Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 41 (2000), No. 1, 33-47.

Towards a Classification of Homogeneous Integral Table Algebras of Degree Five

M. R. Darafsheh **A. Rahnamai Barghi**

Center for Theoretical Physics and Mathematics, AEOI, Tehran, Iran
Department of Mathematics, University of Tehran, Tehran, Iran
Institute for Advanced Studies in Basic Sciences, Gava Zang, Zanjan, Iran
Institute for Studies in Theoretical Physics and Mathematics, Tehran, Iran

Abstract. In this paper we classify some of the homogeneous integral table algebras of degree 5 containing a faithful element of width 3 such that in a given basis of the algebra the basis elements are standard. This work is towards classifying homogeneous integral table algebras of degree 5.

MSC 1991: 20C05, 20C99, 16P10

1. Introduction

The notation of a table algebra in its new formulation is defined in [2]. Most of related notions and properties of C-algebras and table algebras are defined and proved in [4]. Of particular importance is the classification of homogeneous integral table algebras of small degrees. The homogeneous integral table algebras of degree 1 are $(\mathbb{C}G, G)$ where G is a finite abelian group [6]. The homogeneous integral table algebras of degree 2 which contain a faithful element are classified in [5]. Also the homogeneous integral table algebras of degree 3 which contain a faithful real element are classified in [7]. In [3] one can also find the classification of homogeneous integral table algebras of degree 4 which contain a faithful element. In [1] there exists an open problem asking for the classification of homogeneous standard integral table algebras of degree 5 with a faithful element.

The main purpose of this paper is to state classification theorems about some of the homogeneous standard integral table algebras of degree 5. This work solves part of the stated open problem.

In what follows we remind some basic definitions from [2] and [4]. Note that throughout this paper $\mathbb{C}$ denotes the field of complex numbers and $\mathbb{R}^+$ the positive reals.

Definition 1.1. *Let $B = \{b_1, b_2, \ldots, b_n\}$ be a basis of a finite-dimensional associative and commutative algebra A over the complex field $\mathbb{C}$ with identity element $b_1 = 1_A$. Then (A, B) is called a table algebra (and B is said to be a table basis) if and only if the following conditions hold:*

(i) *For all i, j, m,* $\quad b_i b_j = \sum_{m=1}^{n} \lambda_{ijm} b_m$ *with* $\lambda_{ijm} \in \mathbb{R}^+ \cup \{0\}$.

(ii) *There is an algebra automorphism (denoted by $^-$) of A whose order divides 2, such that $b_i \in B$ implies that $\overline{b_i} \in B$ (then $\overline{i}$ is defined by $b_{\overline{i}} = \overline{b_i}$, and $b_i \in B$ is called real if $\overline{b_i} = b$ or equivalently $\overline{i} = i$).*

(iii) *For all i, j,* $\quad \lambda_{ij1} \neq 0$ *if and only if $j = \overline{i}$.*

By [1, Lemma 2.9], there is an algebra homomorphism $f : A \longrightarrow \mathbb{C}$ such that $f(b_i) = f(\overline{b_i}) \in \mathbb{R}^+$ for all i and such a map f is uniquely determined. The positive real numbers $f(b_i), 1 \leq i \leq n$, are called the degrees of (A, B).

In this paper when we refer to a table algebra (A, B) we assume $f : A \longrightarrow \mathbb{C}$ is the unique linear character associated to (A, B).

Definition 1.2. *An integral table algebra (abbreviated ITA) is a table algebra (A, B) such that all the structure constants λ_{ijm} are non-negative integers and all the degrees $f(b_i)$ are rational integers.*

For example any finite group G yields two examples of ITA's; $Z(\mathbb{C}G,\mathrm{Cla}(G))$, the center of the group algebra of a finite group G with the table algebra basis the set of sums $\widehat{C}$ of G-conjugacy classes C of G, with automorphism $^-$ extended linearly from inversion in G, and with degrees $f(\widehat{C}) = |C|$ for all $\widehat{C} \in \mathrm{Cla}(G)$, and $(\mathrm{Ch}(G), \mathrm{Irr}(G))$, the ring of complex valued class functions on G, with table basis the set of irreducible characters of G, with automorphism $^-$ extended linearly from complex conjugation of characters, and with degrees $f(\chi) = \chi(1)$ for all $\chi \in \mathrm{Irr}(G)$.

Definition 1.3. [2] *Let (A, B) be a table algebra. For $a \in A$ we define*

$$\mathrm{Supp}_B(a) = \{b_i \mid a = \sum_{i=1}^{n} \lambda_i b_i \ \text{ where } b_i \in B \text{ and } \lambda_i \neq 0\}.$$

Let (A, B) be a table algebra. A nonempty subset $N \subseteq B$ is called a *table subset* (or a *C-subset*) of B iff $\mathrm{Supp}_B(b_i b_j) \subseteq N$ for all $b_i, b_j \in N$.

Any table subset is stable under $^-$ and contains 1_A by [1, Proposition 2.7]. For any $c \in B$, the set B_c defined by

$$B_c \;=\; \bigcup_{n=1}^{\infty} \mathrm{Supp}(c^n)$$

is easily seen to be a table subset of B, called the table subset generated by c. Any element c of B is called *faithful* iff $B_c = B$.

Definition 1.4. [5, Definition 1.3] *A table algebra (A, B) is called homogeneous (of degree λ) iff $|B| \geq 2$ and, for some fixed $\lambda \in \mathbb{R}^+$, $f(b) = \lambda$ for all $b \in B - \{1\}$.*

For any table algebra (A, B), there is a positive definite Hermitian form $(\ ,\ )$ on A with the following properties [4, Proposition 2.4]: B is an orthogonal set with respect to $(\ ,\ )$; for all $b \in \mathbb{R}B$ (the real span of B) and all $a, c \in A$,

$$(ab, c) = (a, \bar{b}c);$$

and for all $b_i \in B$, and $\lambda_{i\bar{i}1}$ the constants in Definition 1.1,

$$(b_i, b_i) = \lambda_{i\bar{i}1}.$$

It follows that, for all $a, b, c \in B$,

$$c \in \operatorname{Supp}_B(ab) \Longleftrightarrow a \in \operatorname{Supp}_B(\bar{b}c).$$

Definition 1.5. *Let (A, B) be a table algebra and $b \in B$. Then b is called standard iff $(b, b) = f(b)$. Furthermore, B is called standard iff every element of B is standard.*

Any table algebra may be rescaled (replacing each table basis element by a positive scalar multiple) to one which is homogeneous, and any ITA can be rescaled to a homogeneous ITA ([6, Theorem 1]). Therefore a classification theorem for all homogeneous integral table algebras (HITA) is an impossible mission.

