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On the Symmetric Difference Metric for Convex Bodies

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Abstract. Explicit inequalities are proved that relate the Hausdorff metric for convex bodies with the symmetric difference metric and a corresponding surface area deviation measure.

Let $\mathcal{K}^n$ denote the class of all nonempty compact convex subsets of the euclidean n-dimensional space $\mathbf{R}^n$. The members of $\mathcal{K}^n$ will be referred to as *convex bodies* or, more specifically, as *convex bodies in* $\mathbf{R}^n$. One of the indispensable tools in the theory of convex sets is the concept of a 'distance' or a 'deviation' between two convex bodies. In this note we concern ourselves with the following three notions of this kind.

(a) *The Hausdorff Metric.* To define it denote for any $M \in \mathcal{K}^n$ its support function by $h_M(u)$, that is, $h_M(u) = \sup\{u \cdot x : x \in M\}$, where the dot denotes the inner product. Then the *Hausdorff distance* between two bodies $K, L \in \mathcal{K}^n$ is defined by

$$\delta(K,L) = \sup\{|h_L(u) - h_K(u)| : u \in S^{n-1}\},$$

where S^{n-1} denotes the unit sphere in $\mathbf{R}^n$ (centered at the origin of $\mathbf{R}^n$). This is the most often used distance concept for convex bodies.

(b) *The symmetric difference metric.* This is defined by assigning to $K, L \in \mathcal{K}^n$ the distance

$$\Delta_v(K,L) = v(K \cup L) - v(K \cap L),$$

where v denotes the volume in $\mathbf{R}^n$. Equivalently, it can be written in the occasionally more convenient form $\Delta_v(K,L) = (v(K) - v(K \cap L)) + (v(L) - v(K \cap L))$.

(c) *The symmetric surface area deviation.* This is defined similarly as the symmetric difference metric but with the volume v replaced by the surface area s. Thus, we write

$$\Delta_s(K,L) = s(K \cup L) - s(K \cap L),$$

which can also be expressed as $\Delta_s(K,L) = (s(K) - s(K \cap L)) + (s(L) - s(K \cap L))$. Δ_s does not, in general, define a metric since it does not satisfy the triangle inequality. Nevertheless it is an interesting deviation measure. See Florian [2] for further comments regarding this and closely related deviation measures.

There exist many other metrics and deviation measures that have been investigated. In particular, we mention the L_p-metric. For any $p \geq 1$ it is defined by

$$\delta_p(K,L) = \left(\int_{S^{n-1}} |h_L(u) - h_K(u)|^p d\sigma(u) \right)^{1/p},$$

where $\sigma(u)$ refers to the surface area on S^{n-1}. Note that if $n = 2$ and $K \subset L$ then $\delta_1(K,L) = \Delta_s(K,L)$. For further discussions of metrics and deviation measures on $\mathcal{K}^n$ see Gruber [3]. It is often necessary or desirable to make a transition from one metric to another. For this purpose one needs conversion formulas which, in their most practical forms, can be expressed as inequalities between these metrics. For example, Vitale [6] proved inequalities of this kind relating the L_p-metric to the Hausdorff metric. It is the aim of the present article to establish similar relations for the Hausdorff metric on the one hand, and the symmetric difference metric and surface area deviation on the other hand. In this connection we also mention the work of Shephard and Webster [5] where the topological equivalence between the Hausdorff metric and the symmetric difference metric is proved. Our objective, however, is to establish explicit inequalities.

The following theorem contains our principal result. In this theorem κ_n denotes the volume, and σ_n the surface area of the n-dimensional unit ball; $\operatorname{diam} X$ denotes the diameter and $\operatorname{int} X$ the interior of any $X \subset \mathbf{R}^n$.

Theorem. *Assume that $K, L \in \mathcal{K}^n$, and let $D = \max\{\operatorname{diam} K, \operatorname{diam} L\}$. Furthermore, if $\operatorname{int}(K \cap L) \neq \emptyset$ let r denote the inradius of $K \cap L$. Then the following statements hold.*

(i) *For all $n \geq 2$*

$$\Delta_v(K,L) \leq c_1 \delta(K,L), \tag{1}$$

where $c_1 = \frac{2\kappa_n}{2^{1/n}-1} \left(\frac{D}{2}\right)^{n-1}$.

