal{E}_1) \times K_n(\mathcal{E}) \to K_n(\mathcal{E})$.

If $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$ and the pairing $\mathcal{E} \times \mathcal{E}$ is naturally associative (and commutative), then the associated pairing $K_0(\mathcal{E}) \times K_0(\mathcal{E}) \to K_0(\mathcal{E})$ turns $K_0(\mathcal{E})$ into an associative (and commutative) ring which may not contain the identity.

Now, suppose that there is a pairing $\mathcal{E} \circ \mathcal{E}_1 \to \mathcal{E}_1$ which is naturally associative with respect to the pairing $\mathcal{E} \circ \mathcal{E} \to \mathcal{E}$, then the pairing $K_0(\mathcal{E}) \times K_n(\mathcal{E}_1) \to K_n(\mathcal{E}_1)$ turns $K_n(\mathcal{E}_1)$ into a $K_0(\mathcal{E})$-module which may or may not be unitary. However, if $\mathcal{E}$ contains a unit i.e. an object $E$ such that $E \circ M = M \circ \mathcal{E}$ are naturally isomorphic to $M$ for each $\mathcal{E}$-object $M$, then the pairing $K_0(\mathcal{E}) \times K_n(\mathcal{E}_1) \to K_n(\mathcal{E}_1)$ turns $K_n(\mathcal{E}_1)$ into a unitary $K_0(\mathcal{E})$-module.

*Proof of* 2.5. (i) It is clear from the definition of $K_n^G(G/H, \mathcal{C})$ that for any $G/H \in \mathcal{A}(G)$, $K_n^G(G/H, \mathcal{C}) \in \underline{\mathbb{Z}\text{-mod}}$. Now suppose that $(G/H \xleftarrow{\alpha} G/L \xrightarrow{\beta} G/K) \in \mathcal{A}(G)(G/H, G/K)$, then $G/H \xleftarrow{\alpha} G/L \xrightarrow{\beta} G/K$ goes to

$$K_n^G(G/H, \mathcal{C}) \longleftarrow K_n^G(G/L, \mathcal{C}) \longleftarrow K_n^G(G/K, \mathcal{C})$$

in $\underline{R\text{-mod}}(K_n^G(G/K, \mathcal{C})$. If we write $K_n^G$ for $K_n^G(-, \mathcal{C})$, then

$$K_n^G(G/H \leftarrow (G/L)\dot{\cup}(G/L') \to G/K)$$

$$= K_n^G(G/H \leftarrow G/L \to G/K) + K_n^G(G/H \leftarrow G/L' \to G/K)\ .$$

Hence $K_n^G(-, \mathcal{C})$ is a Mackey functor.

(ii) From the discussion in 2.6, it is clear that if we put $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E} = [G/H, \mathcal{C}]$, then

$$K_0^G(G/H, \mathcal{C}) \times K_0^G(G/H, \mathcal{C}) \to K_0^G(G/H, \mathcal{C})$$

turns $K_0^G(G/H, \mathcal{C})$ into a commutative ring with identity. Also,

$$K_0^G(G/H, \mathcal{C}) \times K_n^G(G/H, \mathcal{C}) \to K_n^G(G/H, \mathcal{C})$$

turns

$$K_n^G(G/H, \mathcal{C}) \quad \text{into} \quad K_0^G(G/H, \mathcal{C})\text{-modules}$$

Hence the result.

**2.7. Examples**. (i) In general $[G/H, \mathcal{C}]$ = category of $H$-representations in $\mathcal{C}$. Hence $[G/G, \mathcal{C}]$ = category of $G$ representations in $\mathcal{C}$. If $\mathcal{C} = \underline{M}(\mathbb{C})$, the category of finite dimensional vector spaces over the complex numbers $\mathbb{C}$, $K_0^G(G/G, \underline{M}(\mathbb{C}))$ is the complex representation ring denoted by $R_{\mathbb{C}}(G)$ or simply $R(G)$ in the literature.

(ii) If $\mathcal{C} = \underline{M}(R) :=$ category of finitely generated $R$-modules, where $R$ is a Noetherian ring compatible with the topological structure of $G$, then $K_n^G(G/H, \underline{M}(R) \simeq G_n(RH)$.