Definition 1.6. [5] *Two table algebras (A, B) and (C, D) are called isomorphic (denoted by $B \cong D$) if there exists an algebra isomorphism $\phi : A \longrightarrow C$ such that $\phi(B)$ is a rescaling of D; and the algebras are called exactly isomorphic (denoted by $B \cong_x D$) if $\phi(B) = D$. So $B \cong_x D$ means that B and D yield the same structure constants.*

2. Main results

If (A, B) is a homogeneous standard integral table algebra degree of 5 which contains a faithful element of width 2, then [6, Theorem 3] implies the complete classification of (A, B). The main purpose of this section is to state a classification theorem about homogeneous standard ITA of degree 5 with the additional hypothesis that the basis B contains a faithful element of width 3. The notion of width is defined in [6] as follows:

Definition 2.1. *Let (A, B) be a table algebra and $b \in B$. The width of b is defined to be $|\operatorname{Supp}_B(b\bar{b})|$.*

It is easy to check that if B contains an element b of width 3 then one of the following possibilities hold:
If b is a real element, i.e., $b = \bar{b}$, then

(i) $b^2 = 5.1 + 3b + d, \quad d = \overline{d},$
(ii) $b^2 = 5.1 + 3c + b, \quad c = \overline{c},$
(iii) $b^2 = 5.1 + 3c + d, \quad c = \overline{c}, \quad d = \overline{d},$
(iv) $b^2 = 5.1 + 2c + 2\overline{c},$
(v) $b^2 = 5.1 + 2c + 2d, \quad c = \overline{c}, \quad d = \overline{d},$
(vi) $b^2 = 5.1 + 2b + 2d, \quad d = \overline{d},$

and if b is a nonreal element, then

(vii) $b\overline{b} = 5.1 + 2c + 2d, \quad c = \overline{c}, \quad d = \overline{d},$
(viii) $b\overline{b} = 5.1 + 2b + 2\overline{b},$
(ix) $b\overline{b} = 5.1 + 2c + 2\overline{c},$
(x) $b\overline{b} = 5.1 + 3c + d, \quad c = \overline{c}, \quad d = \overline{d},$

for some distinct elements b, c, d of B.

As far as one can see considering all the above cases is a very difficult task and we classify the cases (i),(ii),(iii) and (x) in this paper. Therefore in all of our statement we assume that (A, B) is a HITA of degree 5 where B is standard, i.e., $f(b) = (b, b)$=5 for all $b \in B - \{1\}$.

Proposition 2.2. *Let (A, B) be a homogeneous standard integral table algebra of degree 5. If B contains a real faithful element b such that $b^2 = 5.1 + 3b + d$ then (A, B) is one of the following:*

$$B = \{1, b, c, d\}, \quad b^2 = 5.1 + 3b + d, \quad c^2 = 5.1 + 2b + 2d,$$

$$d^2 = 5.1 + 3c + b, \quad bc = 3d + 2c, \quad dc = 3b + 2c;$$

or

$$B = \{1, b, d\}, \quad b^2 = 5.1 + 3b + d, \quad d^2 = 5.1 + 4b, \quad bd = b + 4d.$$

Proof. Let $b \in B$ such that $b^2 = 5.1 + 3b + d$ for some $d \in B$ and $d = \overline{d}$, $d \neq b$. Since $f(bd) = 25$ and $(bd, b) = (b^2, d) = 5$ we have one of the following cases:

$$\begin{array}{rcll}
bd &=& b + e_1 + \overline{e_1} + e_2 + \overline{e_2}, \quad b, e_1, e_2 \text{ are distinct elements of } B, & (1)\\
bd &=& b + 3e_1 + e_2, \quad e_1 = \overline{e_1}, \quad e_2 = \overline{e_2}, \quad b, e_1, e_2 \text{ are distinct elements of } B, & (2)\\
bd &=& b + 2e + 2\overline{e}, \quad b, e \text{ are distinct elements of } B, & (3)\\
bd &=& b + 4e, \quad e = \overline{e}, \quad b, e \text{ are distinct elements of } B, & (4)\\
bd &=& b + e_1 + e_2 + e_3 + e_4, \quad b, e_1, e_2, e_3, e_4 \text{ are distinct real elements of } B, & (5)\\
bd &=& b + e_1 + e_2 + e_3 + \overline{e_3}, e_1 = \overline{e_1}, e_2 = \overline{e_2}, \quad b, e_1, e_2, e_3 \text{ are distinct elements of } B, & (6)\\
bd &=& b + 2e_1 + e_2 + e_3, \quad b, e_1, e_2, e_3 \text{ are distinct real elements of } B, & (7)\\
bd &=& b + 2e_1 + e_2 + \overline{e_2}, e_1 = \overline{e_1}, \quad b, e_1, e_2 \text{ are distinct elements of } B, & (8)\\
bd &=& b + 2e_1 + 2e_2, \quad b, e_1, e_2 \text{ are distinct real elements of } B. & (9)
\end{array}$$

If (1) holds then by the equality $b(bd) = b^2 d$ we obtain

$$d^2 + 4d + 3e_1 + 3\overline{e_1} + 3e_2 + 3\overline{e_2} = 5.1 + be_1 + b\overline{e_1} + be_2 + b\overline{e_2}. \qquad (*)$$

Now we claim that e_1 does not appear in the decomposition of be_1. If $(be_1, e_1) \neq 0$ then it is easy that $(be_1, e_1) = 5, 10$ or 15. We deal with these cases separately. First we assume $(be_1, e_1) = 5$ and consider the following cases:

case (i) $be_1 = d + e_1 + x_1 + x_2 + x_3$, where x_1, x_2, x_3 are distinct real elements of B,

case (ii) $be_1 = d + e_1 + 2x_1 + x_2$, where x_1, x_2 are distinct real elements of B,

case (iii) $be_1 = d + e_1 + 3x$, where x is real elements of B.

In the case (i) we have:

$$25 = (be_1, be_1) = (b^2, e_1\overline{e_1}) = (5.1 + 3b + d, 5.1 + b + \cdots) \geq 40,$$

which is a contradiction.
In the case (ii) we have

$$35 = (be_1, be_1) = (b^2, e_1\overline{e_1}) = (5.1 + 3b + d, 5.1 + b + \cdots) \geq 40,$$

again a contradiction.
Let case (iii) hold, and $e_1\overline{e_1} = 5.1 + \alpha b + \beta d + \cdots$ for some non-negative integers α, β. Hence

$$55 = (be_1, be_1) = (b^2, e_1\overline{e_1}) = 25 + 15\alpha + 5\beta.$$

Thus $3\alpha + \beta = 6$. This implies that $\alpha = 1, \beta = 3$ or $\alpha = 2, \beta = 0$.

If $\alpha = 1, \beta = 3$ then $e_1\overline{e_1} = 5.1 + 3d + b$. So

$$(de_1, e_1) = (e_1\overline{e_1}, d) = 15.$$

This shows that e_1 appears thrice in the decomposition of de_1. On the other hand, d and b don't appear in the decompositon of d^2 by $(*)$. This implies that $(d^2, e_1\overline{e_1}) = 25$. Therefore by $(de_1, de_1) \geq 45$, we have a contradiction.

If $\alpha = 2$ and $\beta = 0$, then $e_1\overline{e_1} = 5.1 + 2b + \cdots$. Hence $(be_1, e_1) = (b, e_1\overline{e_1}) = 10$. This implies that e_1 does appear twice in the decomposition of be_1, that is a contradiction. This proves that e_1 does not appear once in the decomposition of be_1.

