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The Root Space Decomposition of the Quadratic Lie Superalgebras

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Abstract. A quadratic Lie superalgebra is a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with a non-degenerate, supersymmetric, even and $\mathfrak{g}$-invariant bilinear form B, B is called an invariant scalar product of $\mathfrak{g}$. In this paper, we study properties of the decomposition of a quadratic Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ relative to a fixed Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$. Finally, we give two characterizations of the basic classical Lie superalgebras among the quadratic Lie superalgebras.

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0. Introduction

In this work, we consider finite dimensional Lie superalgebras over an algebraically closed commutative field $\mathbb{K}$ of characteristic 0.

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra over $\mathbb{K}$. We denote by $\mathrm{Der}(\mathfrak{g})$ the Lie superalgebra of superderivations of $\mathfrak{g}$, by $\mathfrak{Z}(\mathfrak{g})$ the center of $\mathfrak{g}$, and by $\mathfrak{R}(\mathfrak{g})$ the greatest solvable graded ideal of $\mathfrak{g}$ called the radical of $\mathfrak{g}$.

A quadratic Lie superalgebra is a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with a non-degenerate, supersymmetric, even and $\mathfrak{g}$-invariant bilinear form B, B is called an invariant scalar product of $\mathfrak{g}$. In [5], we showed that a Lie superalgebra $\mathfrak{g}$ is quadratic if and only if $\mathfrak{g}_{\bar{0}}$ is a quadratic Lie algebra with an invariant scalar product B_0 and on the $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ there exists a bilinear antisymmetric non-degenerate $\mathfrak{g}_{\bar{0}}$-invariant form B_1 such that

$$B_0([X,Y],Z) = B_1(X,[Y,Z]) \quad \forall X,Y \in \mathfrak{g}_{\bar{1}},\ \forall Z \in \mathfrak{g}_{\bar{0}}.$$

The semisimple Lie algebras and the basic classical Lie superalgebras(see [12], [13]) are quadratic. But, many solvable Lie superalgebras also belong to this class (see [4], [5], [14]). The quadratic Lie superalgebras appear, in particular, in the notion of Lie bi-superalgebras (see [1] and [11]) and in physical models based on Lie superalgebras (see [9], [10], [15]). In [4] and [5], we studied the structure of quadratic Lie superalgebras and we obtained inductive classifications of some important quadratic Lie superalgebras. Recall that in [14], using the notion of double extension, A. Medina and Ph. Revoy gave the inductive classification of the quadratic Lie algebras.

The information about the root space decomposition of a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ relative to a fixed Cartan subalgebra H of $\mathfrak{g}_{\bar{0}}$ is essential in the understanding of the structure of $\mathfrak{g}$. This decomposition is defined as follows:

$$\mathfrak{g} = \bigoplus_{\lambda \in H^*} \mathfrak{g}^{\lambda}, \text{ where } \mathfrak{g}^{\lambda} = \{x \in \mathfrak{g} : \exists n \in \mathbb{N},\ (\mathrm{ad}_{\mathfrak{g}}(h) - \lambda(h)\,\mathrm{id}_{\mathfrak{g}})^n(x) = 0\ \forall h \in H\}.$$

The aim of the second section of this paper is to study the root decomposition of an arbitrary quadratic Lie superalgebra $\mathfrak{g}$. We shall exploit the existence of the invariant scalar product to get informations on the roots of $\mathfrak{g}$ as well as this root space decompsition. Recall that other authors studied root space decompositions of Lie superalgebras which belong to subsets of the set of quadratic Lie superalgebras : 1) V. Kac in [12], [13] and M. Scheunert in [16] studied the root space decomposition of the basic classical Lie superalgebras, 2) in [16], M. Scheunert gave certain properties of the root space decomposition of an arbitrary quadratic Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ such that $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebras and the $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ is semisimple, and 3) in [2], V. V. Astrakhantsev studied the root space decomposition of the quadratic Lie algebra.

Using the results obtained in second section, we give in the third section two characterizations of the basic classical Lie superalgebra among the quadratic Lie superalgebras, these characterizations are:

1) Theorem 3.1. *Let $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ be a B-irreducible quadratic Lie superalgebra such that the $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ is semisimple , H a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$ and Δ the set of the roots of $\mathfrak{g}$ relative to H. Then $\mathfrak{g}$ is a basic classical Lie superalgebra if and only if Δ generates the linear dual H^* of H.*

2) Theorem 3.2. *Let $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ be a quadratic B-irreducible Lie superalgebra, H a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$. Then $\mathfrak{g}$ is a basic classical Lie superalgebras if and only if $\mathfrak{z}(\mathfrak{g}) = \{0\}$ and $\mathrm{ad}_{\mathfrak{g}}\, h$ is semisimple for all $h \in H$.*

1. Definitions and preliminary results

We recall in this section some definitions and results needed in this paper.

Definition 1.1. *Let $\mathfrak{g}$ be a Lie superalgebra and let B be a bilinear form on $\mathfrak{g}$.*

i) *B is called supersymmetric if $B(X,Y) = (-1)^{xy}B(Y,X),\ \forall X \in \mathfrak{g}_x,\ \forall Y \in \mathfrak{g}_y$.*

ii) *B is called $\mathfrak{g}$-invariant if $B([X,Y],Z) = B(X,[Y,Z]),\ \forall X,Y,Z \in \mathfrak{g}$.*

iii) *B is called even if* $B(X,Y)=0,\ \forall X\in\mathfrak{g}_{\bar 0},\ \forall Y\in\mathfrak{g}_{\bar 1}$.

iv) *Let D be homogeneous superderivation of $\mathfrak{g}$ of degree d. D is called superantisymmetric if* $B(D(X),Y)=-(-1)^{dx}B(X,D(Y)),\ \forall X\in\mathfrak{g}_x,\ \forall Y\in\mathfrak{g}$.

We denote by $\mathrm{Der}_a(\mathfrak{g})$ the vector subspace of $\mathrm{Der}(\mathfrak{g})$ generated by all superantisymmetric superderivations of $\mathfrak{g}$.

Definitions 1.2.

i) *Let $\mathfrak{g}$ be a Lie superalgebra with bilinear form B. $(\mathfrak{g},B)$ is called quadratic if B is supersymmetric, even, non-degenerate and $\mathfrak{g}$-invariant. In this case, B is called an invariant scalar product on $\mathfrak{g}$.*

ii) *Let $(\mathfrak{g},B)$ be a quadratic Lie superalgebra. A graded ideal $\mathfrak{I}$ of $\mathfrak{g}$ is called non-degenerate (resp. degenerate) if the restriction of B to $\mathfrak{I}\times\mathfrak{I}$ is a non-degenerate (resp. degenerate) bilinear form.*

iii) *We say that a quadratic Lie superalgebra $(\mathfrak{g},B)$ is B-irreducible if $\mathfrak{g}$ contains no non-trivial non-degenerate graded ideal.*

The following proposition reduces the study of quadratic Lie superalgebras to those which have no non-trivial non-degenerate ideal.

Proposition 1.1. [5] *Let $(\mathfrak{g},B)$ be a quadratic Lie superalgebra. Then, $\mathfrak{g}=\bigoplus_{i=1}^{n}\mathfrak{g}_i$ such that:*

(i) *$\mathfrak{g}_i$ is a non-degenerate graded ideal of $\mathfrak{g}$, for all $i\in\{1,\dots,n\}$,*

(ii) *$(\mathfrak{g}_i, B_i=B_{|\mathfrak{g}_i\times\mathfrak{g}_i})$ is B_i-irreducible, for all $i\in\{1,\dots,n\}$,*

(iii) *$B(\mathfrak{g}_i,\mathfrak{g}_j)=\{0\}$ (i. e. $\mathfrak{g}_i$ and $\mathfrak{g}_j$ are B-orthogonal), for all $i,j\in\{1,\dots,n\}$ such that $i\neq j$.*

Definition 1.3. i) *A simple Lie superalgebra $\mathfrak{g}=\mathfrak{g}_{\bar 0}\oplus\mathfrak{g}_{\bar 1}$ is called classical if the $\mathfrak{g}_{\bar 0}$-module $\mathfrak{g}_{\bar 1}$ is semisimple.*
ii) *A quadratic simple classical Lie superalgebra is called a basic classical Lie superalgebra.*

Now, we recall some extensions of Lie superalgebras introduced in [8] in order to study the Lie superalgebras $\mathfrak{g}=\mathfrak{g}_{\bar 0}\oplus\mathfrak{g}_{\bar 1}$ with $\mathfrak{g}_{\bar 0}$ reductive and completely reducible action of $\mathfrak{g}_{\bar 0}$ in $\mathfrak{g}_{\bar 1}$.

Definition 1.4. *Let $\mathfrak{g}=\mathfrak{g}_{\bar 0}\oplus\mathfrak{g}_{\bar 1}$ be a Lie superalgebra, A a vector space, and let μ be a bilinear symmetric map from $\mathfrak{g}_{\bar 1}\times\mathfrak{g}_{\bar 1}$ to A satisfying*

$$\mu([z,x],y)+\mu(x,[z,y])=0,\ \forall\ z\in\mathfrak{g}_{\bar 0},\ x,y\in\mathfrak{g}_{\bar 1}.$$

On the $\mathbb{Z}_2$-graded vector space $\mathfrak{L}=\mathfrak{L}_{\bar 0}\oplus\mathfrak{L}_{\bar 1}$, where $\mathfrak{L}_{\bar 0}=\mathfrak{g}_{\bar 0}\oplus A$ and $\mathfrak{L}_{\bar 1}=\mathfrak{g}_{\bar 1}$, we define a Lie superalgebra structure by:

$[\mathfrak{L},A]=\{0\}$;

$[u,v] = [u,v]_{\mathfrak{g}}$, where $[,]_{\mathfrak{g}}$ is the multiplication on $\mathfrak{g}$, for any $u \in \mathfrak{g}_\alpha$, $v \in \mathfrak{g}_\beta$ such that $\alpha = \bar{0}$ or $\beta = \bar{0}$;

$[u,v] = [u,v]_{\mathfrak{g}} + \mu(u,v)$, for any $u, v \in \mathfrak{g}_{\bar{1}}$.

The Lie superalgebra $(\mathfrak{L}, [,])$ will be called an *elementary even extension* of $\mathfrak{g}$ by (A, μ).

Proposition 1.2. [8] *Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra with $\mathfrak{g}_{\bar{0}} = A \oplus [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$ (direct sum of ideals) and let $\mathfrak{U} = \mathfrak{U}_{\bar{0}} \oplus \mathfrak{U}_{\bar{1}}$ be another Lie superalgebra with abelian $\mathfrak{U}_{\bar{0}}$, $[\mathfrak{U}_{\bar{1}}, \mathfrak{U}_{\bar{1}}] = \{0\}$ and completely reducible action of $\mathfrak{U}_{\bar{0}}$ in $\mathfrak{U}_{\bar{1}}$, so that $\mathfrak{U}_{\bar{1}} = \oplus_{\lambda \in \Lambda} \mathfrak{U}^\lambda$, $\Lambda \subset (\mathfrak{U}_{\bar{0}})^\star$, and $\mathfrak{U}^\lambda = \{u \in \mathfrak{U}_{\bar{1}} : [x,u] = \lambda(x)u \ \forall x \in \mathfrak{U}_{\bar{0}}\}$. Let each $\mathfrak{U}^\lambda$ be equipped with an A-module structure, so that on the $\mathbb{Z}_2$-graded vector space $\mathfrak{L} = \mathfrak{L}_{\bar{0}} \oplus \mathfrak{L}_{\bar{1}}$, where $\mathfrak{L}_{\bar{0}} = \mathfrak{U}_{\bar{0}} \oplus \mathfrak{g}_{\bar{0}}$ and $\mathfrak{L}_{\bar{1}} = \mathfrak{U}_{\bar{1}} \oplus \mathfrak{g}_{\bar{1}}$, we define the Lie superalgebra structure by:*

$[\mathfrak{g}_{\bar{1}}, \mathfrak{U}] = [\mathfrak{U}_{\bar{0}}, \mathfrak{g}_{\bar{0}}] = [[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}], \mathfrak{U}] = \{0\}$.