(iii) If $\mathcal{C} = \underline{P}(R)$ = category of finitely generated projective $R$-modules, we have

$$K_n^G(G/H, \underline{P}(R)) = G_n(H, R) \text{ and}$$

when $R$ is regular, $G_n(R, H) \simeq G_n(RH)$.

## 3. Induction theory for equivariant higher $K$-functors

In this section, we discuss the induction properties of the equivariant $K$-functors constructed in Section 2 leading to the proof of Theorem 3.10 below.

**3.1. Definition**. *Let $G$ be a compact Lie group. A finite family $\Sigma = (G/H)_{j\in J}$ is called an inductive system. Such a system yields two homomorphisms $p(\Sigma)$ (induction map) and $i(\Sigma)$ (restriction maps) defined by*

$$\begin{array}{rcl} p(\Sigma):\underset{j\in J}{\oplus} M(G/H_j) & \to & M(G/G) \\ (x_j|_{j\in J}) & \mapsto & \underset{j\in J}{\Sigma} p(H_j)_*(x_j) \\ i(\Sigma): M(G/G) & \to & \underset{j\in J}{\oplus} M(G/H_j) \\ x & \mapsto & (p(H_j)^* x | j\in J) \end{array}$$

*Note that $p(H)$ denotes the unique morphism $G/H \to G/G$. $\Sigma$ is said to be projective if $p(\Sigma)$ is surjective and $\Sigma$ is said to be injective if $i(\Sigma)$ is injective. Note that the identity* [id] *of $\mathcal{U}(G, G/K \times G/H)$ has the form* (see 1.3)

$$[\mathrm{id}] = \Sigma_\alpha n_\alpha[\alpha : G/L_\alpha \to G/K \times G/H]. \tag{I}$$

**3.2.** Let $S(K,H)$ be the set of $\alpha$ over which the summation (I) is taken and let $\alpha = (\alpha(1), \alpha(2))$ be the component of $\alpha$, where $\alpha(1): G/L_\alpha \to G/K$; $\alpha(2): G/L_\alpha \to G/H$.

Define induction map

$$p(\Sigma, G/H): \oplus_{j\in J}(\oplus_{\alpha\in S(H_j,H)} M(G/L_\alpha)) \to M(G/H)$$

by

$$(x(j,\alpha)) \mapsto (\Sigma_{j\in J}(\Sigma_{\alpha\in S(H_j,H)} n_\alpha \alpha(2)_* x(j,\alpha))$$

and restriction maps

$$i(\Sigma, G/H): M(G/H) \to \oplus_{j\in J}(\oplus_{\alpha\in S(H_j,H)} M(G/L_\alpha)$$

by

$$x \to (\alpha(2)^* x | (\alpha \in (S(H_j, H) \quad j\in J)\ .$$

**3.3. Theorem.** *Let $M = K_n^G(-, \mathcal{C})$, $V = K_0^G(-,\mathcal{C})$ be respectively Mackey and Green functors $\mathcal{A}(G) \to \mathbb{Z}$-mod defined in 2.5. If $\Sigma$ is projective for $V$, then for each homogeneous space $G/H$, the induction map $p(\Sigma, G/H)$ is split surjective and the restriction map $i(\Sigma, G/H)$ is split injective.*

*Proof.* Since $p(\Sigma)$ is surjective for $V$, there exist elements $x_j \in V(G/H_j)$ such that $\Sigma(H_j)_* x_j = 1$. Define a homomorphism

$$q(\Sigma, G/H): M(G/H) \to \oplus_j(\oplus_\alpha M(G/L_\alpha))$$

by

$$q(\Sigma, G/H)x = \alpha(1)^* x_j \cdot \alpha(2)^* x \text{ such that } \alpha \in S(H_j, h),$$

$j \in J$. Then $p(\Sigma, G/H)q(\Sigma, G/H)$

$$\begin{aligned} &= \Sigma_j(\Sigma_\alpha n_\alpha \alpha(2)_*(\alpha(1)^* x_j \cdot \alpha(2)^* x)) \\ &= \Sigma_j(\Sigma_\alpha n_\alpha \alpha(2)_* \alpha(1)^* x_j)x \\ &= \Sigma p(H)^* p(H_j)_* x_j) \cdot x \\ &= \Sigma p(H)^* (\Sigma p(H_j)_* x_j)x \\ &= p(H)^*(1)x = 1 \cdot x = x \ . \end{aligned}$$