(ii) *If $\operatorname{int}(K \cap L) \neq \emptyset$, then for all $n \geq 2$*

$$\delta(K,L) \leq c_2 \Delta_v(K,L)^{1/n} \tag{2}$$

with $c_2 = \left(\frac{n}{\kappa_{n-1}}\right)^{1/n} \left(\frac{D}{r}\right)^{(n-1)/n}$.

(iii) *If $\operatorname{int}(K \cap L) \neq \emptyset$, then for all $n \geq 2$*

$$\Delta_s(K,L) \leq c_3 \delta(K,L), \tag{3}$$

where $c_3 = \sigma_n \left(\frac{D}{2}\right)^{n-2} \left(\frac{5D}{2r}\right)^{n-1}$. (If $n = 2$ one may choose $c_3 = 4\pi D/r$.)

(iv) *If $\operatorname{int}\ (K \cap L) \neq \emptyset$, then for all $n \geq 3$*

$$\delta(K,L) \leq c_4 \Delta_s(K,L)^{1/(n-1)} \tag{4}$$

with $c_4 = \left(\frac{\sqrt{5}+1)}{2\kappa_{n-1}}\right)^{1/(n-1)} \left(\frac{D}{r}\right)^{(n-2)/(n-1)}$, *and if* $K \cap L \neq \emptyset$ *and* $n = 2$, *then*

$$\delta(K, L) \leq c_5 \Delta_s(K, L)^{1/2}, \tag{5}$$

where $c_5 = \left(\frac{\sqrt{2}+1}{2} D\right)^{1/2}$.

These five inequalities cannot be improved in the sense that the exponents of $\Delta_v(K, L)$ *in* (2) *and* $\Delta_s(K, L)$ *in* (4) *and* (5) *cannot be replaced by any larger numbers (for any choice of* c_2 , c_4 , c_5 *depending on* n, D, *and* r *only). Similarly, in* (1) *and* (3) $\delta(K, L)$ *cannot be replaced by any* $\delta(K, L)^\alpha$ *with* $\alpha > 1$ *(for any appropriate choice of the coefficients).*

Before we turn to the proof of this theorem we add several remarks.

1. The proof of the theorem will show that if $K \subset L$ then c_1 in (1) can be replaced by $c_1/2$, and c_3 in (3) by $\sigma_n((3^{n-1} - 1)/2^{n-1})D^{n-2}$.

2. Inequality (1) can be combined with (4) or (5), and (2) can be combined with (3) to yield inequalities relating the symmetric difference metric with the surface area deviation, namely

$$\Delta_v(K, L) \leq c_1 c_4 \Delta_s(K, L)^{1/(n-1)} \quad (n \geq 3),$$

$$\Delta_v(K, L) \leq c_1 c_5 \Delta_s(K, L)^{1/2} \quad (n = 2),$$

and

$$\Delta_s(K, L) \leq c_2 c_3 \Delta_v(K, L)^{1/n}.$$

Particularly if $K \subset L$ these inequalities are of some interest and then the improvements of the coefficients mentioned in the previous remark are possible.

3. Inequalities (2), (4), and (5) can be viewed as quantitative versions of the fact that $K \subset L$ implies $v(K) \leq v(L)$ and $s(K) \leq s(L)$. Indeed these inequalities show that if $K \subset L$ and if K and L have in some direction support planes of distance at least ϵ, then $v(K) \leq v(L) - c_2^{-n}\epsilon^n$, $s(K) \leq s(L) - c_4^{1-n}\epsilon^{n-1}$ if $n \geq 3$, and $s(K) \leq s(L) - c_5^{-2}\epsilon^2$ if $n = 2$.

4. If $n = 2$ and $K \subset L$, then (5) is essentially the same as a special case of an estimate of Vitale [6] for the L_1-metric.

Proof of the Theorem. In all parts of this proof the following conventions will be used. Hyperplanes will be referred to simply as 'planes'. The unit ball in $\mathbf{R}^n$ (centered at o) will be denoted by B, and for any $M \in \mathcal{K}^n$ and $u \subset S^{n-1}$ we let $H_M(u)$ denote the support plane of M in the direction u. Furthermore, we frequently write only δ, Δ_v, and Δ_s instead of $\delta(K, L)$, $\Delta_v(K, L)$, and $\Delta_s(K, L)$, respectively.