The product $[a,u]$ for $a \in A$ and $u \in \mathfrak{U}^\lambda$ is given by the corresponding A-module structure in $\mathfrak{U}^\lambda$;

$[u,v] = [u,v]_{\mathfrak{U}}$, where $[,]_{\mathfrak{U}}$ is the multiplication on $\mathfrak{U}$, for any $u, v \in \mathfrak{U}$.
The Lie superalgebra $(\mathfrak{L}, [,])$ will be called a nice extension of $\mathfrak{g}$ by $\mathfrak{U}$.
$\mathfrak{g}$ is a Lie subsuperalgebra of $\mathfrak{L}$ and $\mathfrak{U}$ is a graded ideal of $\mathfrak{L}$.

The following is the main result of [8].

Theorem 1.1. [8] *Let $\mathfrak{L} = \mathfrak{L}_{\bar{0}} \oplus \mathfrak{L}_{\bar{1}}$ be a Lie superalgebra with $\mathfrak{L}_{\bar{0}}$ reductive and completely reducible action of $\mathfrak{L}_{\bar{0}}$ in $\mathfrak{L}_{\bar{1}}$. Then, $\mathfrak{L}$ is an elementary even extension of a nice extension of a semisimple Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with completely reducible action of $\mathfrak{g}_{\bar{0}}$ in $\mathfrak{g}_{\bar{1}}$.*

Lemma 1.1. [4] *Let $(\mathfrak{g}, B)$ be a quadratic Lie superalgebra.*

i) *If $\mathfrak{I}$ is a graded ideal of $\mathfrak{g}$, then $\mathfrak{I}^\perp$ is a graded ideal of $\mathfrak{g}$.*
ii) $[\mathfrak{g}, \mathfrak{g}]^\perp = \mathfrak{z}(\mathfrak{g})$.
iii) *If $\mathfrak{H}$ is a semisimple graded ideal of $\mathfrak{g}$, then $\mathfrak{H}$ is non-degenerate and $[\mathfrak{H}, \mathfrak{H}] = \mathfrak{H}$.*

Proof. i) Let $x \in \mathfrak{I}^\perp$, $y \in \mathfrak{g}$ and $z \in \mathfrak{I}$. By invariance of B, we have $B([x,y],z) = B(x,[y,z]) = 0$ and so $[x,y] \in \mathfrak{I}^\perp$. It follows that $\mathfrak{I}^\perp$ is an ideal of $\mathfrak{g}$. Since $\mathfrak{I}$ is graded and B is even, then $\mathfrak{I}^\perp$ is graded.

ii) The invariance of B implies that $B(\mathfrak{z}(\mathfrak{g}), [\mathfrak{g}, \mathfrak{g}]) = B([\mathfrak{z}(\mathfrak{g}), \mathfrak{g}], \mathfrak{g}) = \{0\}$, then $\mathfrak{z}(\mathfrak{g})$ is a subset of $[\mathfrak{g}, \mathfrak{g}]^\perp$. Let $x \in [\mathfrak{g}, \mathfrak{g}]^\perp$, and let $y, z \in \mathfrak{g}$, then $B([x,y],z) = B(x,[y,z]) = 0$. Consequently $[x,y] = 0$ for all $y \in \mathfrak{g}$, so $x \in \mathfrak{z}(\mathfrak{g})$. We conclude that $[\mathfrak{g}, \mathfrak{g}]^\perp = \mathfrak{z}(\mathfrak{g})$.

iii) $B([\mathfrak{H} \cap \mathfrak{H}^\perp, \mathfrak{H} \cap \mathfrak{H}^\perp], \mathfrak{g}) = B(\mathfrak{H} \cap \mathfrak{H}^\perp, [\mathfrak{H} \cap \mathfrak{H}^\perp, \mathfrak{g}]) = \{0\}$, then $[\mathfrak{H} \cap \mathfrak{H}^\perp, \mathfrak{H} \cap \mathfrak{H}^\perp] = \{0\}$. Therefore $\mathfrak{H} \cap \mathfrak{H}^\perp = \{0\}$ (because $\mathfrak{H}$ is semisimple), thus $\mathfrak{H}$ is a non-degenerate graded ideal of $\mathfrak{g}$. Since $\mathfrak{z}(\mathfrak{H}) = \{0\}$, then, by ii), $[\mathfrak{H}, \mathfrak{H}]^\perp \cap \mathfrak{H} = \{0\}$. It follows that $\dim \mathfrak{H} = \dim [\mathfrak{H}, \mathfrak{H}]$, so $[\mathfrak{H}, \mathfrak{H}] = \mathfrak{H}$. □

In the following theorem, we gave a characterization of the basic classical Lie superalgebras among the quadratic Lie superalgebras $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ such that $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra and the $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ is semisimple.

Theorem 1.2. [4] *Let $(\mathfrak{g}, B)$ be a B-irreducible quadratic Lie superalgebra such that $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra and the $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ is semisimple. Then, $\mathfrak{g}$ is simple if and only if $\mathfrak{z}(\mathfrak{g}) = \{0\}$.*

Proof. If $\mathfrak{g}$ is simple, then $\mathfrak{z}(\mathfrak{g}) = \{0\}$. Conversely, suppose that $\mathfrak{z}(\mathfrak{g}) = \{0\}$. If $\mathfrak{g}_{\bar{1}} = \{0\}$, then $\mathfrak{g}$ is a semisimple Lie algebra. Since every ideal of $\mathfrak{g}$ is semisimple, then by Lemma 1.1, $\mathfrak{g}$ is simple because $\mathfrak{g}$ is B-irreducible.

Assume that $\mathfrak{g}_{\bar{1}} \neq \{0\}$. First, we show that $\mathfrak{g}_{\bar{0}} = [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$. Since $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$ is an ideal of $\mathfrak{g}_{\bar{0}}$, then $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$ is reductive and it follows that $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = \mathfrak{S} \oplus \mathfrak{z}([\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}])$ where $\mathfrak{S}$ is the greatest semisimple ideal of $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$. The equality $[\mathfrak{S}, \mathfrak{S}] = \mathfrak{S}$ implies that $\mathfrak{S}$ is an ideal of $\mathfrak{g}_{\bar{0}}$, and by Lemma 1.1, $\mathfrak{S}$ is a non-degenerate ideal of $\mathfrak{g}_{\bar{0}}$.

Let $x \in \mathfrak{z}([\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}])$ such that $B(x, \mathfrak{z}([\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]) = \{0\}$. By invariance of B, $B(x, \mathfrak{S}) = \{0\}$ because $[\mathfrak{S}, \mathfrak{S}] = \mathfrak{S}$, consequently $B(x, [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]) = \{0\}$. Thus $B([x, \mathfrak{g}_{\bar{1}}], \mathfrak{g}_{\bar{1}}) = \{0\}$, so $[x, \mathfrak{g}_{\bar{1}}] = \{0\}$. The inclusion $\mathfrak{z}([\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]) \subset \mathfrak{z}(\mathfrak{g}_{\bar{0}})$ implies that $x \in \mathfrak{z}(\mathfrak{g}) = \{0\}$. We conclude that $\mathfrak{z}([\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}])$ is a non-degenerate ideal of $\mathfrak{g}_{\bar{0}}$. It follows that $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$ is a non-degenerate ideal of $\mathfrak{g}_{\bar{0}}$ because $\mathfrak{S}$ and $\mathfrak{z}([\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}])$ are non-degenerate ideals of $\mathfrak{g}_{\bar{0}}$, $B(\mathfrak{S}, \mathfrak{z}([\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}])) = \{0\}$ and $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = \mathfrak{S} \oplus \mathfrak{z}([\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}])$. Since $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$ is a non-degenerate ideal of $\mathfrak{g}_{\bar{0}}$, we obtain that $\mathfrak{I} = [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \oplus \mathfrak{g}_{\bar{1}}$ is a non-degenerate ideal of $(\mathfrak{g}, B)$, consequently $\mathfrak{g} = \mathfrak{I}$ because $(\mathfrak{g}, B)$ is B-irreducible and $\mathfrak{I} \neq \{0\}$. We conclude that $\mathfrak{g}_{\bar{0}} = [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$.

By Theorem 1.1, $\mathfrak{g}$ is an elementary even extension of a nice extension of a semisimple Lie superalgebra $\mathfrak{H} = \mathfrak{H}_{\bar{0}} \oplus \mathfrak{H}_{\bar{1}}$ such that the $\mathfrak{H}_{\bar{0}}$-module $\mathfrak{H}_{\bar{1}}$ is semisimple. Since $\mathfrak{z}(\mathfrak{g}) = \{0\}$, then $\mathfrak{g}$ is a nice extension of $\mathfrak{H}$. The fact that $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = \mathfrak{g}_{\bar{0}}$ implies that $\mathfrak{g} = \mathfrak{H} \oplus \mathfrak{U}$ where $\mathfrak{U}$ is a Lie superalgebra which is a graded ideal of $\mathfrak{g}$ and $\mathfrak{H}$ is a semisimple Lie sub-superalgebra of $\mathfrak{g}$ such that the $\mathfrak{H}_{\bar{0}}$-module $\mathfrak{H}_{\bar{1}}$ is semisimple and $[\mathfrak{H}, \mathfrak{U}] = \{0\}$, so $\mathfrak{H}$ is a graded ideal of $\mathfrak{g}$. Since $\mathfrak{H}$ is semisimple, then by Lemma 1.1, $\mathfrak{H}$ is non-degenerate and $[\mathfrak{H}, \mathfrak{H}] = \mathfrak{H}$. The invariance of B implies that $\mathfrak{U}$ is a non-degenerate graded ideal of $\mathfrak{g}$ because $[\mathfrak{H}, \mathfrak{U}] = \{0\}$ and $[\mathfrak{H}, \mathfrak{H}] = \mathfrak{H}$. The fact that $\mathfrak{g}$ is B-irreducible implies that $\mathfrak{g} = \mathfrak{H}$ or $\mathfrak{g} = \mathfrak{U}$. If $\mathfrak{g} = \mathfrak{U}$, recall that $[\mathfrak{U}_{\bar{0}}, \mathfrak{U}_{\bar{0}}] = [\mathfrak{U}_{\bar{1}}, \mathfrak{U}_{\bar{1}}] = \{0\}$ (see Proposition 1.2). Since $B([\mathfrak{U}_{\bar{0}}, \mathfrak{U}_{\bar{1}}], \mathfrak{U}_{\bar{1}}) = B(\mathfrak{U}_{\bar{0}}, [\mathfrak{U}_{\bar{1}}, \mathfrak{U}_{\bar{1}}]) = \{0\}$, it follows that $[\mathfrak{U}_{\bar{0}}, \mathfrak{U}_{\bar{1}}] = \{0\}$, so $\mathfrak{U}_{\bar{1}} \subset \mathfrak{z}(\mathfrak{g}) = \{0\}$, which contradicts the fact that $\mathfrak{g}_{\bar{1}} \neq \{0\}$. We conclude that $\mathfrak{g} = \mathfrak{H}$, then $\mathfrak{g}$ is semisimple Lie superalgebra and $[\mathfrak{g}, \mathfrak{g}] \neq \{0\}$. If we suppose that $\mathfrak{g}$ is not simple, then there exists a graded ideal $\mathfrak{I}$ of $\mathfrak{g}$ such that $\mathfrak{I} \neq \{0\}$ and $\mathfrak{I} \neq \mathfrak{g}$. The fact that $\mathfrak{g}$ is B-irreducible implies that $\mathfrak{I}$ is a degenerate ideal of $\mathfrak{g}$ and that $\mathfrak{I} \cap \mathfrak{I}^{\perp} \neq \{0\}$. We have $B([\mathfrak{I} \cap \mathfrak{I}^{\perp}, \mathfrak{I} \cap \mathfrak{I}^{\perp}], \mathfrak{g}) = B(\mathfrak{I} \cap \mathfrak{I}^{\perp}, [\mathfrak{g}, \mathfrak{I} \cap \mathfrak{I}^{\perp}]) = \{0\}$ because $\mathfrak{I}^{\perp}$ is a graded ideal of $\mathfrak{g}$. Consequently, the non-degeneration of B shows that $\mathfrak{I} \cap \mathfrak{I}^{\perp}$ is an abelian graded ideal of $\mathfrak{g}$, which is a contradiction with the semisimplicity of $\mathfrak{g}$. We conclude that $\mathfrak{g}$ is simple. □

In the following theorem we recall the notion of double extension of a quadratic Lie superalgebra. This notion is a tool for construction of new quadratic Lie superalgebras.