So, $p(\Sigma, G/H)q(\Sigma, G/H)$ is the identity. Hence $q(\Sigma, G/H)$ is a splitting for $p(\Sigma, G/H)$. We can also define a splitting $j(\Sigma, G/H)$ for $i(\Sigma, G/H)$ by $j(\Sigma, G/H) : \oplus_j(\oplus_\alpha(MG/L_\alpha)) \to M(G|H)$ where

$$x(j,\alpha) \mapsto \Sigma_j(\Sigma_\alpha n_\alpha \alpha(2)_* \alpha(1)^* x_j \circ x(j,\alpha)) \ .$$

**3.4. Remarks**. As will be seen below (3.6), $V = K_0^G(-,\mathcal{C})$ has "defect sets" $D(V)$ and so $\Sigma = \{G/H | H \in D(V)\}$ is projective for $V$. It would then mean that an induction theorem for $K_0^G(-,\mathcal{C})$ implies a similar theorem for $K_n^G(-,\mathcal{C})$.

**3.5. Definition**. *A finite set $E$ of conjugacy classes $(H)$ is an induction set for a Green functor $V$ if $\oplus V(G/H) \to V(G/G)$ given by*

$$(x(H)) \to \Sigma p(H)_* x(H) \textit{ is surjective .}$$

*Define $E \leq F$ iff for each $(H) \in E$, there exists $(K) \in F$ such that $(H) \leq (K)$ i.e. $H$ is subconjugate to $K$. Then $\leq$ is a partial ordering on induction sets.*

**3.6. Lemma**. *Every Green functor $V$ possesses a minimal induction set $D(V)$, called the defect set of $V$.*

For proof see [4]. Hence $K_0^G(-,\mathcal{C})$ has defect sets.

**3.7.** Let $V$ be a Green functor and $M$ be a $V$-module. Define homomorphisms

$$p_1, p_2 : \bigoplus_{i,j\in J} \bigoplus_{\alpha\in S(i,j)} M(G/L_\alpha) \to \bigoplus_{k\in J} M(G/H_i)$$

by

$$p_2(x(i,j,\alpha)) = \sum_{i\in J} \sum_{\alpha \in S(i,j)} \eta_\alpha \alpha(2)_* x(i,j,\alpha)$$

and

$$p_1(x(i,j,\alpha)) = \sum_{i\in J} \sum_{\alpha \in S(i,j)} \eta_\alpha \alpha(1)_* x(i,j,\alpha)$$

where $S(i,j) = S(H_i, H_j)$ and $\alpha \in S(i,j)$ is in the decomposition of $[id] \in \mathcal{U}(G, G/H_i \times G/H_j)$.

**3.8. Theorem.** *Let $M = K_n^G(-,\mathcal{C})$. Then there exists an exact sequence*

$$\oplus_{i,j\in J}(\oplus_{\alpha\in S(i,j)} M(G/L_\alpha) \xrightarrow{p_2 - p_1} \oplus_{k\in J} M(G/H_k) \xrightarrow{p} M(G/G) \to 0.$$

*Proof.* We have seen in 3.3 that $p$ is surjective through the construction of a splitting homomorphism $q$ such that $pq$=identity. We now construct a homomorphism $q_1$ such that $(p_2 - p_1)q_1 + qp = \text{id}$ from which exactness follows.

Since $p_2$ is defined as $\oplus_{k\in J}p(\Sigma G|H_k)$ we define $q_1 = \oplus_{k\in J}q(G/H_k)$ and obtain as in the proof of 3.3 that $p_2q_1 = $ identity. One can also show that $p_1q_1 = qp$. Hence the result.

**3.9. Definition**. *A subgroup $C$ of $G$ is said to be cyclic if powers of a generator of $C$ are dense in $G$. If $p$ is a rational prime, then a subgroup $K$ of $G$ is called $p$-hyperelementary if there exists an exact sequence $1 \mapsto C \to K \to P \to 1$ where $P$ is a finite $p$-group and $C$ a cyclic group such that the order of $C/C_0$ is prime to $p$. Here $C_0$ is the component of the identity in $C$. It is called hyperelementary if it is $p$-hyperelementary for some $p$.*

Let $\mathcal{H}$ be the set of hyperelementary subgroups of $G$. We now have the following result which is the goal of this section and typifies results that can be obtained.