Now if e_1 appears twice in the decomposition of be_1, then we have $be_1 = d + 2e_1 + \cdots$. Hence $(be_1, be_1) \leq 45$ and

$$(b^2, e_1\overline{e_1}) = (5.1 + 3b + d, 5.1 + 2b + \cdots) \geq 55.$$

Since $(be_1, be_1) = (b^2, e_1\overline{e_1})$, we obtain a contradiction. Now if e_1 appears thrice in the decomposition of be_1, then $be_1 = d + 3e_1 + \cdots$. Hence $(be_1, be_1) = 55$ and

$$(b^2, e_1\overline{e_1}) = (5.1 + 3b + d, 5.1 + 3b + \cdots) \geq 70.$$

Since $(be_1, be_1) = (b^2, e_1\overline{e_1})$ we have a contradiction.
Therefore e_1 does not appear in the decomposition of be_1. Similarly we can prove that e_2 does not appear in the decomposition of be_2.

Now we can write $be_1 = d + \alpha e_2 + \beta\overline{e_2} + \gamma\overline{e_1} + \cdots$ for some non-negative integers α, β, γ. Hence

$$(bd, be_1) = (b + e_1 + \overline{e_1} + e_2 + \overline{e_2}, d + \alpha e_2 + \beta\overline{e_2} + \gamma\overline{e_1} + \cdots) = 5(\alpha + \beta + \gamma).$$

Also, $(b^2, de_1) = (5.1 + 3b + d, b + \delta d + \cdots) = 15 + 5\delta$. Since $(be_1, bd) = (b^2, de_1)$, we have $\alpha + \beta + \gamma = 3 + \delta$. Now we show that every elements $\overline{e_1}, e_2$ and $\overline{e_2}$ appear at least once in the decomposition of be_1.

Otherwise, if $be_1 = d + 2e_2 + \cdots$, then $be_2 = d + 2e_1 + \cdots$ and $\overline{e_1}e_2 = 2b + \cdots$. Since $(be_1, be_2) = (d + 2e_2 + \cdots, d + 2e_1 + \cdots) \le 25$ and $(b^2, \overline{e_1}e_2) = (5.1 + 3b + d, 2b + \cdots) \ge 30$, we have a contradiction, by the equality $(be_1, be_2) = (b^2, \overline{e_1}e_2)$.

If $be_1 = d + 2\overline{e_1} + \cdots$ then $b\overline{e_1} = d + 2e_1 + \cdots$ and $e_1^2 = 2b + \cdots$. Since

$$(be_1, b\overline{e_1}) = (d + 2\overline{e_1} + \cdots, d + 2e_1 + \cdots) \le 25,$$

and $(b^2, e_1^2) = (5.1 + 3b + d, 2b + \cdots) \ge 30$, by the equality $(be_1, b\overline{e_1}) = (b^2, e_1^2)$ we have a contradiction.

If $be_1 = d + 2\overline{e_2} + \cdots$ then $b\overline{e_2} = d + 2e_1 + \cdots$, and $e_1e_2 = 2b + \cdots$. Since $(be_1, b\overline{e_2}) = (d + 2\overline{e_2} + \cdots, d + 2e_1 + \cdots) \le 25$ and $(b^2, e_1e_2) = (5.1 + 3b + d, 2b + \cdots) \ge 30$, by the equality $(b^2, e_1e_2) = (be_1, b\overline{e_2})$ we obtain a contradiction.

Hence we must have the following possibilities by $(*)$

$$\begin{aligned} be_1 &= d + \overline{e_1} + e_2 + \overline{e_2} + x, \\ be_2 &= d + \overline{e_1} + e_1 + \overline{e_2} + y, \\ b\overline{e_1} &= d + \overline{e_2} + e_2 + e_1 + \overline{x}, \\ b\overline{e_2} &= d + \overline{e_1} + e_2 + e_1 + \overline{y}, \end{aligned}$$

for some $x, y \in B - \{1\}$ such that $x \notin \{d, \overline{e_1}, e_2, \overline{e_2}\}$ and $y \notin \{d, e_1, \overline{e_1}, \overline{e_2}\}$. Now by the above and equality $(*)$ we obtain $d^2 = 5.1 + x + \overline{x} + y + \overline{y}$. Since $(b^2, d^2) = (bd, bd) = 25$, we have $\mathrm{Supp}_B(b^2) \cap \mathrm{Supp}_B(d^2) = \{1\}$. It follows that $x \ne b, d$. Now, by the equality $b(be_1) = b^2 e_1$ we have

$$e_1 + 3x + de_1 = \overline{x} + \overline{y} + y + b + bx.$$

If $x = \overline{x} = y$, then $d^2 = 5.1 + 4x$ and $e_1 + de_1 = b + bx$. Hence $bx = e_1 + e_2 + \overline{e_1} + \overline{e_2} + k$ and $de_1 = b + \overline{e_1} + \overline{e_2} + e_2 + k$ where $k = \overline{k},\ k \in B - \{1, b, d, x, e_1, e_2, \overline{e_1}, \overline{e_2}\}$. Since $b^2x = b(bx)$, we can see that $4d + bk = x + 3k + dx$. It follows that there is a basis element s such that $bk = s + x + 3k$ and $dx = s + 4d$. If $s \ne k$ then $(bk, bk) \le 65$, $(b^2, k^2) = (5.1 + 3b + d, 5.1 + 3b + \cdots) \ge 70$. Since $(bk, bk) = (b^2, k^2)$, we get a contradiction. If $s = k$, then $bk = x + 4k$ and $dx = 4d + k$. Let x appear α times in x^2. Since $(dx, dx) = 85$, $(d^2, x^2) = (5.1 + 4x, 5.1 + \alpha x + \cdots)$, we get that $x^2 = 5.1 + 3x + l$ for some $l \in B$. But $d(dx) = 20.1 + 16x + dk$, $d^2x = 5x + 4x^2$, so we have $dk = x + 4l$. Hence $(dk, dk) = 85, (d^2, k^2) = 25$, because $b \ne x$. Since $(dk, dk) = (d^2, k^2)$, we obtain a contradiction.

If $x = \overline{x}$, $x \ne y, \overline{y}$, then $e_1 + 2x + de_1 = b + y + \overline{y} + bx$. Hence $bx = e_1 + \overline{e_1} + \alpha x + \cdots$, where $\alpha \ge 2$ is an integer number. If $\alpha = 2$, then $(bx, bx) = 35$ and $(b^2, x^2) \ge 55$. Since $(bx, bx) = (b^2, x^2)$, we get a contradiction. If $\alpha = 3$, then $(bx, bx) = 55$ and $(b^2, x^2) \ge 70$. Since $(bx, bx) = (b^2, x^2)$, we have a contradiction.

If $x = y, x \neq \overline{x}, y \neq \overline{y}$, then $e_1+2x+de_1 = 2\overline{x}+b+bx$ implies that $bx = e_1+e_2+\beta x+\cdots$, where $\beta \geq 2$. We obtain a contradiction by the equality $(bx, bx) = (b^2, x^2)$.