Theorem 1.3. [5] *Let $(\mathfrak{g}_1, B_1)$ be a quadratic Lie superalgebra, $\mathfrak{g}_2$ a Lie superalgebra and $\psi : \mathfrak{g}_2 \to \mathrm{Der}_a(\mathfrak{g}_1) \subset \mathrm{Der}(\mathfrak{g}_1)$ a morphism of Lie superalgebras. Let φ be the map from $\mathfrak{g}_1 \times \mathfrak{g}_1$ to $\mathfrak{g}_2{}^*$, defined by*

$$\varphi(X, Y)(Z) = (-1)^{(x+y)z} B_1(\psi(Z)(X), Y) \quad \forall X \in (\mathfrak{g}_1)_x,\ \forall Y \in (\mathfrak{g}_1)_y,\ \forall Z \in (\mathfrak{g}_2)_z.$$

Let π be the coadjoint representation of $\mathfrak{g}_2$. Then the vector space $\mathfrak{g} = \mathfrak{g}_2 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2{}^*$ with the product

$$\begin{aligned}[X_2 + X_1 + f, Y_2 + Y_1 + g] =& [X_2, Y_2]_{\mathfrak{g}_2} + [X_1, Y_1]_{\mathfrak{g}_1} + \psi(X_2)(Y_1) - (-1)^{xy}\psi(Y_2)(X_1) \\ &+ \pi(X_2)(g) - (-1)^{xy}\pi(Y_2)(f) + \varphi(X_1, Y_1),\end{aligned}$$

where $X_2 + X_1 + f$ (resp. $Y_2 + Y_1 + g$) is homogeneous of degree x (resp. y) in $\mathfrak{g}$, is a Lie superalgebra. Moreover, if γ is an invariant supersymmetric bilinear form on $\mathfrak{g}_2$, then the bilinear form T defined on $\mathfrak{g}$ by

$$T(X_2 + X_1 + f, Y_2 + Y_1 + g) = B_1(X_1, Y_1) + \gamma(X_2, Y_2) + f(Y_2) + (-1)^{xy} g(X_2)$$

where X_2+X_1+f and Y_2+Y_1+g are homogeneous of respective degree x, y, is an invariant scalar product on $\mathfrak{g}$. The Lie superalgebra $\mathfrak{g}$ is called the double extension of $(\mathfrak{g}_1, B_1)$ by $\mathfrak{g}_2$ by means of ψ.

2. The root space decomposition of a quadratic Lie superalgebra relative to a Cartan subalgebra of the even part

This section is devoted to studying certain properties of the roots and the root space decomposition of a quadratic Lie superalgebra.

Let $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ be a quadratic Lie superalgebra, H a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$, λ a linear form on H, we set

$$\mathfrak{g}^\lambda = \{x \in \mathfrak{g} : \ \exists n \in \mathbb{N}, \ (\mathrm{ad}_{\mathfrak{g}}(h) - \lambda(h)\,\mathrm{id}_{\mathfrak{g}})^n(x) = 0 \ \forall h \in H\}.$$

It's clear that $\mathfrak{g}^\lambda$ is a graded subspace of $\mathfrak{g}$. By [5] (Propositions 8 and 9, ch. 7, §1, p. 14), we have the root space decompostion of $\mathfrak{g}$ relative to $H:\ \mathfrak{g} = \bigoplus_{\lambda \in H^*} \mathfrak{g}^\lambda$.
Let us define:

$$\begin{aligned}\Delta_{\bar{0}} &= \{\lambda \in H^* : \ \lambda \neq 0, \ (\mathfrak{g}^\lambda)_{\bar{0}} \neq \{0\}\}, \\ \Delta_{\bar{1}} &= \{\lambda \in H^* : \ (\mathfrak{g}^\lambda)_{\bar{1}} \neq \{0\}\}, \\ \Delta &= \Delta_{\bar{0}} \cup \Delta_{\bar{1}}.\end{aligned}$$

The elements of Δ are called the roots of $\mathfrak{g}$ relative to H, more precisely, a root is called even (resp. odd) if it is an element of $\Delta_{\bar{0}}$ (resp. $\Delta_{\bar{1}}$). Remark that some roots could be even and odd.

We begin with a preliminary lemma, whose proof is straightforward.

Lemma 2.1. *Let $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ be a Lie superalgebra, H a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$, $x \in H$, $y \in \mathfrak{g}$, $z \in \mathfrak{g}$, and $\alpha, \beta \in \mathbb{K}$. Then for all $n \geq 1$:*

$$(\mathrm{ad}\,x - (\alpha+\beta)\,\mathrm{id}_{\mathfrak{g}})^n([y,z]) = \sum_{p=0}^{p=n} C_n^p [(\mathrm{ad}\,x - \alpha\,\mathrm{id}_{\mathfrak{g}})^p y, (\mathrm{ad}\,x - \beta\,\mathrm{id}_{\mathfrak{g}})^{n-p} z].$$

Proposition 2.1. *Let $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ be a quadratic Lie superalgebra, H a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$ and Δ the set of the roots of $\mathfrak{g}$ relative to H. Then:*

i) $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \subset \mathfrak{g}^{\alpha+\beta}$, $\forall \alpha, \beta \in \Delta \cup \{0\}$.

ii) *If $\alpha, \beta \in \Delta \cup \{0\}$ such that $\alpha + \beta \neq 0$, then $B(\mathfrak{g}^\alpha, \mathfrak{g}^\beta) = \{0\}$.*

iii) *$B_{|_{H \times H}}$ is non-degenerate.*

iv) *If $\alpha \in \Delta_i$, then $-\alpha \in \Delta_i$ (where $i = E\bar{0}, \bar{1}$).*

v) $\dim(\mathfrak{g}^\alpha)_{\bar{0}} = \dim(\mathfrak{g}^{-\alpha})_{\bar{0}}$ *and* $\dim(\mathfrak{g}^\alpha)_{\bar{1}} = \dim(\mathfrak{g}^{-\alpha})_{\bar{1}}$ $\forall \alpha \in \Delta$.

Proof. The assertion i) one derives easily from Lemma 2.1. Proposition 9 of [5] implies the assertion ii).

iii) $\mathfrak{g}^0 = H \oplus (\mathfrak{g}^0)_{\bar{1}}$ because $H = (\mathfrak{g}^0)_{\bar{0}}$. By ii), $B(\mathfrak{g}^0, \mathfrak{g}^\alpha) = \{0\}$ $\forall \alpha \in \Delta \backslash \{0\}$, consequently $B_{|_{\mathfrak{g}^0 \times \mathfrak{g}^0}}$ is non-degenerate. Since B is even then $B_{|_{H \times H}}$ is non-degenerate.

iv) Let $\alpha \in \Delta$, then by ii), $B(\mathfrak{g}^\alpha, \mathfrak{g}^\beta) = \{0\}$ for all $\beta \in \Delta$ such that $\beta \neq -\alpha$. Consequently, $\mathfrak{g}^{-\alpha} \neq \{0\}$ because B is non-degenerate, it follows that $-\alpha \in \Delta$. If $\alpha \in \Delta_i$ $(i = \bar{0}$ *or* $\bar{1})$, then $-\alpha \in \Delta_i$ because B is even.

v) Let $\alpha \in \Delta$, because $B_{|_{\mathfrak{g}^\alpha \times \mathfrak{g}^{-\alpha}}}$ is non-degenerate and B is even, then $B_{|_{(\mathfrak{g}^\alpha)_i \times (\mathfrak{g}^{-\alpha})_i}}$ is non-degenerate where $i \in \{\bar{0}, \bar{1}\}$. We conclude that $\dim(\mathfrak{g}^\alpha)_i = \dim(\mathfrak{g}^{-\alpha})_i$ $\forall \alpha \in \Delta$ and $\forall\ i \in \{\bar{0}, \bar{1}\}$. □

Let us consider $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ a quadratic Lie superalgebra and H a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$. The fact that $B_{H \times H}$ is non-degenerate implies that if $\varphi \in H^*$ be any linear form on H then there exists a unique $h_\varphi \in H$ such that $\varphi(h) = B(h_\varphi, h)$ $\forall\ h \in H$.

If φ, $\phi \in H^*$, we define:

$$(\varphi | \phi) = B(h_\varphi, h_\phi) = \varphi(h_\phi) = \phi(h_\varphi).$$

Proposition 2.2. *Let $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ be a quadratic Lie superalgebra, H a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$, Δ the set of the roots of $\mathfrak{g}$ relative to H and $\alpha \in \Delta \backslash \{0\}$.*

i) *If $\{e_i, 1 \leq i \leq n\}$ is a basis of $(\mathfrak{g}^\alpha)_{\bar{0}}$ relative to which the elements of H act in upper triangular form, and $\{r_i, 1 \leq i \leq n\}$ is the dual basis in $(\mathfrak{g}^{-\alpha})_{\bar{0}}$ of $\{e_i, 1 \leq i \leq n\}$, then $[e_i, r_i] = h_\alpha$ and $[e_i, r_j] = 0$ if $i < j$, $\forall i, j \in \{1, \ldots, n\}$.*

ii) *If $\{f_i, 1 \leq i \leq m\}$ is a basis of $(\mathfrak{g}^\alpha)_{\bar{1}}$ relative to which the elements of H act in upper triangular form, and $\{s_i, 1 \leq i \leq m\}$ is the dual basis in $(\mathfrak{g}^{-\alpha})_{\bar{1}}$ of $\{f_i, 1 \leq i \leq m\}$, then $[f_i, s_i] = -h_\alpha$ and $[f_i, s_j] = 0$ if $i < j$, $\forall i, j \in \{1, \ldots, m\}$.*

Proof. Let $h \in H$, $i, j \in \{1, \ldots, n\}$ and $k, p \in \{1, \ldots, m\}$, then:

$$\operatorname{ad} h(e_i) = \alpha(h) e_i + \sum_{l=1}^{i-1} a_i^l e_l, \text{ where } a_i^l \in \mathbb{K}, \quad \forall l \in \{1, \ldots, i-1\},$$

$$\operatorname{ad} h(f_k) = \alpha(h) f_k + \sum_{q=1}^{k-1} b_k^q f_q, \text{ where } b_k^q \in \mathbb{K}, \quad \forall q \in \{1, \ldots, k-1\}.$$

By the invariance of B, we have

$$B([e_i,r_j],h) = B(r_j,[h,e_i]) = \alpha(h)B(r_j,e_i) + \sum_{l=1}^{i-1} a_i^l B(r_j,e_l),$$
$$B([f_k,s_p],h) = -\,B(s_p,[h,f_k]) = -\alpha(h)B(s_p,f_k) - \sum_{q=1}^{k-1} b_k^q B(s_p,f_q).$$

It follows that:

a) If $i<j$ and $k<p$, then $B([e_i,r_j],h) = B([f_k,s_p],h) = 0$,

b) $B([e_i,r_i],h) = \alpha(h) = B(h_\alpha,h)$ and $B([f_k,s_k],h) = -\alpha(h) = -B(h_\alpha,h)$.