**3.10. Theorem**. *Let $M = K_n^G(-, \underline{M}(\mathbb{C}))$. Then $\oplus_{\mathcal{H}}M(G/H) \to M(G/G)$ is surjective (i.e. $M$ satisfies hyper-elementary induction) i.e. $M(G/G)$ can be computed in terms of $p$-hyperelementary subgroups of $G$).*

*Proof.* It suffices to show that if $V = K_0^G(-, \underline{M}(\mathbb{C}))$ then $\bigoplus_{H\in\mathcal{H}} V(G/H) \to V(G/G)(I)$ is surjective since $K_n^G(-, \underline{M}(\mathbb{C})$ is a $V$-module.

Now it is clear from example 2.7 that $K_0^G(G/G, \underline{M}(\mathbb{C})$ is the complex representation ring $R(G)$. Moreover, it is known (see [11]) that $R(G)$ is generated as an Abelian group by modules induced from hyperelementary subgroups of $G$. This is equivalent to the surjectivity of $\bigoplus_{H\in\mathcal{H}} V(G/H) \to V(G/G)$.

## 4. Remarks on possible generalizations

**4.1.** Let $\mathcal{B}$ be a category with finite sums, a final object and finite pull-backs (and hence finite products).

A Mackey functor $M : \mathcal{B} \to \mathbb{Z}$-mod is a bifunctor $M = (M_*, M^*)$, $M_*$ covariant, $M^*$ contravariant such that $M(X) = M_*(X) = M^*(X)$ for all $X \in \mathcal{B}$ and
(i) For any pullback diagram

$$\begin{array}{ccc} A' & \xrightarrow{p_2} & A_2 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle f_2} \quad \text{in } \mathcal{B} \\ A_1 & \xrightarrow{f_1} & A \end{array}$$

the diagram

$$\begin{array}{ccc} M(A') & \xrightarrow{p_{2*}} & M(A_i) \\ \uparrow{\scriptstyle p_1^*} & & \uparrow{\scriptstyle f_2^*} \\ M(A_i) & \xrightarrow{f_{2*}} & M(A) \end{array}$$

(ii) $M^*$ transforms finite coproducts in $\mathcal{B}$ over finite products in $\underline{\mathbb{Z}\text{-mod}}$.

**4.2. Example**. Now suppose that $G$ is a compact Lie group. Let $\mathcal{B}$ be the category of $G$-spaces of the $G$-homotopy type of G-CW-complexes (e.g. G-ENR spaces, see [4] or [8]). Then $\mathcal{B}$ is a category with finite coproducts (topological sums), final object and finite pullbacks (fibred products) (see [1]). Hence a Mackey functor is defined on $\mathcal{B}$ along the lines of 4.1.

Hence, in a way analogous to what was done in [2] or [3] we could define for $X, Y \in \mathcal{B}$, the notion of $Y$-exact sequences in the exact category $[\underline{X}, \mathcal{C}]$ (where $\mathcal{C}$ is an exact category) and obtain $K_n^G(X, \mathcal{C}, Y)$ as the $n^{th}$ algebraic $K$-group of $[\underline{X}, \mathcal{C}]$ with respect to $Y$-exact sequences.

We could also have the notion of an element $\zeta \in [\underline{X}, \mathcal{C}]$ being $Y$ projective and obtain a full subcategory $[\underline{X}, \mathcal{C}]_Y$ of $Y$-projective functors in $[\underline{X}, \mathcal{C}]$ so that we could obtain $P_n^G(X, \mathcal{C}, Y)$ as the $n^{th}$ algebraic $K$-group of $[X, \mathcal{C}]_Y$ with respect to split exact sequences and then show that $K_n^G(-, \mathcal{C}, Y)$, $P_n^G(-, \mathcal{C}, Y)$: $\mathcal{B} \to \mathbb{Z}$-mod are Mackey functors and that $K_0^G(-, \mathcal{C}, Y) : \mathcal{B} \to \mathbb{Z}$-mod is a Green functor and $K_n^G(-, \mathcal{C}, Y)$, $P_n^G(-, \mathcal{C}, Y)$ are $K_0^G(-, \mathcal{C}, T)$-modules in a way analogous to what was done in [2], [3].

It is hoped to explore these possibilities for further results in a future paper.

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