If $x = \overline{y}, x \neq \overline{x}, y \neq \overline{y}$, then $e_1+2x+de_1 = 2y+b+bx$. Hence we have $bx = e_1+\overline{e_2}+\gamma x+\cdots$, for some $\gamma \geq 2$. It is easy to check that this case cannot hold.

If x is different from $\overline{x}, \overline{y}, x$, then $e_1 + 3x + de_1 = \overline{x} + \overline{y} + y + b + bx$. Hence $bx = e_1 + 4x$ or $bx = e_1 + 3x + z$. If $bx = e_1 + 4x$, then $de_1 = \overline{x} + \overline{y} + y + b + x$. Since $b(bx) = b^2x$, we can see that $dx = e_1 + e_2 + \overline{e_1} + \overline{e_2} + d$. Also, since $d(bx) = 4dx + de_1, (bd)x = d(bx)$, we have

$$4x + e_1 + e_1x + e_2x + \overline{e_1}x + \overline{e_2}x = b + y + x + \overline{y} + 4e_1 + 4e_2 + 4\overline{e_1} + 4\overline{e_2} + 4d.$$

Since the intersection parameters of table algebras are non-negative, x appears at least 4 times on the left-hand side in the above equality. But x appears exactly once on the right-hand side in the above equality, which is a contradiction. If $bx = 3x + e_1 + z$, for some $z \in B - \{b, y, \overline{x}, \overline{y}\}$. This implies that b appears thrice in the decompositoion of $x\overline{x}$. Also we have

$$(bx, bx) = (e_1 + 3x + z, e_1 + 3x + z) = 55,\ (b^2, x\overline{x}) = (5.1 + 3b + d, 5.1 + 3b + \cdots) \geq 70.$$

But since $(b^2, x\overline{x}) = (bx, bx)$ we get a contradiction. This proves that case (1) cannot occur.

If (2) holds then $bd = b+3e_1+e_2$ where $e_1 = \overline{e_1}$ and $e_2 = \overline{e_2}$. Since $(b, de_1) = (e_1, bd) = 15$, b appears thrice in the decomposition of de_1. Let $d^2 = 5.1 + \alpha b + \beta d + \cdots$, for some non-negative integers α, β. Hence $(b^2, d^2) = 25 + 15\alpha + 5\beta$. By the equality $(b^2, d^2) = (bd, bd) = (b + 3e_1 + e_2, b + 3e_1 + e_2) = 55$, we have $3\alpha + \beta = 6$. This implies that $\alpha = 2, \beta = 0$ or $\alpha = 1, \beta = 3$.

If $\alpha = 2, \beta = 0$ then $d^2 = 5.1 + 2b + \cdots$. Thus $10 = (d^2, b) = (d, bd)$ that is d appears twice in the decomposition of bd, a contradiction. If $\alpha = 1, \beta = 3$ then $d^2 = 5.1+3d+b$. Since $5 = (d^2, b) = (bd, d)$ we have $e_2 = d$, i.e., $bd = 3e_1+b+d$. By the equality $b(bd) = b^2d$ we have $be_1 = 3d + 2e_1$ and by the equality $(bd)d = bd^2$ we obtain that $de_1 = 3b + 2e_1$. This shows that b appears twice in the decomposition of e_1^2 and so does d. Therefore $e_1^2 = 5.1 + 2b + 2d$, and $B_b = \{1, b, d, e_1\}$. Since b is a faithful element we obtain that $B = \{1, b, d, e_1\}$.

If (3) holds then $bd = b + 2e + 2\overline{e}$ such that $b \neq e$. We have $(bd, bd) = 45$. Let $d^2 = 5.1 + \alpha b + \beta d + \cdots$, for some non-negative integers α, β. Since $(b^2, d^2) = (5.1 + 3b + d, 5.1 + \alpha b + \beta d + \cdots) = 25 + 15\alpha + 5\beta$ and $(bd, bd) = (b^2, d^2)$, we have $3\alpha + \beta = 4$. This implies that $\alpha = 1,\ \beta = 1$ or $\alpha = 0,\ \beta = 4$. If $\alpha = 1,\ \beta = 1$ then $d^2 = 5.1 + b + d + \cdots$. Thus $(bd, d) = (d^2, b) = 5$ which implies that d appears once in the decomposition of bd which is a contradiction to the form of bd. If $\alpha = 0, \beta = 4$ then $d^2 = 5.1 + 4d$. Thus $bd^2 = 5b + 4bd$ and $(bd)d = bd + 2de + 2d\overline{e}$, therefore by the equality $bd^2 = (bd)d$ we obtain

$$5b + 3bd = 2de + 2d\overline{e}.$$

Hence by the above equality there are two cases as follow:

$$de = 2b + 3e, \quad d\overline{e} = 2b + 3\overline{e} \qquad (**)$$

$$de = 2b + 3\overline{e}, \quad d\overline{e} = 2b + 3e \qquad (***)$$

If $(**)$ holds then $(de, de) = 65$ and $(e\overline{e}, d) = (d\overline{e}, \overline{e}) = 15$ implies that d appears thrice in the decomposition of $e\overline{e}$. Since $65 = (de, de) = (d^2, e\overline{e})$ and

$$(d^2, e\overline{e}) = (5.1 + 4d, 5.1 + 3d + \cdots) = 80,$$

we have a contradiction.

If $(***)$ holds then $(d^2, e^2) = (de, d\overline{e}) = 20$ and $(e^2, d) = (d\overline{e}, e) = 15$ implies that d appears thrice in the decomposition of e^2. Since $(e^2, d^2) = 60$ we have a contradiction. This proves that case (3) cannot occur.

If (4) holds then $bd = b + 4e$, such that e is a real element. Thus $(bd, bd) = 85$. On the other hand, let $d^2 = 5.1 + \alpha b + \beta d + \cdots$, for some non-negative integers α, β. Then $(b^2, d^2) = (5.1 + 3b + d, 5.1 + \alpha b + \beta d + \cdots) = 25 + 5(3\alpha + \beta)$. Since $f(d^2) = 25$ and $(bd, bd) = (b^2, d^2)$ we have

$$3\alpha + \beta = 12, \quad \alpha + \beta \leq 4.$$

Therefore $\alpha = 4, \beta = 0$, i.e., $d^2 = 5.1 + 4b$. Moreover, $(bd, d) = (d^2, b) = 20$. Hence d appears 4 times in the decomposition of bd, i.e., $bd = b + 4d$. This shows that $B_b = \{1, b, d\}$. Since b is faithful, $B = \{1, b, d\}$.