Since $B_{|_{H\times H}}$ is non-degenerate, then

$$[e_i,r_i] = h_\alpha, [f_k,s_k] = -h_\alpha,\ \ [e_i,r_j] = 0 \text{ if } i<j \text{ and } [f_k,s_p] = 0 \text{ if } k<p. \qquad \square$$

Proposition 2.3. *Let $(\mathfrak{g} = \mathfrak{g}_{\bar 0}\oplus\mathfrak{g}_{\bar 1}, B)$ be a quadratic Lie superalgebra, H a Cartan subalgebra of $\mathfrak{g}_{\bar 0}$, Δ the set of the roots of $\mathfrak{g}$ relative to H and $\alpha\in\Delta_{\bar 0}$. If $(\alpha|\alpha) = 0$, then $(\alpha|\beta) = 0,\ \ \forall\beta\in\Delta$.*

Proof. Let $\alpha\in\Delta_{\bar 0}$ such that $(\alpha|\alpha) = 0$ and let $\beta\in\Delta$. Consider the following vector subspace of $\mathfrak{g}$: $V = \sum_{n\in\mathbb{Z}}\mathfrak{g}^{\beta+n\alpha}$. By Proposition 2.2, there exists an element e_α of $(\mathfrak{g}^\alpha)_{\bar 0}$ and an element $e_{-\alpha}$ of $(\mathfrak{g}^{-\alpha})_{\bar 0}$ such that $[e_\alpha, e_{-\alpha}] = h_\alpha$. The fact that V is invariant by $\mathrm{ad}_{\mathfrak{g}}\, h_\alpha$, by $\mathrm{ad}_{\mathfrak{g}}\, e_\alpha$ and by $\mathrm{ad}_{\mathfrak{g}}\, e_{-\alpha}$ implies that $\mathrm{trace}((\mathrm{ad}_{\mathfrak{g}}\, h_\alpha)_{|_V} = 0$. It follows that $\sum_{n\in\mathbb{Z}}(\beta(h_\alpha)+n\alpha(h_\alpha))\dim(\mathfrak{g}^{\beta+n\alpha}) = 0$, then $\beta(h_\alpha)\sum_{n\in\mathbb{Z}}\dim(\mathfrak{g}^{\beta+n\alpha}) = 0$, we conclude that $\beta(h_\alpha) = (\alpha|\beta) = 0$. $\square$

Remark 2.1. Proposition 2.2 (resp. Proposition 2.3) is a generalization of Proposition 5 (resp. the corollary of Proposition 6) of [2] obtained by V. V. Astrakhantsev in the case of quadratic Lie algebras.

The following proposition is a main result of [2].

Proposition 2.4. *Let $(\mathfrak{g}, B)$ be a quadratic Lie algebra, H a Cartan subalgebra of $\mathfrak{g}_{\bar 0}$, Δ the set of the roots of $\mathfrak{g}$ relative to H and $\alpha\in\Delta$. If $(\alpha|\alpha)\neq 0$, then* $\dim\mathfrak{g}^\alpha = 1$.

Proposition 2.5. *Let $(\mathfrak{g} = \mathfrak{g}_{\bar 0}\oplus\mathfrak{g}_{\bar 1}, B)$ be a quadratic Lie superalgebra, H a Cartan subalgebra of $\mathfrak{g}_{\bar 0}$, Δ the set of the roots of $\mathfrak{g}$ relative to H and $\alpha\in\Delta$ such that $(\alpha|\alpha)\neq 0$.*

i) *If $\alpha\in\Delta_{\bar 0}$, then* $\dim(\mathfrak{g}^\alpha)_{\bar 0} = \dim(\mathfrak{g}^{-\alpha})_{\bar 0} = 1$.

ii) *If $\alpha\in\Delta_{\bar 1}$, then $2\alpha\in\Delta_{\bar 0}$ and* $\dim(\mathfrak{g}^\alpha)_{\bar 1} = \dim(\mathfrak{g}^{-\alpha})_{\bar 1} = 1$.

Proof. Let $\alpha\in\Delta$ such that $(\alpha|\alpha)\neq 0$.

i) If $\alpha\in\Delta_{\bar 0}$, Proposition 2.4 implies that $\dim(\mathfrak{g}^\alpha)_{\bar 0} = \dim(\mathfrak{g}^{-\alpha})_{\bar 0} = 1$.

ii) Now, suppose that $\alpha \in \Delta_{\bar{1}}$, then by the assertion iv) of Proposition 2.1 we have $\dim(\mathfrak{g}^{\alpha})_{\bar{1}} = \dim(\mathfrak{g}^{-\alpha})_{\bar{1}} = m$. Moreover, by Proposition 2.2, there exist a basis $\{f_i, 1 \leq i \leq m\}$ of $(\mathfrak{g}^{\alpha})_{\bar{1}}$ relative to which the elements of H act in upper triangular form, and a basis $\{s_i, 1 \leq i \leq m\}$ of $(\mathfrak{g}^{-\alpha})_{\bar{1}}$ such that $[f_i, s_i] = -h_\alpha$ and $[f_i, s_j] = 0$ if $i < j$, $\forall i, j \in \{1, \dots, m\}$. Let $i \in \{1, \dots, m\}$. Since

$$[[f_i, f_i], s_i] = [f_i, [f_i, s_i]] + [f_i, [s_i, f_i]],$$

then $[[f_i, f_i], s_i] = 2\ [h_\alpha, f_i] = 2\alpha(h_\alpha)f_i + \sum_{j=1}^{i-1} a_i^j f_j \neq 0$ because $\{f_1, \dots, f_m\}$ is a basis of $(\mathfrak{g}^{\alpha})_{\bar{1}}$ and $\alpha(h_\alpha) \neq 0$. Consequently, $[f_i, f_i] \neq 0$, it follows that $(\mathfrak{g}^{2\alpha})_{\bar{0}} \neq \{0\}$. We conclude that $2\alpha \in \Delta_{\bar{0}}$.

In the following, we will show that $\dim(\mathfrak{g}^{\alpha})_{\bar{1}} = 1$. Suppose that $\dim(\mathfrak{g}^{\alpha})_{\bar{1}} = m > 1$. Since $2\alpha(h_{2\alpha}) = 2\alpha(2h_\alpha) = 4\alpha(h_\alpha) \neq 0$, then, by the assertion i), $\dim(\mathfrak{g}^{2\alpha})_{\bar{0}} = 1$. Consequently there exists $\lambda \in \mathbb{K}$ such that $[f_2, f_2] = \lambda[f_1, f_1]$ because $[f_1, f_1] \in (\mathfrak{g}^{2\alpha})_{\bar{0}}$ and $[f_2, f_2] \in (\mathfrak{g}^{2\alpha})_{\bar{0}}$. Therefore, $[[f_2, f_2], s_2] = 2\lambda[f_1, [f_1, s_2]] = 0$ because $[f_1, s_2] = 0$, which contradicts the fact that

$$[[f_2, f_2], s_2] = 2[h_\alpha, f_2] = 2af_1 + 2\alpha(h_\alpha)f_2 \neq 0 \quad \text{where} \quad a \in \mathbb{K},$$

because $\alpha(h_\alpha) \neq 0$ and $\{f_i, 1 \leq i \leq m\}$ is a basis of $(\mathfrak{g}^{\alpha})_{\bar{1}}$. We conclude that $\dim(\mathfrak{g}^{\alpha})_{\bar{1}} = 1$.□

Corollary. *Let $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ be a quadratic Lie superalgebra such that $\mathfrak{g}_{\bar{0}}$ is reductive, H a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$, Δ the set of the roots of $\mathfrak{g}$ relative to H. If $\alpha \in \Delta_{\bar{1}}$, then $(\alpha|\alpha) \neq 0$ if and only if $2\alpha \in \Delta_{\bar{0}}$.*

Proof. By Proposition 2.5, the fact that $(\alpha|\alpha) \neq 0$ implies that $2\alpha \in \Delta_{\bar{0}}$. Conversely, if $2\alpha \in \Delta_{\bar{0}}$ then 2α is the root of the Lie reductive Lie algebra $\mathfrak{g}_{\bar{0}}$, consequently, $(2\alpha|2\alpha) \neq 0$, it follows that $(\alpha|\alpha) \neq 0$. □

Remark 2.2. In the case of a quadratic Lie superalgebra $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ such that $\mathfrak{g}_{\bar{0}}$ is not a reductive Lie algebra, the implication

$$2\alpha \in \Delta_{\bar{0}} \Rightarrow (\alpha|\alpha) \neq 0,$$

where Δ is the set of the roots of $\mathfrak{g}$ relative to a fixed Cartan subalgebra of H of $\mathfrak{g}_{\bar{0}}$ and $\alpha \in \Delta_{\bar{1}}$, is not true in general. Indeed, consider the two-dimensional simple $\mathfrak{sl}(2)$-module $\mathfrak{U} = \mathfrak{D}(\frac{1}{2})$, denote by π the representation of $\mathfrak{sl}(2)$ on $\mathfrak{U}$ associated to this module. Recall that there exists a basis $\{Y, F, G\}$ of $\mathfrak{sl}(2)$ such that $[F, G] = 2Y$, $[Y, F] = F$, $[Y, G] = -G$, and there exists a basis $\{e_1, e_2\}$ of $\mathfrak{U}$ such that $F.e_1 = e_2$, $F.e_2 = 0$, $Y.e_1 = -\frac{1}{2}e_1$, $Y.e_2 = \frac{1}{2}e_2$, $G.e_1 = 0$, $G.e_2 = e_1$.

On $\mathfrak{U}$ we consider the structure of abelian Lie superalgebra such that $\mathfrak{U} = \mathfrak{U}_{\bar{1}}$ and $\mathfrak{U}_{\bar{0}} = \{0\}$. Let $B : \mathfrak{U} \times \mathfrak{U} \to \mathbb{K}$ be the bilinear form on $\mathfrak{U}$ defined by: $B(e_1, e_1) = B(e_2, e_2) = 0$ and $B(e_1, e_2) = -B(e_2, e_1) = 1$, then $(\mathfrak{U}, B)$ is a quadratic Lie superalgebra. It's easy to see that $\pi(\mathfrak{sl}(2))$ is contained in $\mathrm{Der}_a(\mathfrak{U})$ and, by Theorem 1.3, the double extension $\mathfrak{g} = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)^* \oplus \mathfrak{U}$ of $\mathfrak{U}$ by $\mathfrak{sl}(2)$ by means of π is a quadratic Lie superalgebra such that

$\mathfrak{g}_{\bar{1}} = \mathfrak{U}$ is an irreducible ($\mathfrak{g}_{\bar{0}} = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)^*$)-module, $\mathfrak{z}(\mathfrak{g}) = \{0\}$ and $\mathfrak{g}$ isn't simple because $\mathfrak{sl}(2)^*$ is an ideal of $\mathfrak{g}$. Moreover, $\mathfrak{g}$ is a T-irreducible quadratic superalgebra where T is an invariant scalar product of $\mathfrak{g}$. More precisely, T is defined as follows (see Theorem 1.3):

$$T(X_2 + X_1 + f, Y_2 + Y_1 + l) = B(X_1, Y_1) + f(Y_2) + l(X_2),$$

where $X_1, Y_1 \in \mathfrak{U}$, $X_2, Y_2 \in \mathfrak{sl}(2)$ and $f, l \in \mathfrak{sl}(2)^*$. It's clear that $H = \mathbb{K}Y \oplus \mathbb{K}f$, where $f \in H^*$ defined by: $f(Y) = 1$, $f(G) = f(F) = 0$, is a Cartan subalgebra of ($\mathfrak{g}_{\bar{0}} = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)^*$). It's easy to verify that if Δ is the set of the roots of $\mathfrak{g}$ relative to H, then $\Delta_{\bar{0}} = \{\pm\alpha\}$ and $\Delta_{\bar{1}} = \{\pm\beta\}$, where α and β are defined as follows:

$$\alpha(Y) = 1 \text{ and } \alpha(f) = 0, \quad \beta(Y) = \frac{1}{2} \text{ and } \beta(f) = 0. \text{ Then } \alpha = 2\beta.$$

Consequently $h_\alpha = f$ and $h_\beta = \frac{1}{2}f$, it follows that $(\alpha|\alpha) = (\beta|\beta) = 0$. We conclude that $2\beta \in \Delta_{\bar{0}}$ and $(\beta|\beta) = 0$.