The proof of (1) is a consequence of Steiner's formula for the volume of a parallel body. If W_i denotes the i-th projection integral (Quermaßintegral) in $\mathbf{R}^n$ this formula states that for any $\lambda \geq 0$

$$v(K + \lambda B) = \sum_{i=0}^{n} \binom{n}{i} W_i(K)\lambda^i.$$

Letting $\lambda = \delta(K, L)$ and noting that $W_0(K) = v(K)$ and

$$v(L) - v(K \cap L) = v(K \cup L) - v(K) \leq v(K + \delta B) - v(K) \tag{6}$$

we obtain

$$v(L) - v(K \cap L) \leq \sum_{i=1}^{n} \binom{n}{i} W_i(K)\delta^i.$$

Since, among all convex bodies of given diameter, W_i attains its maximum for balls (see Bonnesen-Fenchel [1, Sec. 54]) it follows that for all $M \in \mathcal{K}^n$

$$W_i(M) \leq 2^{i-n}\kappa_n(\operatorname{diam} M)^{n-i}. \tag{7}$$

Hence, for any $\alpha > 0$ we have

$$v(L) - v(K \cap L) \leq 2^{-n}\kappa_n D^{n-1}\delta\frac{1}{\alpha}\sum_{i=1}^{n}\binom{n}{i}(2\alpha)^i\left(\frac{\delta}{\alpha D}\right)^{i-1}.$$

Letting $\alpha = \left(2^{1/n} - 1\right)/2$ and assuming that $\delta/\alpha D \leq 1$ we deduce that

$$v(L) - v(K \cap L) \leq \kappa_n \frac{(1+2\alpha)^n - 1}{2\alpha}\left(\frac{D}{2}\right)^{n-1}\delta = \frac{\kappa_n}{2^{1/n} - 1}\left(\frac{D}{2}\right)^{n-1}\delta.$$

But the same estimate holds if $\delta/\alpha D > 1$ since another application of (7) (with $i = 0$) shows that

$$v(L) - v(K \cap L) \leq 2^{-n}\kappa_n D^n \leq 2^{-n}\kappa_n D^{n-1}\frac{\delta}{\alpha} = \frac{\kappa_n}{2^{1/n} - 1}\left(\frac{D}{2}\right)^{n-1}\delta.$$

Combining this with the corresponding inequality for $v(K) - v(K \cap L)$ we obtain (1).

If $K \subset L$, then $\Delta_v(K, L) = v(L) - v(K \cap L)$ and it is clear that c_1 can be replaced by $c_1/2$ as stated in our first remark.

If $\lambda \geq 0$ and $K = B$, $L = (1+\lambda)B$, then $\delta(K, L) = \lambda$ and $\Delta_v(K, L) = \kappa_n((1+\lambda)^n - 1) \geq n\kappa_n\lambda$ which shows that (3) cannot be improved in the sense stated in the theorem.

The analogue of (1) for the surface area, that is inequality (3), is more difficult to prove. The problem is that (6) rests on the fact that the inclusion $K \cup L \subset K + \delta B$ implies $v(K \cup L) \leq v(K + \delta B)$; but the corresponding inference for the surface area is not valid since, in general, $K \cup L$ is not convex. To surmount this difficulty we use the following concepts. If $M \in \mathcal{K}^n$, $o \in M$, and $u \subset S^{n-1}$ let $\rho_M(u)$ denote the *radial function* of M, that is the length of the line segment $M \cap R(u)$, where $R(u)$ denotes the ray $\{\tau u : \tau \geq 0\}$. Moreover, if $o \in K \cap L$ we define the *radial distance* between K and L by

$$\delta_\rho(K, L) = \sup\{|\rho_L(u) - \rho_K(u)| : u \in S^{n-1}\}.$$

For the proof of (3) let us now assume, as we may, that $rB \subset K \cap L$. If $u_* \in S^{n-1}$ is such that δ is the distance between the respective support planes $H_K(u_*)$ and $H_L(u_*)$, then, considering the line segment from o to a point in $\partial K \cap H_K(u_*)$ or $\partial L \cap H_L(u_*)$ we see immediately that