If (5) holds, then by the equality $b(bd) = b^2 d$ we obtain

$$5.1 + d + be_1 + be_2 + be_3 + be_4 = 5d + 3e_1 + 3e_2 + 3e_3 + 3e_4 + d^2.$$

Since $(b^2, d^2) = (bd, bd) = 25$, we have $\mathrm{Supp}_B(d^2) \cap \mathrm{Supp}_B(b^2) = \{1\}$. Now we claim that e_1 does not appear in the decomposition of be_1. If $(be_1, e_1) \neq 0$, then $(be_1, e_1) = 5, 10, 15$ or 20. We assume $(be_1, e_1) = 5$. Hence e_2, e_3 or e_4 may appear in the decomposition of be_1. For example, let e_2 appear α times in be_1 for some $\alpha \in \mathbb{Z}^+$. Thus $be_1 = d + e_1 + \alpha e_2 + \cdots$. If $\alpha = 1$, then $35 \geq (be_1, be_1) = (b^2, e_1^2) = (5.1 + 3b + d, 5.1 + b + \cdots) \geq 40$ which is a contradiction. If $\alpha = 2$, then $be_1 = d + e_1 + 2e_2 + \cdots$ and $35 \geq (be_1, be_1) = (b^2, e_1^2) = (5.1 + 3b + d, 5.1 + b + \cdots) \geq 40$ which is a contradiction. If $\alpha = 3$, then $be_1 = d + e_1 + 3e_2$ and $be_2 = d + 3e_1 + \cdots$. Thus $35 \geq (be_1, be_2) = (b^2, e_1 e_2) \geq 45$, which is a contradiction. Hence, we have $(be_1, e_1) \neq 5$. By the same way the other cases are proved and so e_1 cannot occur in the decomposition of be_1. Similarly, we can prove that e_i cannot occur in the decomposition of be_i, for $i = 2, 3, 4$. Now, it is easy to check that

$$be_1 = d + e_2 + e_3 + e_4 + x,$$
$$be_2 = d + e_1 + e_3 + e_4 + y,$$
$$be_3 = d + e_1 + e_2 + e_4 + z,$$
$$be_4 = d + e_1 + e_2 + e_3 + w,$$

where $d^2 = 5.1 + x + y + z + w$. Since $b(be_1) = bd + be_2 + be_3 + be_4 + bx$, $b^2 e_1 = 5e_1 + 3be_1 + de_1$ and $b(be_1) = b^2 e_1$, we have $e_1 + 3x + de_1 = b + y + z + w + bx$. The obvious analogue of the proof of case (1) shows that $bx = e_1 + 3x + k$ and $de_1 = b + y + z + w + k$ where $k \in B - \{1, e_1, x\}$. Hence $x^2 = 5.1 + 3b + \cdots$ which implies that

$$55 = (bx, bx) = (b^2, x^2) = (5.1 + 3b + d, 5.1 + 3b + \cdots) \geq 70,$$

that is contradiction. Therefore the case (5) cannot hold.

If (6) holds, then we can prove that

$$be_1 = d + e_2 + e_3 + \overline{e_3} + x, \quad x = \overline{x}$$

$$be_2 = d + e_1 + e_3 + \overline{e_3} + y, \quad y = \overline{y}$$

$$be_3 = d + e_1 + e_2 + \overline{e_3} + z,$$

$$be_4 = d + e_1 + e_2 + e_3 + \overline{z},$$

where x, y, z are elements (not necessarily distinct) of $B - \{1\}$ and $d^2 = 5.1 + x + y + z + \overline{z}$. Since $b(be_1) = b^2e_1, b(be_1) = bd + be_1 + be_2 + be_3 + b\overline{e_3} + bx$ and $b^2e_1 = 5e_1 + 3be_1 + de_1$, we have

$$e_1 + 3x + de_1 = b + y + z + \overline{z} + bx$$

The obvious analogue of the proof of case (1) shows $bx = e_1 + 3x + k$ and $de_1 = b + y + z + \overline{z} + k$ such that $k \neq e_1, x$. It follows that $55 = (bx, bx) = (b^2, x^2) = (5.1 + 3b + d, 5.1 + 3b + \cdots) \geq 70$ which is a contradiction. Therefore the case (6) cannot occur.

Now, we consider the case (7). Let b, d appear α, β times in d^2 respectively. Since $35 = (bd, bd) = (b^2, d^2)$, we obtain $\alpha = 0, \beta = 2$, and so $d^2 = 5.1 + 2d + x + y$, for some (not necessarily distinct) elements $x, y \in B - \{1, d\}$. Since $b(bd) = b^2 + 2be_1 + be_2 + be_3$, $b^2d = 5d + 3bd + d^2$ and $b(bd) = b^2d$ we obtain

$$2be_1 + be_2 + be_3 = 6e_1 + 3e_2 + 3e_3 + 6d + x + y. \tag{I}$$

The obvious analogue of (5) shows that e_1 does not appear in the decomposition of be_1. Now, from (I) e_1 appears at least thrice in be_2. In fact, if e_1 appears 4 times in be_2, then we have $be_2 = d + 4e_1$. Hence we have $(be_1, e_2) = (be_2, e_1) = 20, (be_1, d) = 10$. It follows that $|\mathrm{Supp}_B(be_1)| \geq 6$ which is a contradiction. Thus e_1 appears thrice in the decomposition of be_2. It implies that $be_2 = d + 3e_1 + \cdots$ and $be_3 = d + 3e_1 + \cdots$. Hence e_2, e_3 both appear thrice in be_1 which is a contradiction. Therefore the case (7) cannot occur.

We can observe exactly in the same way that the cases (8) and (9) cannot occur. Now the proof of proposition is complete. □

Proposition 2.3. *Let (A, B) be a homogeneous standard integral table algebra of degree 5 such that B contains a real faithful element b so that $b^2 = 5.1 + 3c + b$. Then $B = \{1, b, c\}$ and*

$$c^2 = 5.1 + 2b + 2c, \quad bc = 3b + 2c, \quad b^2 = 5.1 + 3c + b.$$

Proof. Let $b \in B$ such that $b^2 = 5.1 + 3c + b$ and b, c are two real elements. Since $(bc, b) = (b^2, c) = 15$, b appears thrice in the decomposition of bc. Thus $bc = 3b + e_1 + e_2$, for some $e_1, e_2 \in B - \{1, b\}$. Let $c^2 = 5.1 + \gamma c + \delta b + \cdots$, for some non-negative integers δ, γ. Hence we have $\delta + \gamma \leq 4$ by the equality $f(c^2) = 25$. Moreover,

$$(b^2, c^2) = (5.1 + 3c + b, 5.1 + \gamma c + \delta b + \cdots) = 25 + 15\gamma + 5\delta. \tag{1}$$

If $e_1 = e_2 = d$ then $bc = 3b + 2d$ and so $(bc, bc) = 65$. Since $(bc, bc) = (c^2, b^2)$, we have $3\gamma + \delta = 8$ and therefore $\gamma = 2 = \delta$. This shows that $c^2 = 5.1 + 2c + 2b$ and so $(bc, c) = (c^2, b) = 10$. Hence c appears twice in the decomposition of bc. Thus $bc = 3b + 2c$. Therefore $B_b = \{1, b, c\}$. Since b is faithful, we obtain $B = \{1, b, c\}$.

If $e_1 \neq e_2$, then by (1)

$$(bc, bc) = (b^2, c^2) = (5.1 + 3c + b, 5.1 + \gamma c + \delta b + \cdots).$$

Hence $25 + 15\gamma + 5\delta = 55$. This shows that $\gamma = 1,\ \delta = 3$ or $\gamma = 2,\ \delta = 0$.
If $\gamma = 2,\ \delta = 0$ then $c^2 = 5.1 + 2c + x + y$ for some $x, y \in B - \{1, b, c\}$ and $bc = 3b + e_1 + e_2$.