Proposition 2.6. *Let* $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ *be a quadratic Lie superalgebra,* H *a Cartan subalgebra of* $\mathfrak{g}_{\bar{0}}$*,* Δ *the set of the roots of* $\mathfrak{g}$ *relative to* H*. Let* α *and* β *be two elements of* $\Delta_{\bar{1}}$ *such that* $\alpha \neq \pm\beta$*. If* $(\alpha|\beta) \neq 0$*, then* $\alpha + \beta \in \Delta_{\bar{0}}$ *or* $\alpha - \beta \in \Delta_{\bar{0}}$*.*

Proof. Let α and β be two elements of $\Delta_{\bar{1}}$ such that $\alpha \neq \pm\beta$. By Proposition 2.2, there exist a basis $\{f_{\alpha,i}, 1 \leq i \leq m\}$ of $(\mathfrak{g}^\alpha)_{\bar{1}}$ relative to which the elements of H act in upper triangular form, and a basis $\{s_{\alpha,i}, 1 \leq i \leq m\}$ of $(\mathfrak{g}^{-\alpha})_{\bar{1}}$ such that $[f_{\alpha,i}, s_{\alpha,i}] = -h_\alpha$ and $[f_{\alpha,i}, s_{\alpha,j}] = 0$ if $i < j$, $\forall i, j \in \{1, \ldots, m\}$, and there exist a basis $\{f_{\beta,i}, 1 \leq i \leq n\}$ of $(\mathfrak{g}^\beta)_{\bar{1}}$ relative to which the elements of H act in upper triangular form, and a basis $\{s_{\beta,i}, 1 \leq i \leq n\}$ of $(\mathfrak{g}^{-\beta})_{\bar{1}}$ such that $[f_{\beta,i}, s_{\beta,i}] = -h_\beta$ and $[f_{\beta,i}, s_{\beta,j}] = 0$ if $i < j$, $\forall i, j \in \{1, \ldots, n\}$.

Since $[[f_{\beta,1}, f_{\alpha,1}], s_{\alpha,1}] + [[s_{\alpha,1}, f_{\beta,1}], f_{\alpha,1}] = [f_{\beta,1}, [f_{\alpha,1}, s_{\alpha,1}]]$ and $[f_{\alpha,1}, s_{\alpha,1}] = -h_\alpha$, then $[[f_{\beta,1}, f_{\alpha,1}], s_{\alpha,1}] + [[s_{\alpha,1}, f_{\beta,1}], f_{\alpha,1}] = [h_\alpha, f_{\beta,1}] = (\alpha|\beta) f_{\beta,1} \neq 0$. Consequently, $[f_{\beta,1}, f_{\alpha,1}] \neq 0$ or$[s_{\alpha,1}, f_{\beta,1}] \neq 0$. It follows that $(\mathfrak{g}^{\alpha+\beta})_{\bar{0}} \neq \{0\}$ or $(\mathfrak{g}^{\alpha-\beta})_{\bar{0}} \neq \{0\}$. We conclude that $\alpha + \beta \in \Delta_{\bar{0}}$ or $\alpha - \beta \in \Delta_{\bar{0}}$. $\square$

Remark 2.3. The assertion ii) of Proposition 2.5 and Proposition 2.6 generalize the results obtained in [16] by M. Scheunert in case of the quadratic Lie superalgebras $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ such that $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra and the $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ is semisimple.

Lemma 2.2. *Let* $(\mathfrak{g}, B)$ *be a quadratic Lie algebra,* H *a Cartan subalgebra of* $\mathfrak{g}$*,* Δ *the set of the roots of* $\mathfrak{g}$ *relative to* H *and* $\alpha \in \Delta$ *such that* $(\alpha|\alpha) \neq 0$*. Then,* $k\alpha \in \Delta$ *(where* $k \in \mathbb{K}\backslash\{0\}$*) if and only if* $k = \pm 1$*.*

Proof. Let $\alpha \in \Delta$ such that $(\alpha|\alpha) \neq 0$. Consider $A = \{k \in \mathbb{Z} : \ k\alpha \in \Delta\}$, it is clear that $\pm 1 \in A$. Let p be the greatest element of A and $V_\alpha = \sum_{n=-1}^{p} \mathfrak{g}^{n\alpha}$. By Proposition 2.2, there exists an element e_α of $(\mathfrak{g}^\alpha)_{\bar{0}}$ and an element $e_{-\alpha}$ of $(\mathfrak{g}^{-\alpha})_{\bar{0}}$ such that $[e_\alpha, e_{-\alpha}] = h_\alpha$. It's clear that V_α is invariant by $\mathrm{ad}_{\mathfrak{g}}\, h_\alpha$ and by $\mathrm{ad}_{\mathfrak{g}}\, e_\alpha$. The fact that $(\alpha|\alpha) \neq 0$ implies that

$(-\alpha|-\alpha) \neq 0$, and by Proposition 2.4 we have $\dim \mathfrak{g}^{-\alpha} = 1$. Consequently, V_α is invariant by $\mathrm{ad}_{\mathfrak{g}}\, e_{-\alpha}$, it follows that $\mathrm{trace}((\mathrm{ad}_{\mathfrak{g}}\, h_\alpha)_{|V_\alpha} = 0$. Then $\sum_{n=-1}^{p} n(\alpha|\alpha) \dim \mathfrak{g}^{n\alpha} = 0$. Since $(n\alpha|n\alpha) = n^2(\alpha|\alpha) \neq 0$, then $\dim \mathfrak{g}^{n\alpha} = 1$, so $(\sum_{n=-1}^{p} n)(\alpha|\alpha) = 0$. The fact that $(\alpha|\alpha) \neq 0$ implies that $p = 1$. Consequently, $A = \{-1, 1\}$ because $-\Delta = \Delta$. □

Proposition 2.7. *Let $(\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}, B)$ be a quadratic Lie superalgebra, H a Cartan subalgebra of $\mathfrak{g}_{\bar 0}$, Δ the set of the roots of $\mathfrak{g}$ relative to H and $\alpha \in \Delta$ such that $(\alpha|\alpha) \neq 0$.*

i) *If $\alpha \in \Delta_{\bar 0}$, $k\alpha \in \Delta_{\bar 0}$ (where $k \in \mathbb{K}\backslash\{0\}$) if and only if $k = \pm 1$.*
ii) *If $\alpha \in \Delta_{\bar 0}$ and $k\alpha \in \Delta$ where $k \in \mathbb{K}\backslash\{0\}$, then $k \in \{-1, -\frac{1}{2}, \frac{1}{2}, 1\}$.*
iii) *If $\alpha \in \Delta_{\bar 1}$, $k\alpha \in \Delta$ (where $k \in \mathbb{K}\backslash\{0\}$) if and only if $k \in \{-1, -2, 1, 2\}$.*

Proof. Let $\alpha \in \Delta$ such that $(\alpha|\alpha) \neq 0$, and let $k \in \mathbb{K}\backslash\{0\}$ such that $k\alpha \in \Delta$.

i) If $\alpha \in \Delta_{\bar 0}$, by Lemma 2.2, $k\alpha \in \Delta_{\bar 0}$ if and only if $k = \pm 1$.

ii) Suppose that $\alpha \in \Delta_{\bar 0}$. If $k\alpha \in \Delta_{\bar 0}$, then, by the assertion i), $k = \pm 1$. If $k\alpha \in \Delta_{\bar 1}$, then, by Proposition 2.5, $2k\alpha \in \Delta_{\bar 0}$ because $(k\alpha|k\alpha) \neq 0$, it follows that $2k = \pm 1$, so $k = \pm\frac{1}{2}$. We conclude that if $\alpha \in \Delta_{\bar 0}$ and $k\alpha \in \Delta$ where $k \in \mathbb{K}\backslash\{0\}$, then $k \in \{-1, -\frac{1}{2}, \frac{1}{2}, 1\}$.

iii) Now, suppose that $\alpha \in \Delta_{\bar 1}$. Because $(\alpha|\alpha) \neq 0$, then, by Proposition 2.5, $2\alpha \in \Delta_{\bar 0}$. If $k\alpha \in \Delta_{\bar 1}$, then the fact that $(k\alpha|k\alpha) \neq 0$ implies that $2k\alpha = k(2\alpha) \in \Delta_{\bar 0}$, consequently, by the assertion i), $k = \pm 1$.

If $k\alpha \in \Delta_{\bar 0}$, then the fact that $\frac{1}{k}(k\alpha) = \alpha \in \Delta_{\bar 1}$ implies that $\frac{1}{k} = \pm\frac{1}{2}$, it follows that $k = \pm 2$. We conclude that if $\alpha \in \Delta_{\bar 1}$ and $k\alpha \in \Delta$ where $k \in \mathbb{K}\backslash\{0\}$, then $k \in \{-1, -2, 1, 2\}$.

Conversely, if $\alpha \in \Delta_{\bar 1}$, by Proposition 2.1, $-\alpha \in \Delta_{\bar 1}$. Since $(\alpha|\alpha) \neq 0$, then by Proposition 2.5, $2\alpha \in \Delta$ and by Proposition 2.1, $-2\alpha \in \Delta$. □

Corollary. *Let $(\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}, B)$ be a quadratic Lie superalgebra, H a Cartan subalgebra of $\mathfrak{g}_{\bar 0}$, Δ the set of the roots of $\mathfrak{g}$ relative to H. If $\mathfrak{g}_{\bar 0}$ is a reductive Lie algebra, then $\Delta_{\bar 0} \cap \Delta_{\bar 1} = \emptyset$.*

Proof. Assume that $\mathfrak{g}_{\bar 0}$ is a reductive Lie algebra. Suppose that $\Delta_{\bar 0} \cap \Delta_{\bar 1} \neq \emptyset$, then there exists $\alpha \in H^*$ such that $\alpha \in \Delta_{\bar 0} \cap \Delta_{\bar 1}$. The fact that $\mathfrak{g}_{\bar 0}$ is a reductive Lie algebra implies that $(\alpha|\alpha) \neq 0$, it follows by Proposition 2.5, that $2\alpha \in \Delta_{\bar 0}$ which contradicts the fact that $\alpha \in \Delta_{\bar 0}$ (see the the first assertion of Proposition 2.7). □

Remark 2.4. The converse of the corollary above is not true. Indeed, take the quadratic Lie superalgebra $\mathfrak{g} = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)^* \oplus \mathfrak{D}(\frac{1}{2})$ of Remark 2.2. We proved in Remark 2.2 that if Δ is the set of the roots of $\mathfrak{g}$ relative to the cartan subalgebra $H = \mathbb{K}Y \oplus \mathbb{K}f$ of $\mathfrak{g}_{\bar 0}$, then $\Delta_{\bar 0} = \{\pm\alpha\}$ and $\Delta_{\bar 1} = \{\pm\beta\}$, where α and β are defined as follows:

$$\alpha(Y) = 1 \text{ and } \alpha(f) = 0, \quad \beta(Y) = \frac{1}{2} \text{ and } \beta(f) = 0,$$

so $\Delta_{\bar 0} \cap \Delta_{\bar 1} = \emptyset$. The multiplication on $\mathfrak{g}$ (see Theorem 1.3) implies that $\mathfrak{g}_{\bar 0}$ is not a reductive Lie algebra because $\mathfrak{sl}(2)$ is a Levi component of $\mathfrak{g}_{\bar 0}$, $\mathfrak{sl}(2)^*$ is the solvable radical of $\mathfrak{g}_{\bar 0}$ and $[\mathfrak{sl}(2), \mathfrak{sl}(2)^*] \neq \{0\}$.

