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Koszul Cohomology and k-Normality of a Projective Variety

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Abstract. Let X be a smooth projective variety and let L be a very ample divisor of X embedding it in $\mathbb{P}^N$. In this paper we use the Koszul groups of X to get information about the k-normality of X (i.e. the surjectivity of the map $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k)) \to H^0(X, kL)$ via an upper bound for the degree of the generators of $\oplus_{t\geq 0} H^0(X, tL)$. The above idea is applied to some scrolls over curves and surfaces and to some other varieties, by using also results due to Green and Butler.

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1. Introduction

It is well-known that "there are fewer ways to compute Koszul cohomology groups than reasons to compute them" (see [4]). In this paper we want to give another reason to compute them: almost every work on Koszul cohomology of a smooth projective subvariety X of $\mathbb{P}^N$ considers only the case in which X is projectively normal, (p.n.) (see [6], [7], [8], [13]) in which case Koszul groups give immediately a free resolution of the ideal sheaf $\mathcal{I}_X$ of X in $\mathbb{P}^N$ which is the main interest of the above papers. We know only Birkenhake's works [10] and [11] treating the case in which X is not linearly normal. When X is not p.n. Koszul groups give only an upper bound for the degree of the generators of the ring $R(X) = \oplus_{t\geq 0} H^0(X, tL)$ where L is the very ample line bundle of X giving the embedding of X in $\mathbb{P}^N$. Note that $R(X)$

is the coordinate ring of X if X is projectively normal. In some cases, e.g. for some scrolls, the information given by Koszul groups on $R(X)$ is sufficient to establish the k-normality of X.

More precisely in this paper we prove that for a scroll X over a smooth curve, whose dimension is at least three, the ring $R(X)$ is generated in degree 2 if a condition weaker than Butler's one (see [9]) is satisfied. The same fact is true for varieties, 4-dimensional at least, which are fibered in hypersurfaces of degree 2 and 3 over a smooth curve. Hence these varieties are projectively normal if and only if they are 2-normal, moreover, this fact is true for scrolls over a genus 2 curve without any other assumptions.

As a consequence of a suitable use of corollary 1.d.4 of [4], we get that for a regular surface (X, L) such that there exists a smooth curve in $|L - K_X|$, $R(X)$ is generated in degree 2 and 3. We also obtain some conditions assuring the projective normality of scrolls on surfaces.

The paper is organized as follows: In Section 2 we fix notation and recall some facts about Koszul cohomology; in Section 3 we use Butler's work to compute some Koszul vanishings for scrolls and varieties which are fibered in hypersurfaces; in Section 4 we show some other vanishings for Koszul cohomology of scrolls; in Section 5 we consider another method to compute vanishings and we apply it to regular surfaces and to scrolls over surfaces.

2. Notation and background material

$\mathbb{P}^N$	N-dimensional projective space over $\mathbb{C}$
S	$\mathbb{C}[x_0, x_1, \dots, x_N]$ the coordinate ring of $\mathbb{P}^N$
$S(a)$	the graded ring S twisted by the integer a
X	smooth n-dimensional projective subvariety of $\mathbb{P}^N$
K_X	canonical divisor of X
L	very ample line bundle embedding X in $\mathbb{P}^N$ via $H^0(X, L)$
I_X	the homogeneous ideal of X in the ring S
$\mathcal{I}_X$	the ideal sheaf of X in $\mathbb{P}^N$
$\mathcal{O}_X$	the structural sheaf of X
Ω_X	cotangent bundle of X
$R(X)$	the graded ring $\oplus_{t\geq 0} H^0(X, tL)$ which is an S-module
C	smooth algebraic curve of genus g
E	rank r vector bundle over a smooth variety X
E^*	its dual
$\mu(E)$	slope of E
$\mu^-(E)$	minimal slope of a quotient vector bundle of E over C
$\mu^+(E)$	maximal slope of a subbundle of E over C
$\mathbb{P}(E)$	projectivized of E
p	natural projection from $\mathbb{P}(E)$ to X
T	tautological line bundle of $\mathbb{P}(E)$
F	numerical class of a fibre in $\mathbb{P}(E)$ or generic fibre of p
$\sim$	linear equivalence among divisors
$\equiv$	numerical equivalence among divisors

Let (X, L) be as above, i.e. a smooth, linearly normal subvariety of $\mathbb{P}^N$, embedded by $H^0(X, L)$, where $N = h^0(X, L) - 1$. $R_t = H^0(X, tL)$, $R_0 = H^0(X, \mathcal{O}_X)$, then $R(X) = \oplus_{t\geq 0} R_t$ is a graded S-module having a minimal free resolution $\cdots \to E_{p+1} \to E_p \to \cdots \to E_1 \to E_0 \to R \to 0$ in which $E_0 = \oplus_{q\geq 0}(B_{0,q} \otimes S(-q))$, $E_1 = \oplus_{q\geq 0}(B_{1,q} \otimes S(-q))$ and so on, where $B_{p,q}$ are $\mathbb{C}$ vector spaces whose dimensions $b_{p,q}$ keep track of how many $S(-q)$ appear in E_p; the $b_{p,q}$ do not depend on the choice of the minimal free resolution, (see [6]). We will write $b_{p,q}$ instead of $b_{p,q}(X)$, R instead of $R(X)$, when any confusion is impossible.