First we assume that $x \neq y$. Since $(bc)c = bc^2$ we have

$$be_1 + be_2 = e_1 + e_2 + 2c + 3x + 3y. \tag{2}$$

Hence we have

$$(be_1, c) = (bc, e_1) = 5 \text{ and } (be_2, c) = (bc, e_2) = 5. \tag{3}$$

This implies that c appears once in the decomposition of be_1 and also be_2. Moreover from (2) we know that only one of e_1 or e_2 appears in be_2. Let $e_1\overline{e_2} = \alpha' c + \beta' b \cdots$ for some positive integers α', β'. Thus

$$(be_1, be_2) = (b^2, e_1\overline{e_2}) = 5(3\alpha' + \beta'). \tag{4}$$

Now from (2) we must have one of the following cases:

$$(*) \qquad \begin{cases} be_1 = c + e_1 + \alpha x + \beta y, & \text{(I)} \\ be_2 = c + e_2 + \alpha y + \beta x. & \text{(II)} \end{cases}$$

$$(**) \qquad \begin{cases} be_1 = c + e_2 + \alpha x + \beta y, & \text{(I)} \\ be_2 = c + e_1 + \alpha y + \beta x, & \text{(II)} \end{cases}$$

such that $\alpha + \beta = 3$.

Moreover, from (4) and the above eqalities we have

$$3\alpha' + \beta' = 1 + 2\alpha\beta. \tag{5}$$

Assume that $x, y \notin \{e_1, e_2\}$. If the case $(*)$ holds, then from (5) we have one of the following cases:

$$\alpha = 0,\ \beta = 3 \text{ and } \alpha' = 0,\ \beta' = 1, \tag{i}$$
$$\alpha = 1,\ \beta = 2 \text{ and } \alpha' = 1,\ \beta' = 2 \text{ or } \alpha' = 0,\ \beta' = 5, \tag{ii}$$
$$\alpha = 2,\ \beta = 1 \text{ and } \alpha' = 1,\ \beta' = 2 \text{ or } \alpha' = 0,\ \beta' = 5, \tag{iii}$$
$$\alpha = 3,\ \beta = 0 \text{ and } \alpha' = 0,\ \beta' = 1. \tag{iv}$$

If the case (i) holds, then $e_1\overline{e_2} = b + \cdots$. So $(be_2, e_1) = (e_1\overline{e_2}, b) = 5$. But this shows that e_1 appears in the decomposition of be_2, a contradiction .

If the case (ii) holds, then $be_1 = c + e_1 + 2y + x,\ be_2 = c + e_2 + 2x + y$ and $e_1\overline{e_2} = c + 2b + \cdots$. So $(be_2, e_1) = (e_1\overline{e_2}, b) = 10$. Thus e_1 appears twice in the decomposition of be_2, a contradiction. Also if (ii) holds for case $\alpha' = 0,\ \beta' = 5$ then $e_1\overline{e_2} = 5b$. This shows that $(be_2, e_1) = (b, e_1\overline{e_2}) = 25$, that is, e_1 appears 5 times in the decomposition of be_2, a contradiction.

If (iii) holds, for case $\alpha' = 1,\ \beta' = 2$ then

$$e_1\overline{e_2} = c + 2b + \cdots, \qquad be_1 = c + e_1 + 2x + y, \quad be_2 = c + e_2 + x + 2y.$$

Hence $(be_2, e_1) = (e_1\overline{e_2}, b) = 10$. This shows that e_1 appears in the decomposition of be_2, a contradiction. Also, if (iii) holds for case $\alpha' = 0,\ \beta' = 5$, then $e_1\overline{e_2} = 5b$. This shows that $(be_2, e_1) = (b, e_1\overline{e_2}) = 25$, that is, e_1 appears 5 times in be_2, a contradiction.

If the case (iv) holds, it is easy to check that e_1 appears in the decomposition of be_2 that is a contradiction. This proves that the case $(*)$ cannot occur.

Now we assume that the case $(**)$ holds. Since $(e_1\overline{e_2}, b) = (be_1, e_2) = 5$, b appears once in $e_1\overline{e_2}$. Assume that c appears α' times in $e_1\overline{e_2}$. Thus $e_1\overline{e_2} = b + \alpha' c + \cdots$ and so $(b^2, e_1\overline{e_2}) = (5.1 + 3c + b, b + \alpha' c + \cdots) = 5 + 15\alpha'$. Since

$$(b^2, e_1\overline{e_2}) = (be_1, be_2) = 5 + 10\alpha\beta,$$

we have $2\alpha\beta = 3\alpha'$. Hence one of the following cases hold:

$$\alpha = 0,\ \beta = 3 \ \text{ and } \alpha' = 0, \tag{v}$$

$$\alpha = 3,\ \beta = 0 \ \text{ and } \alpha' = 0. \tag{vi}$$

If the case (v) holds then $be_1 = e_2 + c + 3y$ and $be_2 = e_2 + c + 3x$ by $(**)$. Since $b(be_1) = b^2 e_1$ we have

$$by + b + x = e_1 c + y + e_1. \tag{6}$$

Obviously $by = 3e_1 + y + z$ and $e_1 c = b + x + z + 2e_1$ for some $z \in B - \{1, e_1\}$, by (6) and the decomposition of be_1. Since $(\overline{y}e_1, b) = (by, e_1) = 15$, b appears thrice in the decomposition of $\overline{y}e_1$. Therefore $(e_1\overline{y}, bc) \geq 45$. Also $(by, e_1c) = (3e_1 + y + z, b + 2e_1 + x + z) \leq 40$. Since $(\overline{y}e_1, bc) = (by, e_1c)$ we have a contradiction.

If the case (vi) holds, then $be_1 = e_2 + c + 3x$ and $be_2 = e_1 + c + 3y$. Since $b(be_1) = b^2 e_1$ we have $bx + b + y = e_1 + x + e_1 c$. Hence

$$bx = x + z + 3e_1, \qquad e_1 c = b + 2e_1 + y + z,$$

for some $z \in B - \{1, e_1\}$. Thus $(bx, e_1c) = (x + z + 3e_1, b + 2e_1 + y + z) \leq 40$. On the other hand, $(bc, \overline{e_1}x) = (bx, e_1c)$ leading to contradiction.

If $e_1 = x$ then $c^2 = 5.1 + 2c + e_1 + y$ and $bc = 3b + e_1 + e_2$. Since $b(bc) = b^2 c$, we have

$$be_1 + be_2 = 2c + 4e_1 + 3y + e_2. \tag{7}$$

Hence

$$be_1 = c + e_2 + \alpha e_1 + \beta y \quad \text{and} \quad be_2 = c + \alpha' e_1 + \beta' y, \tag{8}$$

such that $\alpha' + \alpha = 4$, $\beta' + \beta = 3$, or we must have

$$be_2 = c + e_2 + \alpha' e_1 + \beta' y \quad \text{and} \quad be_1 = c + \alpha e_1 + \beta y, \tag{9}$$

such that $\alpha' + \alpha = 4, \beta' + \beta = 3$.