$$\delta_\rho(K, L) \geq \delta(K, L). \tag{8}$$

Next we show that

$$\delta(K, L) \geq \frac{r}{D}\delta_\rho(K, L). \tag{9}$$

Interchanging, if necessary, the roles of K and L one can choose a $u_o \in S^{n-1}$ such that

$$\delta_\rho(K, L) = \rho_L(u_o) - \rho_K(u_o).$$

Let E_K denote the support plane of K at the point $\partial K \cap R(u_o)$ and let E_L be the plane parallel to E_K and containing $\partial L \cap R(u_o)$. Now, if h_o denotes the distance of o from E_L, and δ_o the distance between E_K and E_L, then elementary geometry shows that

$$\delta_\rho(K, L) = \frac{\rho_L(u_o)}{h_o}\delta_o \leq \frac{D}{r}\delta_o \leq \frac{D}{r}\delta(K, L)$$

and this proves (9).

Using (8), (9), and the obvious fact that $|\rho_L(u) - \rho_K(u)| \geq |\rho_L(u) - \rho_{K\cap L}(u)|$ (for all $u \subset S^{n-1}$) we deduce that

$$\begin{aligned}\delta &\geq \frac{r}{D}\delta_\rho(K, L) = \frac{r}{D}\sup\{|\rho_L(u) - \rho_K(u)| : u \in S^{n-1}\}\\ &\geq \frac{r}{D}\sup\{|\rho_L(u) - \rho_{K\cap L}(u)| : u \in S^{n-1}\} = \frac{r}{D}\delta_\rho(L, K \cap L)\\ &\geq \frac{r}{D}\delta(L, K \cap L).\end{aligned}$$

Thus, $\delta \geq (r/D)\delta(L, K \cap L)$ and similarly one finds $\delta \geq (r/D)\delta(K, K \cap L)$. It follows that $K \subset (K \cap L) + (D/r)\delta B$ and $L \subset (K \cap L) + (D/r)\delta B$, and this implies

$$\Delta_s \leq 2\left(s\left((K \cap L) + \tfrac{D}{r}\delta B\right) - s(K \cap L)\right). \tag{10}$$

We now use Steiner's formula for the surface area which states that for any $M \in \mathcal{K}^n$ and $\epsilon \geq 0$

$$s(M + \epsilon B) = n\sum_{i=0}^{n-1}\binom{n-1}{i}W_{i+1}(M)\epsilon^i \tag{11}$$

(see Hadwiger [4, p. 214]). Letting $\epsilon = (D/r)\delta$, $M = K \cap L$ and using (10) together with the relation $nW_1(M) = s(M)$ we find that

$$\Delta_s \leq 2n\sum_{i=1}^{n-1}\binom{n-1}{i}W_{i+1}(K \cap L)\left(\frac{D}{r}\delta\right)^i.$$

This, combined with (7) and the obvious estimate $\delta/D \leq 1$, yields

$$\Delta_s \leq \sigma_n 2^{2-n}D^{n-1}\sum_{i=1}^{n-1}\binom{n-1}{i}\left(\frac{2D}{r}\right)^i\left(\frac{\delta}{D}\right)^i \leq \sigma_n 2^{2-n}D^{n-2}\delta\left(\left(\frac{2D}{r}+1\right)^{n-1} - 1\right).$$

Observing also that $1 \le D/2r$, we obtain (3).

If $K \subset L$, then

$$\Delta_s = s(L) - s(K) \le s(K + \delta B) - s(K).$$

Using again (7) and (11) (with $M = K$ and $\epsilon = \delta$), one finds

$$\Delta_s \le \sigma_n 2^{1-n} D^{n-1} \sum_{i=1}^{n-1} \binom{n-1}{i} 2^i \left(\frac{\delta}{D}\right)^i \le \sigma_n \frac{3^{n-1}-1}{2^{n-1}} D^{n-2}\delta,$$

which justifies the corresponding statement in our first remark.