Note that $S = \oplus_{t\geq 0} H^0(X, \mathcal{O}_{\mathbb{P}^N}(t))$, so we have a natural graded map $\rho : S \to R$ and the S-module structure on R is given by $sr = \rho(s)r$. X is p.n. if every graded piece of ρ is surjective.

E_0 is the free S-module corresponding to the generators of R, $b_{0,q}$ is the number of generators of R whose degree is q. Let us be careful: every R_t is also a $\mathbb{C}$ vector space of finite dimension, but we are considering R as an S-module: there is only one generator of degree 0, the multiplicative identity $\mathbf{1}$ of the ring R, which is also the generator of the $\mathbb{C}$ vector space R_0. There are no generators in degree 1 because, as X is linearly normal, every element of R_1 comes from S by ρ, so it is the product of an element of S and the generator $\mathbf{1}$, hence $b_{0,1} = 0$.

If X is p.n., for the same reason we have no other generators for R as an S-module, so that E_0 is isomorphic to S and the kernel of the map $E_0 \to R$ is precisely I_X, in this case $\cdots \to E_p \to \cdots \to E_1 \to I_X \to 0$ is a free resolution for I_X; this is the point of view of [6], [7], [8], [13], but what can we say when X is not p.n.? Let us examine E_0 firstly. We have the following

Proposition 2.1. *Let X be as above, then:*

- $b_{0,0} = 1$,
- $b_{0,1} = 0$,
- *X is 2-normal if and only if $b_{0,2} = 0$,*
- *for $q \geq 3$, if X is q-normal then $b_{0,q} = 0$ (but not vice versa),*
- *X is p.n. if and only if $b_{0,q} = 0$ for any $q \geq 2$.*

Proof. The values of $b_{0,0}$ and $b_{0,1}$ were discussed above. As X is linearly normal, the 2-normality of X is equivalent to the vanishing of $b_{0,2}$: in fact if X is 2-normal then $b_{0,2}=0$ because any element of R_2 is a multiple of $\mathbf{1}$ by an element of S. If there are no degree 2 generators in R, as S-module, every element of R_2 must be an S-linear combination of the generators of R of degree 0 or 1, i.e. it must be a multiple of $\mathbf{1}$ and then it comes from S by ρ. In any case if we consider the $\mathbb{C}$-linear map between $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(2))$ and R_2, $b_{0,2}$ is the $\mathbb{C}$-dimension of the cokernel of this map.

For $q \geq 3$ we have that if X is q-normal then $b_{0,q} = 0$ for the above reason, but not vice versa because $b_{0,q}$ is always the number of the degree q generators of R, but when we consider the $\mathbb{C}$-linear map between $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(q))$ and R_q, $b_{0,q}$ is only less than or equal the $\mathbb{C}$-dimension of the cokernel of this map because in the cokernel there can be also elements which are S-linear combinations of generators of R whose degree is less than q.

If X is p.n., R is generated over S by $\mathbf{1}$, hence $b_{0,q} = 0$ for all $q \geq 2$, the vice versa is obvious. $\square$

Let us consider E_1, i.e. the free S-module of the primitive syzygies among the generators of R, where primitive means, according to Green [4], that every considered degree q syzygy is not an S-linear combination of syzygies whose degree is less than q. Then $b_{1,q}$ is the number of the degree q generators of this S-module. We have the following

Proposition 2.2. *Let X be as above, then:*

- $b_{1,0} = b_{1,1} = 0$,
- $b_{1,2} = h^0(\mathbb{P}^N, \mathcal{I}_X(2))$,
- *if* $b_{1,q} = 0$ *for* $q \geq k+1$ *then* $b_{0,q} = 0$ *for* $q \geq k$.