If (8) holds, then we have

$$be_1 = c + 3e_1 + e_2 \quad \text{and} \quad be_2 = c + e_1 + 3y. \tag{8.1}$$

If (9) holds, then we have

$$be_1 = c + 4e_1 \quad \text{and} \quad be_2 = c + e_2 + 3y. \tag{9.1}$$

First assume that (8.1) holds. Then

$$(be_1, be_1) = (c + 3e_1 + e_2, c + 3e_1 + e_2) = 55.$$

Also, $(b^2, e_1\overline{e_1}) = (5.1 + 3c + b, 5.1 + 3b + \lambda c + \cdots) = 40 + 15\lambda$. Since $(b^2, e_1\overline{e_1}) = (be_1, be_1)$, we must have $\lambda = 1$, that is, $e_1\overline{e_1} = 5.1 + 3b + c$. By the equality $b(be_1) = b^2e_1$ we have $e_1c = b + c + e_1 + e_2 + y$. Moreover

$$(c^2, e_1\overline{e_1}) = (5.1 + 2c + e_1 + y, 5.1 + 3b + c) = 35,$$

and that $(e_1c, e_1c) = 25$. Since $(e_1c, e_1c) = (c^2, e_1\overline{e_1})$ we have a contradiction.

Next if the case (9.1) holds then $(be_1, be_1) = 85$. Moreover, $(b^2, e_1\overline{e_1}) = (5.1+3c+b, 5.1+4b) = 45$. Now by the equality $(b^2, e_1\overline{e_1}) = (be_1, be_1)$ we have a contradiction.

Similary if $e_1 = y$ we will obtain a contradiction and so this cases cannot occure.

Now if $\{e_1, e_2\} = \{x, y\}$ then without loss of generality we can assume that $x = e_1$ and $y = e_2$. Hence $c^2 = 5.1 + 2c + e_1 + e_2$ and $bc = 3b + e_1 + e_2$. By the equality $b(bc) = b^2c$ we must have

$$be_1 + be_2 = 2c + 4e_1 + 4e_2.$$

Hence we have one of the following cases:

$$be_1 = c + e_1 + 3e_2, \qquad be_2 = c + 3e_1 + e_2 \tag{2.1}$$

$$be_1 = be_2 = c + 2e_1 + 2e_2, \tag{2.2}$$

$$be_1 = c + 3e_1 + e_2, \qquad be_2 = c + 3e_2 + e_1. \tag{2.3}$$

If the case (2.1) holds then by the equality $(be_1, be_2) = (b^2, e_1\overline{e_1})$, since $(be_1, be_2) = 35$ and $(b^2, e_2\overline{e_1}) = (5.1+3b+c, 3b+\cdots) \geq 45$, we have a contradiction. If the case (2.2) holds, then $(be_1, be_1) = (c+2e_1+2e_2, c+2e_1+2e_2) = 45$, and $(b^2, e_1\overline{e_1}) = (5.1+3b+c, 5.1+2b+\ldots) \geq 55$. Since $(be_1, be_1) = (b^2, e_1\overline{e_1})$ we have a contradiction.

By similar considerations we see that the case (2.3) cannot hold. Therefore we have proved that $\{x, y\} \cap \{e_1, e_2\} = \emptyset$.

Now we assume that $x = y$. Then $c^2 = 5.1+2c+2x$, $b^2 = 5.1+3c+b$ and $bc = 3b+e_1+e_2$. By the equality $b^2c = b(bc)$ we have

$$be_1 = c + e_2 + 3x \quad \text{and} \quad be_2 = c + e_1 + 3x.$$

Hence $15 = (be_1, x) = (bx, e_1)$ and $15 = (be_2, x) = (bx, e_2)$. This shows that e_1 and e_2 appear thrice in bx. But $f(bx) = 25$ and $f(3e_1 + 3e_2) = 30$, that is a contradiction. Similarly if $\gamma = 1, \delta = 3$ we will obtain a contradiction. The proposition is proved now. □

Proposition 2.4. *There is no homogeneous standard integral table algebra of degree 5 such that its table basis contains a real faithful element, say b, where*

$$b^2 = 5.1 + 3c + d, \quad d \neq c, \ d = \overline{d}, \ c = \overline{c}.$$

Proof. Let $b \in B$ such that $b^2 = 5.1 + 3c + d$, and c, d be two distinct real elements of B. Since $15 = (b^2, c) = (bc, b)$ and $f(bc) = 25$, one of the following cases holds:

$$bc = 3b + 2e, \ e \in B - \{b\}, \ e = \overline{e}, \tag{1}$$

$$bc = 3b + e_1 + e_2, \ e_1, e_2 \in B - \{b\}, \ e_1 \neq e_2. \tag{2}$$

If the case (1) holds, then $bc = 3b + 2e, e = \overline{e}$. Hence

$$(bc, bc) = (3b + 2e, 3b + 2e) = 65.$$

Let $c^2 = 5.1 + \alpha c + \beta d \cdots$ for some non-negative integers α, β. Since $f(c^2) = 25$, we have $\alpha + \beta \leq 4$. Moreover

$$(b^2, c^2) = (5.1 + 3c + d, 5.1 + \alpha c + \beta d + \cdots) = 25 + 15\alpha + 5\beta.$$

By the equality $(b^2, c^2) = (bc, bc)$ we have $3\alpha + \beta = 8$ and $\alpha + \beta \leq 4$. This shows that $\alpha = 2 = \beta$. Hence $c^2 = 5.1 + 2c + 2d$. Since $10 = (c^2, d) = (dc, c)$, c appears twice in the decomposition of dc. Now by the equality $b(bc) = b^2 c$ we must have

$$2be = 2c + 3d + dc \tag{1.1}$$

Obviously, from (1.1) d must appear in the decomposition of dc. Let d appear λ times in dc, so $dc = 2c + \lambda d + \cdots$. By (1.1) we have $be = 2c + \gamma d + \cdots$ such that $2\gamma = 3 + \lambda$. Hence $\lambda = 1, \gamma = 2$ or $\lambda = 3, \gamma = 3$.