Lemma 2.3. *If $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ is a B-irreducible quadratic Lie superalgebra such that the $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ is semisimple and $\mathfrak{g}_{\bar{0}} \neq \{0\}$, then $\mathfrak{z}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{0}}$.*

Proof. Since the $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ is semisimple, then $\mathfrak{g}_{\bar{1}} = {\mathfrak{g}_{\bar{1}}}^{\mathfrak{g}_{\bar{0}}} \oplus [\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}]$ where ${\mathfrak{g}_{\bar{1}}}^{\mathfrak{g}_{\bar{0}}} = \{x \in \mathfrak{g}_{\bar{1}} : [\mathfrak{g}_{\bar{0}}, x] = \{0\}\}$. It's clear that $\mathfrak{I} = \mathfrak{g}_{\bar{0}} \oplus [\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}]$ is a graded ideal of $\mathfrak{g}$. Since B is invariant, then $B([\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}], {\mathfrak{g}_{\bar{1}}}^{\mathfrak{g}_{\bar{0}}}) = \{0\}$, it follows that $\mathfrak{I}$ is a non-degenerate graded ideal of $\mathfrak{g}$. Thus $\mathfrak{g} = \mathfrak{I}$ because $\mathfrak{g}$ is B-irreducible, so $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}] = \mathfrak{g}_{\bar{1}}$ and ${\mathfrak{g}_{\bar{1}}}^{\mathfrak{g}_{\bar{0}}} = \{0\}$. The fact that $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{1}}$ is contained in ${\mathfrak{g}_{\bar{1}}}^{\mathfrak{g}_{\bar{0}}}$ implies that $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{1}} = \{0\}$, consequently $\mathfrak{z}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{0}}$.□

Proposition 2.8. *If $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ is a B-irreducible quadratic Lie superalgebra such that the $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ is semisimple and $\mathfrak{g}_{\bar{0}} \neq \{0\}$, H a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$ and Δ the set of the roots of $\mathfrak{g}$ relative to H and if $\Delta_{\bar{0}} \cap \Delta_{\bar{1}} = \emptyset$, then $0 \notin \Delta_{\bar{1}}$.*

Proof. Suppose that $\Delta_{\bar{0}} \cap \Delta_{\bar{1}} = \emptyset$ and $0 \in \Delta_{\bar{1}}$. Then $(\mathfrak{g}^0)_{\bar{1}} \neq \{0\}$ and by Engel's theorem there exists an element $X \neq 0$ of $(\mathfrak{g}^0)_{\bar{1}}$ such that $[H, X] = \{0\}$. Since $[X, (\mathfrak{g}^\alpha)_{\bar{1}}] \subset (\mathfrak{g}^\alpha)_{\bar{0}}$, $[X, (\mathfrak{g}^\alpha)_{\bar{0}}] \subset (\mathfrak{g}^\alpha)_{\bar{1}}$ for all $\alpha \in \Delta$ then $[X, (\mathfrak{g}^\alpha)_{\bar{0}}] = [X, (\mathfrak{g}^\alpha)_{\bar{1}}] = \{0\}$ for all $\alpha \in \Delta \backslash \{0\}$. Now, by invariance of B we have

$$B([(\mathfrak{g}^0)_{\bar{1}}, X], H) = B((\mathfrak{g}^0)_{\bar{1}}, [X, H]) = \{0\},$$

it follows that $[(\mathfrak{g}^0)_{\bar{1}}, X] = \{0\}$ because $[(\mathfrak{g}^0)_{\bar{1}}, X] \subset H$ and $B_{|_{H \times H}}$ is non-degenerate. Hence, $[X, \mathfrak{g}^\alpha] = \{0\}$ for all $\alpha \in \Delta$, so $X \in \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{1}}$. Then, by Lemma 2.3, $X = 0$. Thus we have arrived at a contradiction and the proposition is proved. □

Remark 2.5. Using the corollary of Proposition 2.7 and Proposition 2.8, we obtain the following known result(the assertion a) of Proposition 1 of [16], page 137): *If $\mathfrak{g}$ is a basic classical Lie superalgebra, then if we denote by Δ the set of the roots of $\mathfrak{g}$ relative to a fixed subalgebra H of $\mathfrak{g}_{\bar{0}}$, then $\Delta_{\bar{0}} \cap \Delta_{\bar{1}} = \emptyset$ and $0 \notin \Delta_{\bar{1}}$* (because, by Theorem 1 of [16] (page 101), $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra).

Remark 2.6. If $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ is a quadratic Lie superalgebra, H a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$ and Δ the set of the roots of $\mathfrak{g}$ relative to H. Then $(\mathfrak{g}_{\bar{0}}, B_{\bar{0}} = B_{|_{\mathfrak{g}_{\bar{0}} \times \mathfrak{g}_{\bar{0}}}})$ is a quadratic Lie algebra and $\Delta_{\bar{0}}$ is the root system of $\mathfrak{g}_{\bar{0}}$ relative to H. Consequently, the results obtained by V. V. Astrakhantsev in [2] on the root system of the quadratic Lie algebras furnish us with other information on the even roots of the quadratic Lie superalgebras.

3. Characterizations of basic classical Lie superalgebras among the quadratic Lie superalgebras

Using the information about the root decomposition of quadratic Lie superalgebras obtained in the second section, we give in this section two characterizations of the basic classical Lie superalgebras among the quadratic Lie superalgebras. We need the two following lemmas.

Lemma 3.1. *Let $(\mathfrak{g}, B)$ be a quadratic Lie algebra, H a Cartan subalgebra of $\mathfrak{g}$, Δ the set of the roots of $\mathfrak{g}$ relative to H. The following assertions are equivalent:*

i) $\mathfrak{g}$ *is reductive;*
ii) H *is commutative and* $(\alpha|\alpha) \neq 0\ \forall \alpha \in \Delta$.

Proof. ii) $\Rightarrow$ i). By Proposition 2.4: $\forall \alpha \in \Delta,\ \dim \mathfrak{g}^\alpha = 1$ because $(\alpha|\alpha) \neq 0$. Let $\alpha \in \Delta$ and let $\{e_\alpha\}$ be a basis of $\mathfrak{g}^\alpha$, then $[h, e_\alpha] = \alpha(h)e_\alpha$. This defines an H-module structure on $\mathfrak{g}^\alpha$. Moreover, the H-module $\mathfrak{g}$ is semisimple because H is commutative. It's clear that if α and β are two elements of Δ, then the H-modules $\mathfrak{g}^\alpha$ and $\mathfrak{g}^\beta$ are isomorphic if and only if $\alpha = \beta$. Consequently $\mathfrak{g}^\beta$ is an isotypic component of the H-module $\mathfrak{g} = H \oplus (\bigoplus_{\alpha \in \Delta} \mathfrak{g}^\alpha)$, for all $\beta \in \Delta$. Moreover, the fact that H is commutative implies that H is an isotypic component of the H-module $\mathfrak{g} = H \oplus (\bigoplus_{\alpha \in \Delta} \mathfrak{g}^\alpha)$.

Let $\mathfrak{R}(\mathfrak{g})$ be the solvable radical of $\mathfrak{g}$. Since $\mathfrak{R}(\mathfrak{g})$ is an H-submodule of $\mathfrak{g}$, then $\mathfrak{R}(\mathfrak{g}) = (\mathfrak{R}(\mathfrak{g}) \cap H) \oplus (\bigoplus_{\alpha \in \Delta} (\mathfrak{R}(\mathfrak{g}) \cap \mathfrak{g}^\alpha))$. Let $\alpha \in \Delta$, since $[h, e_\alpha] = \alpha(h)e_\alpha$, $[h, e_{-\alpha}] = -\alpha(h)e_{-\alpha} \forall h \in H$ and $[e_\alpha, e_{-\alpha}] = kh_\alpha$ where $k \in \mathbb{K}\backslash\{0\}$ (see Proposition 2.2), then $\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha} \oplus \mathbb{K}h_\alpha$ is a Lie subalgebra of $\mathfrak{g}$ which is isomorphic to $sl(2, \mathbb{K})$. Consequently $\mathfrak{R}(\mathfrak{g}) \cap \mathfrak{g}^\alpha = \{0\}$. It follows that $\mathfrak{R}(\mathfrak{g}) = \mathfrak{R}(\mathfrak{g}) \cap H$, so the fact that H is commutative implies that $[\mathfrak{R}(\mathfrak{g}), H] = \{0\}$.

Let $X \in \mathfrak{R}(\mathfrak{g}) \cap H$ and $\alpha \in \Delta$, then $[X, \mathfrak{g}^\alpha] \subset \mathfrak{R}(\mathfrak{g}) \cap \mathfrak{g}^\alpha = \{0\}$. Consequently $[\bigoplus_{\alpha \in \Delta} \mathfrak{g}^\alpha, \mathfrak{R}(\mathfrak{g})] = \{0\}$. We conclude that $\mathfrak{R}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$, it follows that $\mathfrak{g}$ is a reductive Lie algebra.

The implication i) $\Rightarrow$ ii) is straightforward. $\square$

Lemma 3.2. *Let* $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ *be a Lie superalgebra,* H *a Cartan subalgebra of* $\mathfrak{g}$*,* Δ *the set of the roots of* $\mathfrak{g}$ *relative to* H*. If* $\mathfrak{g}$ *is a classical simple Lie superalgebra, then* Δ *generates the linear dual* H^* *of* H*.*

Proof. Suppose that $\mathfrak{g}$ is a classical simple Lie superalgebra, then, by Theorem 1 of [16] (page 101), $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra. If $\mathfrak{g}_{\bar{0}}$ is a semisimple Lie algebra, then, by [6], $\Delta_{\bar{0}}$ generates H^*. Now, assume that $\mathfrak{g}_{\bar{0}}$ is not semisimple, so $\mathfrak{g}_{\bar{0}} = \mathfrak{z}(\mathfrak{g}_{\bar{0}}) \oplus \mathfrak{S}$ such that $\mathfrak{z}(\mathfrak{g}_{\bar{0}}) \neq \{0\}$, where $\mathfrak{z}(\mathfrak{g}_{\bar{0}})$ is the center of $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{S}$ is the greatest semisimple ideal of $\mathfrak{g}_{\bar{0}}$. Moreover, by corollary of [16] (page 107), $\dim \mathfrak{z}(\mathfrak{g}_{\bar{0}}) = 1$, the $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ decomposes into the direct sum of two irreducible $\mathfrak{g}_{\bar{0}}$-modules: $\mathfrak{g}_{\bar{1}} = \mathfrak{g}_{\bar{1}}^1 \oplus \mathfrak{g}_{\bar{1}}^2$, and there exists a unique element $C \in \mathfrak{z}(\mathfrak{g}_{\bar{0}})$ such that:

$$[C, x] = (-1)^r x\ \forall x \in \mathfrak{g}_{\bar{1}}^r;\ r = 1, 2.$$

It's clear that $H = \mathfrak{z}(\mathfrak{g}_{\bar{0}}) \oplus L$ where L is a Cartan subalgebra of $\mathfrak{S}$. By [6], the root system $\mathfrak{R}(\mathfrak{S}, L)$ of $\mathfrak{S}$ relative to L generates the linear dual L^* of L.

Let $\alpha \in \mathfrak{R}(\mathfrak{S}, L)$, consider $\tilde{\alpha} \in H^*$ defined by $\tilde{\alpha}_{|_L} = \alpha$ and $\tilde{\alpha}(C) = 0$, it's easy to verify that $\tilde{\alpha} \in \Delta$. Therefore $\{\tilde{\alpha} :\ \alpha \in \mathfrak{R}(\mathfrak{S}, L)\} \subset \Delta$. In the following, we construct an element $\tilde{\lambda}$ of H^* such that $\{\tilde{\lambda}\} \cup \{\tilde{\alpha} :\ \alpha \in \mathfrak{R}(\mathfrak{S}, L)\}$ generates H^*.

By Lie's theorem, there exists $X \in \mathfrak{g}_{\bar{1}}^1\backslash\{0\}$ and there exists $\lambda \in L^*$ such that $[l, X] = \lambda(l)X,\ \forall l \in L$. Consider $\tilde{\lambda} \in H^*$ defined by $\tilde{\lambda}_{|_L} = \lambda$ and $\tilde{\lambda}(C) = -1$. It follows that

$$[C, X] - \tilde{\lambda}(C)X = [l, X] - \tilde{\lambda}(l)X = 0,\ \forall l \in L,$$

consequently, $[h, X] - \tilde{\lambda}(h)X = 0,\ \forall h \in H$, so $X \in (\mathfrak{g}^{\tilde{\lambda}})_{\bar{1}}$. Therefore $\tilde{\lambda} \in \Delta_{\bar{1}} \subset \Delta$.