The same example as that used in connection with inequality (1) shows that no essential improvement of (3) is possible.

We now prove (2). Without any loss in generality it can be assumed that $rB \subset K \cap L$. There exists a direction u_o such that, with proper designation of K and L,

$$\delta = h_L(u_o) - h_K(u_o).$$

Let $p \in L \cap H_L(u_o)$ and let C be the smallest cone that contains rB, has apex p and whose base is in $H_{(rB)}(-u_o)$. If J denotes the slab bounded by $H_K(u_o)$ and $H_L(u_o)$ then the cone $\tilde{C} = C \cap J$ is similar to C and it is obvious that $\Delta_v \ge v(\tilde{C})$. Let Q denote the largest ball in C with center in $H_K(u_o)$, and let ρ denote the radius of Q. Then we have $\rho/r = \delta/h_L(u_o)$ and therefore $\rho \ge (r/D)\delta$. Also, since the base of $\tilde{C}$ contains the $(n-1)$-dimensional ball $H_K(u_o) \cap Q$, the ($(n-1)$-dimensional) volume of the base of $\tilde{C}$ is at least $\kappa_{n-1}\rho^{n-1}$. Hence,

$$\Delta_v \ge v(\tilde{C}) \ge \frac{\kappa_{n-1}}{n}\rho^{n-1}\delta \ge \frac{\kappa_{n-1}}{n}\left(\frac{r}{D}\right)^{n-1}\delta^n,$$

and this proves (2).

That the exponent of $\Delta_v(K, L)$ in (2) cannot be improved is seen by letting K be a right cone with spherical base, and L the truncated cone obtained by cutting off from K (with a plane parallel to its base) the cone similar to K of height δ.

For the proof of (4) and (5) we first consider two convex bodies M and N in $\mathbf{R}^n$ such that $rB \subset M \subset N$ and derive a lower bound for $\Delta_s(M, N)$ in terms of $\hat{\delta} = \delta(M, N)$, r, and $\hat{D} = \max\{\operatorname{diam} M, \operatorname{diam} N\}$. Clearly, there is a $u_o \in S^{n-1}$ such that

$$\delta(M, N) = h_N(u_o) - h_M(u_o).$$

Let $H_M^+(u_o)$ be the closed half-space bounded by $H_M(u_o)$ and containing M, and let $M' = N \cap H_M^+(u_o)$. Obviously,

$$\delta(M, N) = \delta(M', N). \tag{12}$$

Also, since $\Delta_s(M', N) \le \Delta_s(M, N)$ and $\Delta_s(M', N) = s_o - s(N \cap H_M(u_o))$, where s_o denotes the surface area of the part of ∂N not lying in $\partial M'$, we have

$$\Delta_s(M, N) \ge s_o - s(N \cap H_M(u_o)). \tag{13}$$

Let now ℓ be a line orthogonal to $h_M(u_o)$, and let us perform a spherical symmetrization (Schwarzsche Abrundung, see Bonnesen-Fenchel [1, Sec. 41]) of M' and N with ℓ as axis. It transforms these bodies into rotationally symmetric bodies, say $\bar{M}'$ and $\bar{N}$. Moreover, $N \cap H_M(u_o)$ is transformed into an $(n-1)$-dimensional ball, say G, that lies in the plane $H_M(u_o)$ and has its center in ℓ. Since spherical symmetrization preserves volume and does not increase surface area it follows from (13) that

$$\Delta_s(M,N) \geq \bar{s}_o - s(\bar{N} \cap H_M(u_o)), \tag{14}$$

where $\bar{s}_o$ is defined as s_o but with respect to $\bar{N}$ and $\bar{M}'$ rather than N and M'. Let now C denote the cone with apex $q = H_N(u_o) \cap \ell$ and base G, and let $\tilde{s}$ denote the lateral surface area of C. Then, $\tilde{s} \leq \bar{s}_o$, and it follows from (14) that

$$\Delta_s(M,N) \geq \tilde{s} - s(\bar{N} \cap H_M(u_o)). \tag{15}$$

Furthermore, if G has radius ρ, then

$$\tilde{s} = \kappa_{n-1}\rho^{n-2}\sqrt{\hat{\delta}^2 + \rho^2},$$

and (15) implies therefore

$$\Delta_s(M,N) \geq \kappa_{n-1}\rho^{n-2}(\sqrt{\rho^2 + \hat{\delta}^2} - \rho) = \kappa_{n-1}\frac{\rho^{n-2}\hat{\delta}^2}{(\hat{\delta}^2 + \rho^2)^{1/2} + \rho}. \tag{16}$$