If $\lambda = 1, \gamma = 2$ then $dc = 2c + d + 2x$ and $be = 2c + 2d + x$ for some $x \in B - \{c, d\}$ by (1.1). Hence

$$(dc, dc) = (2c + d + 2x, 2c + d + 2x) = 45.$$

Let $d^2 = 5.1 + \delta d + c + \cdots$, for some non-negative integers δ. We have

$$(d^2, c^2) = (5.1 + c + \delta d + \cdots, 5.1 + 2c + 2d) = 25 + 10(1 + \delta).$$

Since $(dc, dc) = (d^2, c^2)$, we have $\delta = 1$ and so $d^2 = 5.1 + c + d + \cdots$. Since $(bd, bd) = (b^2, d^2) = 45$, $f(bd) = 25$, $10 = (be, d) = (e, bd)$ and $5 = (b^2, d) = (bd, b)$, we have $bd = b + 2e + 2e_1$ for some $e_1 \in B - \{1, b, e\}$. Hence $d^2 = 5.1 + c + d + v + w$ for some $v, w \in B - \{1, c, d\}$. Since $b(bd) = b^2 d$ we must have $be_1 = 2x + 2d + v$ and $d^2 = 5.1 + c + d + 2v$. Let $ec = 2b + \delta c + \varepsilon e_1 + \cdots$ for some non-negative integers δ, ε by the equality $bc = 2c + 2d + x$. Hence $(bd, ec) = (b + 2e + 2e_1, 2b + \delta e + \varepsilon e_1 + \cdots) = 10 + 10(\delta + \varepsilon)$. Since $(bd, ec) = (dc, be) = (2c + 2x + d, 2c + 2d + x) = 40$, we have $\delta + \varepsilon = 3$. Moreover,

$$40 = (c^2, be) = (bc, ec) = (3b + 2e, 2b + \delta e + \varepsilon e_1 + \cdots).$$

Hence $\delta = 1$ and $\varepsilon = 2$. Therefore we have

$$ec = 2b + 2e_1 + e.$$

On the other hand $45 = (ec, ec) = (e^2, c^2) = (5.1 + c + \cdots, 5.1 + 2c + 2d)$. So we must have $e^2 = 5.1 + c + d + \cdots$. This shows that $e^2 = d^2$, by the equality $(bc)^2 = b^2 c^2$. Furthermore, $(bc)e = 3be + 2e^2$ and $(ec)b = 2b^2 + be + 2be_1$ which imply that $e^2 = 5.1 + c + d + v + x$, by the equality $(bc)e = (ec)b$. This implies that $v = x$. Hence

$$(d^2, e^2) = (5.1 + c + d + 2x, 5.1 + c + d + 2x) = 55.$$

But $(de, de) \leq 45$. Since $(d^2, e^2) = (de, de)$ we have a contradiction.

Now we assume that $\lambda = \gamma = 3$. Hence $be = dc = 2c + 3d$. We have $15 = (dc, d) = (d^2, c)$, this implies that c appears thrice in the decomposition of d^2. Also $15 = (be, d) = (bd, e)$ implies that e appears thrice in the decomposition of bd. Hence we have $d^2 = 5.1+3c+g$, $bd = 3e + b + h$, for some $h, g \in B - \{1, b, c\}$. Therefore $(bd, bd) = 55$ and

$$(b^2, d^2) = (5.1 + 3c + d, 5.1 + 3c + g) = \begin{cases} 75 & \text{if } g = d, \\ 70 & \text{if } g \neq d. \end{cases}$$

Since $(b^2, d^2) = (bd, bd)$ we have a contradiction.

If the case (2) holds, then $bc = 3b+e_1+e_2$, $e_1 \neq e_2$. Let $c^2 = 5.1+\lambda c+\gamma d+\cdots$, for some non-negative integers λ, γ. We have $(b^2, c^2) = (5.1+3c+d, 5.1+\lambda c+\gamma d+\cdots) = 25+5(3\lambda+\gamma)$, and $(bc, bc) = 55$. Since $(b^2, c^2) = (bc, bc)$, we have $6 = 3\lambda + \gamma$, such that $\lambda + \gamma \leq 4$ by $f(c^2) = 25$. Hence, $\lambda = 1, \gamma = 3$ or $\lambda = 2, \gamma = 0$.

If $\lambda = 1, \gamma = 3$ then $c^2 = 5.1+3d+c$. This shows that c appears thrice in the decomposition of dc. Since $b^2c = b(bc)$, we must have $be_1 + be_2 + c = 6d + dc$. Hence $be_1 = \alpha' d + \cdots$ and $be_2 = \beta' d+\cdots$, where $\alpha' + \beta' = 6$. Since $5\alpha' = (be_1, d) = (bd, e_1)$ and $5\beta' = (be_2, d) = (bd, e_2)$, we have that e_1, e_2 appear α' and β' times in the decomposition of bd respectively. But by the equalities $f(bd) = 25$ and $f(\alpha' e_1 + \beta' e_2) = 5(\alpha' + \beta') = 30$ that is a contradiction.

If $\lambda = 2, \gamma = 0$, then $bc = 3b+e_1+e_2$ and $c^2 = 5.1+2c+x+y$ for some $x, y \in B-\{1, c, d\}$ By the equality $b(bc) = b^2c$ we have

$$be_1 + be_2 + 3d = dc + 2c + 3x + 3y.$$

By the above equality we can see that d appears δ times in the decomposition of dc such that $3 \leq \delta \leq 5$.

If $\delta = 3$ then $dc = 3d + \cdots$. Hence $d^2 = 5.1 + 3c + x$. Therefore

$$(bd, bd) = (b^2, d^2) = \begin{cases} 75 & \text{if } x = d, \\ 70 & \text{if } x \neq d. \end{cases}$$

On the other hand $(bd, b) = (b^2, d) = 5$. So $bd = b + \cdots$. This implies that (bd, bd) cannot be equal to 70 or 75, a contradiction.

If $\delta = 4$ then $dc = 4d + x$, for some $x \in B - \{1, d\}$. Hence $20 = (dc, d) = (d^2, c)$, and so $d^2 = 5.1+4c$. Therefore $(d^2, c^2) = (5.1+4c, 5.1+c+3d) = 65$, and $(dc, dc) = (4d+x, 4d+x) = 85$. Since $(d^2, c^2) = (dc, dc)$ we have a contradiction.

If $\delta = 5$ then $dc = 5d$. Hence $25 = (dc, d) = (d^2, c)$, i.e., c appears 5 times in the decomposition of d^2. Since $f(d^2) = 25$ we have a contradiction. This completes the proof of the proposition. □

Proposition 2.5. *There is no homogeneous standard integral table algebra (A, B) of degree 5 such that B contains a faithful element b where*

$$b\overline{b} = 5.1 + 3c + d, \quad c = \overline{c}, \quad d = \overline{d}.$$

Proof. Let $b \in B$ such that $b\overline{b} = 5.1 + 3c + d$, for some c, $d \in B - \{1, b\}$ and $c = \overline{c}, d = \overline{d}$. Hence $(b\overline{b}, b\overline{b}) = 75$. Since $(b\overline{b}, b\overline{b}) = (b^2, b^2)$, we have $(b^2, b^2) = 75$. But this is a contradiction, because b^2 has one of the following cases:

$$b^2 = \begin{cases} 5c, & \text{then } (b^2, b^2) = 125. \\ 4c + d, & \text{then } (b^2, b^2) = 85. \\ 3c + 2d, & \text{then } (b^2, b^2) = 65. \\ 3c + d + e, & \text{then } (b^2, b^2) = 55. \\ 2c + 2d + e, & \text{then } (b^2, b^2) = 45. \\ 2c + d + e + g, & \text{then } (b^2, b^2) = 35. \\ c + d + e + g + h, & \text{then } (b^2, b^2) = 25. \end{cases}$$

This completes the proof of the proposition. □

Acknowledgment. Both authors would like to thank the referee for his suggestions. The first author would like to thank the Institute for Studies in Physics and Mathematics for its financial support. The second author would like to thank the Institute for Advanced Studies in Basic Sciences, Gava Zang, Zanjan, for its financial support.

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Received March 9, 1998