The fact that $\tilde{\lambda}(C) \neq 0$ implies that $\tilde{\lambda}$ is not an element of the vector subspace W of H^* generated by $\{\tilde{\alpha} : \ \alpha \in \mathfrak{R}(\mathfrak{S}, L)\}$, it follows that

$$\dim(W + \mathbb{K}\tilde{\lambda}) = \dim W + 1 = \dim L^* + 1 = \dim H^*.$$

Consequently $\{\tilde{\lambda}\} \cup \{\tilde{\alpha} : \ \alpha \in \mathfrak{R}(\mathfrak{S}, L)\}$ generates H^*. We conclude that Δ generates H^* because $\{\tilde{\lambda}\} \cup \{\tilde{\alpha} : \ \alpha \in \mathfrak{R}(\mathfrak{S}, L)\} \subset \Delta$. □

Remarks 3.1. 1) If $\mathfrak{g}$ is a simple Lie algebra, H a Cartan subalgebra of $\mathfrak{g}$ and Δ the set of the roots of $\mathfrak{g}$ relative to H then, by Lemma 3.1, we have: $(\alpha|\alpha) \neq 0\ \forall \alpha \in \Delta$. From the proof of Lemma 3.2, we can see that this is not true in the case of simple Lie superalgebra. More precisely, if $\mathfrak{g}$ is a basic classical Lie superalgebra such that $\mathfrak{g}_{\bar{0}}$ is not a semisimple Lie algebra, then $(\tilde{\lambda}|\tilde{\lambda}) = 0$ where $\tilde{\lambda}$ is the root of $\mathfrak{g}$ constructed in the proof of Lemma 3.2. Indeed, if we suppose that $(\tilde{\lambda}|\tilde{\lambda}) \neq 0$, then, by Proposition 2.5, $2\tilde{\lambda} \in \Delta_{\bar{0}}$ because $\tilde{\lambda} \in \Delta_{\bar{1}}$, so $2\tilde{\lambda}(\mathfrak{z}(\mathfrak{g}_{\bar{0}})) = \{0\}$, which contradicts the fact that $\tilde{\lambda}(C) \neq 0$. Hence $(\tilde{\lambda}|\tilde{\lambda}) = 0$.

Note that there exist basic classical Lie superalgebras $\mathfrak{g}$ such that the even part $\mathfrak{g}_{\bar{0}}$ is a semisimple Lie algebra and $(\alpha|\alpha) \neq 0\ \forall \alpha \in \Delta$ where Δ is the set of the roots of $\mathfrak{g}$ relative to a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$, for example $osp(1, 2)$, and there exist basic classical Lie superalgebras $\mathfrak{g}$ such that $\mathfrak{g}_{\bar{0}}$ is a semisimple Lie algebra with at least one root α relative to a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$ such that $(\alpha|\alpha) = 0$, for example $A(1, 1) = sl(2, 2)/\mathbb{K}.I_4$.

2) Let $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ be a quadratic Lie superalgebra, H a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$, Δ the set of the roots of $\mathfrak{g}$ relative to H. In Proposition 2.3, we proved that if $\alpha \in \Delta_{\bar{0}}$ such that $(\alpha|\alpha) = 0$, then $(\alpha|\beta) = 0\ \forall \beta \in \Delta$. Remark that there exist quadratic Lie superalgebras $\mathfrak{g}$ with $\alpha \in \Delta_{\bar{1}}$ such that $(\alpha|\alpha) = 0$ and there exists $\beta \in \Delta$ such that $(\alpha|\beta) \neq 0$, where Δ consists of the roots of $\mathfrak{g}$ relative to a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$. Indeed, if $\mathfrak{g}$ is a basic classical Lie superalgebra such that $\mathfrak{g}_{\bar{0}}$ is not semisimple, then $(\tilde{\lambda}|\tilde{\lambda}) = 0$ where $\tilde{\lambda}$ is the root constructed in the above proof. Since Δ generates H^* and $\tilde{\lambda} \neq 0$, then there exists $\beta \in \Delta$ such that $(\alpha|\beta) \neq 0$.

Now we are in position to state the first main result of this third section.

Theorem 3.1. *Let $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ be a B-irreducible quadratic Lie superalgebra such that the $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ is semisimple and $\mathfrak{g}_{\bar{0}} \neq \{0\}$, H a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$ and Δ the set of the roots of $\mathfrak{g}$ relative to H. Then $\mathfrak{g}$ is a basic classical Lie superalgebra if and only if Δ generates the linear dual H^* of H.*

Proof. If $\mathfrak{g}$ is a basic classical Lie superalgebra, then by Lemma 3.2, Δ generates the linear dual H^* of H.

Conversely, suppose that Δ generates the linear dual H^* of H.

Claim 1: $\mathfrak{z}(\mathfrak{g}) = \{0\}$.

Proof of Claim 1. Let $x \in \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{0}}$, then $x \in H$ and $[x, e_\alpha] = 0\ \forall \alpha \in \Delta$. Therefore $\alpha(x)e_\alpha = 0\ \forall \alpha \in \Delta$, so $\alpha(x) = 0\ \forall \alpha \in \Delta$, consequently $x = 0$ because Δ generates H^*. It follows that $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{0}} = \{0\}$, then, by Lemma 2.3, $\mathfrak{z}(\mathfrak{g}) = \{0\}$.

Claim 2: If $\alpha \in \Delta_{\bar{0}}$, then $(\alpha|\alpha) \neq 0$.

Proof of Claim 2. Let $\alpha \in \Delta_{\bar{0}}$. If we assume that $(\alpha|\alpha) = 0$, then by Proposition 2.3, $(\alpha|\beta) = 0\ \forall \beta \in \Delta$, consequently $h_\alpha = 0$ because Δ generates H^*, which contradicts the fact that $\alpha \neq 0$. Hence, $(\alpha|\alpha) \neq 0\ \forall \alpha \in \Delta_{\bar{0}}$.

Claim 3: If $\alpha \in \Delta_{\bar{0}}$ or if $\alpha \in \Delta_{\bar{1}}$ such that $(\alpha|\alpha) \neq 0$, then $[h, h_\alpha] = 0\ \forall h \in H$.

Proof of Claim 3. Let $h \in H$ and let $\alpha \in \Delta_{\bar{0}}$. By Claim 2, $(\alpha|\alpha) \neq 0$. The Proposition 2.5 implies that $\dim(\mathfrak{g}^\alpha)_{\bar{0}} = \dim(\mathfrak{g}^{-\alpha})_{\bar{0}} = 1$, so, by Proposition 2.2, $(\mathfrak{g}^\alpha)_{\bar{0}} = \mathbb{K}e_\alpha$ and $(\mathfrak{g}^{-\alpha})_{\bar{0}} = \mathbb{K}e_{-\alpha}$ such that $[e_\alpha, e_{-\alpha}] = h_\alpha$. Then we have:

$$[h, h_\alpha] = [h, [e_\alpha, e_{-\alpha}]] = [e_\alpha, [h, e_{-\alpha}]] - [e_{-\alpha}, [h, e_\alpha]] = (-\alpha(h) + \alpha(h))[e_\alpha, e_{-\alpha}] = 0.$$

Now, let $\alpha \in \Delta_{\bar{1}}$ such that $(\alpha|\alpha) \neq 0$ and let $h \in H$. The assertion ii) of Proposition 2.5 implies that $\dim(\mathfrak{g}^\alpha)_{\bar{1}} = \dim(\mathfrak{g}^{-\alpha})_{\bar{1}} = 1$, Consequently, by Proposition 2.2, $(\mathfrak{g}^\alpha)_{\bar{1}} = \mathbb{K}f_\alpha$ and $(\mathfrak{g}^{-\alpha})_{\bar{1}} = \mathbb{K}f_{-\alpha}$ such that $[f_\alpha, f_{-\alpha}] = -h_\alpha$. So

$$[h_\alpha, h] = [h, [f_\alpha, f_{-\alpha}]] = [f_\alpha, [h, f_{-\alpha}]] + [f_{-\alpha}, [h, f_\alpha]] = (-\alpha(h) + \alpha(h))[f_\alpha, f_{-\alpha}] = 0.$$

Claim 4: If $\alpha, \beta \in \Delta_{\bar{1}}$ such that $\alpha \neq \beta$ and $(\alpha|\alpha) = (\beta|\beta) = 0$, then $[h_\alpha, h_\beta] = 0$.

Proof of Claim 4. First case: $(\alpha|\beta) \neq 0$. Then, by Proposition 2.6, $\alpha + \beta \in \Delta_{\bar{0}}$ or $\alpha - \beta \in \Delta_{\bar{0}}$. By Claim 3, it follows that

$$[h, h_{(\alpha+\beta)}] = 0\ \forall h \in H \text{ or } [h, h_{(\alpha-\beta)}] = 0\ \forall h \in H.$$

Consequently $[h, h_\alpha] = -[h, h_\beta]\ \forall h \in H$ or $[h, h_\alpha] = [h, h_\beta]\ \forall h \in H$, which implies that $[h_\alpha, h_\beta] = 0$.

Second case: $(\alpha|\beta) = 0$. If we assume that $\alpha + \beta \in \Delta_{\bar{0}}$ or $\alpha - \beta \in \Delta_{\bar{0}}$, then, by Claim 2, $(\alpha+\beta|\alpha+\beta) \neq 0$ or $(\alpha-\beta|\alpha-\beta) \neq 0$, it follows that $(\alpha|\beta) \neq 0$, because $(\alpha|\alpha) = (\beta|\beta) = 0$, which contradicts the fact that $(\alpha|\beta) = 0$. Hence, $\alpha + \beta \notin \Delta_{\bar{0}}$ and $\alpha - \beta \notin \Delta_{\bar{0}}$. By the assertion iv) of Proposition 2.1, it follows that $-\alpha - \beta \notin \Delta_{\bar{0}}$ and $-\alpha + \beta \notin \Delta_{\bar{0}}$. Consequently, we have:

$$[(\mathfrak{g}^\alpha)_{\bar{1}}, (\mathfrak{g}^\beta)_{\bar{1}}] = [(\mathfrak{g}^{-\alpha})_{\bar{1}}, (\mathfrak{g}^\beta)_{\bar{1}}] = [(\mathfrak{g}^\alpha)_{\bar{1}}, (\mathfrak{g}^{-\beta})_{\bar{1}}] = [(\mathfrak{g}^{-\alpha})_{\bar{1}}, (\mathfrak{g}^{-\beta})_{\bar{1}}] = \{0\}. \qquad (*)$$

By Proposition 2.2, there exist $f_\alpha \in (\mathfrak{g}^\alpha)_{\bar{1}},\ f_{-\alpha} \in (\mathfrak{g}^{-\alpha})_{\bar{1}},\ f_\beta \in (\mathfrak{g}^\beta)_{\bar{1}},\ f_{-\beta} \in (\mathfrak{g}^{-\beta})_{\bar{1}}$ such that $[f_\alpha, f_{-\alpha}] = -h_\alpha$ and $[f_\beta, f_{-\beta}] = -h_\beta$. So, by $(*)$ and by Jacobi identity, we have:

$$[h_\alpha, h_\beta] = [[f_\alpha, f_{-\alpha}], [f_\beta, f_{-\beta}]] = 0.$$

Then, if $\alpha, \beta \in \Delta_{\bar{1}}$ such that $(\alpha|\alpha) = (\beta|\beta) = 0$, then $[h_\alpha, h_\beta] = 0$ and the Claim 4 is proved.

By Claims 2, 3 and 4, we conclude that $[h_\alpha, h_\beta] = 0\ \forall \alpha, \beta \in \Delta$.