Since $rB \subset N$ the body $\bar{N}$ contains a ball, say O, of radius r with center on ℓ. Consider a line T that contains q and is tangent to O. Let η denote the distance between the center of G and $T \cap G$, and t the distance between q and the point $T \cap O$. Then we deduce, observing also (12), that $t/r = \hat{\delta}/\eta$ and consequently

$$\rho \geq \eta = \frac{r}{t}\hat{\delta} \geq \frac{r}{\hat{D}}\hat{\delta}.$$

Combining this with (16) we infer that

$$\Delta_s(M,N) \geq \kappa_{n-1}\frac{\rho^{n-3}\hat{\delta}^2}{((\hat{D}/r)^2 + 1)^{1/2} + 1}.$$

If $n \geq 3$ then ρ^{n-3} is increasing or constant and therefore

$$\Delta_s(M,N) \geq \kappa_{n-1}\frac{(r/\hat{D})^{n-3}}{((\hat{D}/r)^2 + 1)^{1/2} + 1}\hat{\delta}^{n-1}.$$

Since $1 \leq \hat{D}/2r$ this yields

$$\Delta_s(M,N) \geq \kappa_{n-1}\left(\frac{r}{\hat{D}}\right)^{n-2}\frac{2}{\sqrt{5}+1}\hat{\delta}^{n-1}. \tag{17}$$

The desired inequality (4) is now obtained by adding the two inequalities that result by substituting into (17) $M = K \cap L$ and $N = K$ or $N = L$, and noting that $\hat{D} \leq D$, $\delta(K, K \cap L)^{n-1} + \delta(L, K \cap L)^{n-1} \geq (\max\{\delta(K, K \cap L), \delta(L, K \cap L)\})^{n-1} \geq \delta(K, L)^{n-1}$, and $\Delta_s(K, L) = \Delta_s(K, K \cap L) + \Delta_s(L, K \cap L)$.

That the exponent of $\Delta_s(K, L)$ cannot be improved is seen by the same example as that described in connection with (2).

All conclusions of this proof up to and including inequality (16) remain valid if $n = 2$. Thus, in this case we have

$$\Delta_s(M, N) \geq 2\frac{\hat{\delta}^2}{(\hat{\delta}^2 + \rho^2)^{1/2} + \rho},$$

and since $\hat{\delta} \leq \hat{D}$, $\rho \leq \hat{D}$ it follows that

$$\Delta_s(M, N) \geq \frac{2}{(\sqrt{2} + 1)\hat{D}}\hat{\delta}^2.$$

Letting again $M = K \cap L$ and $N = K$ or $N = L$ and proceeding as before we obtain (5). Note that (5) is trivially satisfied if $\operatorname{int}(M \cap N) = \emptyset$ and $\operatorname{int}(M \cap N) \neq \emptyset$.

It is remarkable that the case $n = 2$ is really exceptional. To show that in (5) the exponent $1/2$ cannot be replaced by a larger number consider the following example. In the usual (x, y)-coordinate system let K be the convex domain bounded by $y = 0 \quad (-1 \leq x \leq 1)$ and the curve $y = 1 - x^m \quad (-1 \leq x \leq 1)$, where m is a positive even integer. Furthermore, assuming that an $\epsilon \in [0, 1]$ is given, let L be defined as the part of K where $y \leq 1 - \epsilon$. Clearly, $\delta(K, L) = \epsilon$ and $\Delta_s(K, L) \geq 2(\sqrt{\epsilon^2 + \epsilon^{2/m}} - \epsilon^{1/m}) = 2\epsilon^2/(\sqrt{\epsilon^2 + \epsilon^{1/m}} + \epsilon^{1/m}) \geq (2/(\sqrt{2} + 1))\epsilon^{2-1/m}$. Since m can be arbitrarily large this justifies our assertion.

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