Because the linear map $\varphi : H \to H^*$ defined by $\varphi(h)(x) = B(h,x)$, $\forall x \in H$ is an isomorphism of vector spaces, then the fact that Δ generates H^* implies that $\{h_\alpha : \alpha \in \Delta\}$ generates H. Since $[h_\alpha, h_\beta] = 0$ $\forall \alpha, \beta \in \Delta$, then H is commutative. Consequently, by Lemma 3.1, $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra because its Cartan subalgebra H is commutative and $(\alpha|\alpha) \neq 0$ $\forall \alpha \in \Delta_{\bar{0}}$. Since $\mathfrak{z}(\mathfrak{g}) = \{0\}$, then by Theorem 1.2, the Lie superalgebra $\mathfrak{g}$ is simple. We conclude that $\mathfrak{g}$ is a basic classical Lie superalgebra. □

An immediate open question is

Question. *Let $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ be a B-irreducible quadratic Lie superalgebra, H a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$ and Δ the set of the roots of $\mathfrak{g}$ relative to H. If Δ generates the linear dual H^* of H, is $\mathfrak{g}$ a basic classical Lie superalgebra?*

In the theorem above, we have shown that the answer is affirmative if the $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ is semisimple.

Remark 3.2. From the proof of Theorem 3.1, we can see that if $\mathfrak{g}$ is a quadratic Lie superalgebra such that the set of the roots of $\mathfrak{g}$ relative to a Cartan subalgebra H of $\mathfrak{g}_{\bar{0}}$ generates the linear dual H^* of H, then $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra. Consequently we get the following result: *Let $\mathfrak{g}$ be a simple quadratic Lie superalgebra, H a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$ and Δ the set of roots of $\mathfrak{g}$ relative to H. Then Δ generates the linear dual H^* of H if and only if $\mathfrak{g}$ is classical.*

Indeed, if $\mathfrak{g}$ is classical then, by Lemma 3.2, Δ generates H^*. Conversely, if Δ generates H^*, then $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra, so, by Theorem 1 of [16] (page 101), $\mathfrak{g}$ is classical.

Corollary 1. *Let $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ be a quadratic Lie superalgebra such that the $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ is semisimple , H a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$ and Δ the set of the roots of $\mathfrak{g}$ relative to H. Then the following assertions are equivalent:*

- i) *Δ generates the linear dual H^* of H;*
- ii) *$\mathfrak{g} = \bigoplus_{i=1}^n \mathfrak{g}_i$, where each $\mathfrak{g}_i$ is a non-degenerate graded ideal of $\mathfrak{g}$ such that $\mathfrak{g}_i$ is a basic classical Lie superalgebra and $B(\mathfrak{g}_i, \mathfrak{g}_j) = \{0\}$ for all $i, j \in \{1, \ldots, n\}$ such that $i \neq j$. Moreover, this decomposition is unique.*

Proof. i) $\Rightarrow$ ii). Since $(\mathfrak{g}, B)$ is a quadratic Lie superalgebra, then, by Proposition 1.1, $\mathfrak{g} = \bigoplus_{i=1}^n \mathfrak{g}_i$ such that:

- (a) $\mathfrak{g}_i$ is a non-degenerate graded ideal, for all $i \in \{1, \ldots, n\}$,
- (b) $(\mathfrak{g}_i, B_i = B_{|(\mathfrak{g}_i \times \mathfrak{g}_i)})$ is B_i-irreducible, for all $i \in \{1, \ldots, n\}$,
- (c) $B(\mathfrak{g}_i, \mathfrak{g}_j) = \{0\}$ for all $i, j \in \{1, \ldots, n\}$ such that $i \neq j$. Moreover, the fact that the $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ is semisimple implies that the $(\mathfrak{g}_i)_{\bar{0}}$-module $(\mathfrak{g}_i)_{\bar{1}}$ is semisimple for all $i \in \{1, \ldots, n\}$.

It is well-known that the Cartan subalgebra H of $\mathfrak{g}_{\bar{0}}$ takes the form $H = \bigoplus_{i=1}^n H_i$, where H_i is a Cartan subalgebra of $(\mathfrak{g}_i)_{\bar{0}}$, $1 \leq i \leq n$, and it's clear that the fact that Δ generates H^* implies that the set Δ_i of the roots of $\mathfrak{g}_i$ relative to a Cartan subalgebra H_i of $(\mathfrak{g}_i)_{\bar{0}}$

generates a linear dual $(H_i)^*$ of H_i, for all $i \in \{1, \dots, n\}$. Consequently, by Theorem 3.1, $\mathfrak{g}_i$ is a basic classical Lie superalgebra for all $i \in \{1, \dots, n\}$.

Now, suppose that $\mathfrak{g} = \bigoplus_{i=1}^{m} \mathfrak{g}'_i$, where each $\mathfrak{g}'_i$ is a graded ideal of $\mathfrak{g}$ such that $\mathfrak{g}'_i$ is a basic classical Lie superalgebra. Let $k \in \{1, \dots, m\}$, since $\mathfrak{g}'_k$ is simple, then $[\mathfrak{g}'_k, \mathfrak{g}'_k] = \mathfrak{g}'_k$. Consequently, $[\mathfrak{g}, \mathfrak{g}'_k] = \mathfrak{g}'_k$, it follows that $\mathfrak{g}'_k = \bigoplus_{i=1}^{n} [\mathfrak{g}_i, \mathfrak{g}'_k]$. Since $\mathfrak{g}'_k$ is simple, then there exists $\alpha(k) \in \{1, \dots, n\}$ such that $\mathfrak{g}'_k = [\mathfrak{g}_{\alpha(k)}, \mathfrak{g}'_k]$. The fact that $\mathfrak{g}_{\alpha(k)}$ and $\mathfrak{g}'_k$ are simple graded ideals of $\mathfrak{g}$ implies that $\mathfrak{g}_{\alpha(k)} = \mathfrak{g}'_k$. We conclude that $n = m$ and for any $k \in \{1, \dots, n\}$ there exists a unique $\alpha(k) \in \{1, \dots, n\}$ such that $\mathfrak{g}_{\alpha(k)} = \mathfrak{g}'_k$. We conclude that the decomposition $\mathfrak{g} = \bigoplus_{i=1}^{n} \mathfrak{g}_i$, where each $\mathfrak{g}_i$ is a graded ideal of $\mathfrak{g}$ such that $\mathfrak{g}_i$ is a basic classical Lie superalgebra, is unique.

The implication ii) $\Rightarrow$ i) is clear. □

Remark 3.3. The following result is the version of the Corollary 1 above in the case of Lie algebras: *Let $(\mathfrak{g}, B)$ be a quadratic Lie algebra, H be a Cartan subalgebra of $\mathfrak{g}$ and Δ be the set of the roots of $\mathfrak{g}$ relative to H. Then, $\mathfrak{g}$ is a semisimple Lie algebra if and only if Δ generates the linear dual H^* of H.*

Remark that using Proposition 13 of [2], we can obtain this result.

Corollary 2. *Let $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ be a quadratic Lie superalgebra, H a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$ and Δ the set of the roots of $\mathfrak{g}$ relative to H. $\Delta_{\bar{0}}$ generates H^* if and only if $\mathfrak{g} = \bigoplus_{i=1}^{n} \mathfrak{g}_i$, where each $\mathfrak{g}_i$ is a non-degenerate graded ideal of $\mathfrak{g}$ such that $\mathfrak{g}_i$ is a basic classical Lie superalgebra with $(\mathfrak{g}_i)_{\bar{0}}$ a semisimple Lie algebra , and $B(\mathfrak{g}_i, \mathfrak{g}_j) = \{0\}$ for all $i, j \in \{1, \dots, n\}$ such that $i \neq j$. Moreover, this decomposition is unique.*

Proof. Suppose that $\Delta_{\bar{0}}$ generates H^*, then, by Remark 3.3, $\mathfrak{g}_{\bar{0}}$ is a semisimple Lie algebra, so the $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ is semisimple. By Corollary 1, it follows that $\mathfrak{g} = \bigoplus_{i=1}^{n} \mathfrak{g}_i$, where each $\mathfrak{g}_i$ is a non-degenerate graded ideal of $\mathfrak{g}$ such that $\mathfrak{g}_i$ is a basic classical Lie superalgebra, and $B(\mathfrak{g}_i, \mathfrak{g}_j) = \{0\}$ for all $i, j \in \{1, \dots, n\}$ such that $i \neq j$. Moreover, this decomposition is unique. The fact that $\mathfrak{g}_{\bar{0}}$ is a semisimple implies that $(\mathfrak{g}_i)_{\bar{0}}$ is a semisimple Lie algebra for all $i \in \{1, \dots, n\}$. The converse is clear. □

In Theorem 3.1 we obtained a characterization of the basic classical Lie superalgebras among the quadratic Lie superalgebras with completely reducible action of even part on odd part. Now, in the following theorem we give a characterization of the basic classical Lie superalgebras among the quadratic Lie superalgebras.

Theorem 3.2. *Let $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ be a quadratic B-irreducible Lie superalgebra, H a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$. Then $\mathfrak{g}$ is a basic classical Lie superalgebra if and only if $\mathfrak{z}(\mathfrak{g}) = \{0\}$ and $\mathrm{ad}_{\mathfrak{g}} h$ is semisimple for all $h \in H$.*

Proof. If $\mathfrak{g}$ is a basic classical Lie superalgebra, then $\mathfrak{z}(\mathfrak{g}) = \{0\}$ and $ad_{\mathfrak{g}}h$ is semisimple for all $h \in H$ (see [12], [13] or [16]). Conversely, suppose that $\mathfrak{z}(\mathfrak{g}) = \{0\}$ and $\mathrm{ad}_{\mathfrak{g}} h$ is semisimple for all $h \in H$, and show that $\mathfrak{g}$ is a basic classical Lie superalgebra. Let $\alpha \in \Delta_{\bar{0}}$. If we suppose that $(\alpha|\alpha) = 0$, then by Proposition 2.3, $(\alpha|\beta) = 0\ \forall \beta \in \Delta$.

The fact that $\mathrm{ad}_{\mathfrak{g}} h_\alpha$ is semisimple implies that $[h_\alpha, H] = \{0\}$ and $[h_\alpha, \mathfrak{g}^\beta] = \{0\}$, it follows that $h_\alpha \in \mathfrak{z}(\mathfrak{g}) = \{0\}$, which contradicts the fact that $\alpha \neq 0$. We conclude that $(\alpha|\alpha) \neq 0 \ \forall \alpha \in \Delta_{\bar{0}}$.

Moreover, H is commutative because $\mathrm{ad}_{\mathfrak{g}} h$ is semisimple for all $h \in H$, then by Lemma 3.1, $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra. Since $\mathfrak{z}(\mathfrak{g}_{\bar{0}}) \subset H$, then $\mathrm{ad}_{\mathfrak{g}} x$ is semisimple for all $x \in \mathfrak{z}(\mathfrak{g}_{\bar{0}})$. By Corollary 1.6.4 of [7], it follows that the $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ is semisimple. By Theorem 1.2, we conclude that $\mathfrak{g}$ is a simple classical Lie superalgebra. □

Corollary. *Let* $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ *be a quadratic Lie superalgebra,* H *a Cartan subalgebra of* $\mathfrak{g}_{\bar{0}}$*. Then the following assertions are equivalent:*

i) $\mathfrak{z}(\mathfrak{g}) = \{0\}$ *and* $\mathrm{ad}_{\mathfrak{g}} h$ *is semisimple for all* $h \in H$*;*

ii) $\mathfrak{g} = \bigoplus_{i=1}^{n} \mathfrak{g}_i$, *where each* $\mathfrak{g}_i$ *is a non-degenerate graded ideal of* $\mathfrak{g}$ *such that* $\mathfrak{g}_i$ *is a basic classical Lie superalgebra, and* $B(\mathfrak{g}_i, \mathfrak{g}_j) = \{0\}$ *for all* $i, j \in \{1, \ldots, n\}$ *such that* $i \neq j$*. Moreover, this decomposition is unique.*

Remark 3.4. The following result is the version of the corollary above in the case of Lie algebras: *Let* $(\mathfrak{g}, B)$ *be a quadratic Lie algebra,* H *a Cartan subalgebra of* $\mathfrak{g}$*. Then* $\mathfrak{g}$ *is a semisimple Lie algebra if and only if* $\mathfrak{z}(\mathfrak{g}) = \{0\}$ *and* $\mathrm{ad}_{\mathfrak{g}} h$ *is semisimple for each element* h *of* H*.*

This result is Theorem II.1. of [3].

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