Proof. The first vanishings are obvious. $b_{1,2}$ is the number of the generators of the S-module of degree 2 syzygies (in this degree every syzygy is primitive). As a degree q syzygy can involve only generators of R whose degree is less than or equal to $q-1$, a degree 2 syzygy is always of the following type: $s\mathbf{1} = 0$ with $s \in S$, because there are no degree 1 generators in R. Moreover, the number of the generators of the S-module of the degree 2 syzygies coincides with the dimension of this S-module viewed as a $\mathbb{C}$ vector space. Note that if $q \geq 3$ this is not longer true: the submodule of the primitive degree q syzygies of type $s\mathbf{1} = 0$ always corresponds to degree q $\mathbb{C}$-independent hypersurfaces of $\mathbb{P}^N$ containing X, but we have only that $b_{1,q}$ is greater than or equal to the $\mathbb{C}$-dimension of the $\mathbb{C}$-vector space of irreducible hypersurfaces of degree q containing X (which is less than or equal to $h^0(\mathbb{P}^N, \mathcal{I}_X(q))$: there can be primitive degree q syzygies which have no links with the hypersurfaces containing X.

Now let us assume that k is the maximal degree for the primitive syzygies among the generators of R: $\mathbf{1}, x_1, x_2, \ldots, x_h$, and, by contradiction, let us assume that one of these generators, say x_h, belongs to R_q with $q \geq k$, and thus there are no primitive syzygies involving x_h. However, it is easy to see that any generator is involved by a syzygy, so we have $\alpha_0\mathbf{1} + \alpha_1 x_1 + \cdots + \alpha_h x_h = 0$ where $\alpha_i \in S$, $\deg(\alpha_i) \geq 1$, and this is not a primitive syzygy. Hence it must be an S-linear combination of primitive syzygies, but this is not possible because no primitive syzygy involves x_h. □

Let us consider the exact sequence $0 \to M \to V \otimes \mathcal{O}_X \to L \to 0$ of vector bundles over X, where M is the kernel of the evaluation map $V \otimes \mathcal{O}_X \to L$ and $V = H^0(X, L)$. Let $q \geq 2$; to estimate $b_{1,q}$ we have

Proposition 2.3. *Let X, L, M be as above, then $b_{1,q} = 0$ if $H^1(X, \Lambda^2 M \otimes L^{q-2}) = 0$.*

Proof. By [7], we have that $b_{1,q}$ is the dimension of the $\mathbb{C}$-vector space which is the homology at the middle level in the following piece of the Koszul complex:

$$\cdots \to \Lambda^2(V) \otimes R_{q-2} \to V \otimes R_{q-1} \to R_q \cdots .$$

Let us call $\alpha_q : \Lambda^2(V) \otimes R_{q-2} \to V \otimes R_{q-1}$ and $\beta_q : V \otimes R_{q-1} \to R_q$. From it we get: $0 \to \Lambda^2 M \to \Lambda^2(V) \otimes \mathcal{O}_X \to V \otimes L \to S^2(L) = L \otimes L \to 0$ which splits as

$$0 \to M \otimes L \to V \otimes L \to L \otimes L \to 0 \quad \text{and} \quad 0 \to \Lambda^2 M \to \Lambda^2(V) \otimes \mathcal{O}_X \to M \otimes L \to 0.$$

Hence we have, for any $m \in \mathbb{Z}$: $0 \to \Lambda^2 M \otimes L^m \to \Lambda^2(V) \otimes L^m \to M \otimes L^{m+1} \to 0$ and $0 \to M \otimes L^m \to V \otimes L^m \to L^{m+1} \to 0$. By choosing $m = q-2$ in the first case we get the following exact sequence:

$$0 \to H^0(X, \Lambda^2 M \otimes L^{q-2}) \to \Lambda^2(V) \otimes R_{q-2} \to H^0(X, M \otimes L^{q-1}) \to H^1(X, \Lambda^2 M \otimes L^{q-2}) \to \cdots .$$

By choosing $m = q - 1$ in the second case we get this exact sequence:

$$0 \to H^0(X, M \otimes L^{q-1}) \to V \otimes R_{q-1} \to R_q \to \cdots .$$

If we call $\gamma_q : \Lambda^2(V) \otimes R_{q-2} \to H^0(X, M \otimes L^{q-1})$, $\delta_q : H^0(X, M \otimes L^{q-1}) \to V \otimes R_{q-1}$ and $\varepsilon_q : V \otimes R_{q-1} \to R_q$, we have that $\beta_q = \varepsilon_q$ and $\alpha_q = \delta_q \circ \gamma_q$.

Hence if $H^1(X, \Lambda^2 M \otimes L^{q-2}) = 0$ we get that γ_q is surjective, $\mathrm{kern}(\beta_q) = \mathrm{Im}(\alpha_q)$ and therefore $b_{1,q} = 0$. □

Remark 2.4. Note that, in the same way, it is possible to get that $b_{0,q} = 0$ if $H^1(X, M \otimes L^{q-1}) = 0$, i.e. the condition $H^1(X, M \otimes L^{q-1}) = 0$ for $q \geq 2$ implies that X is p.n.; in this form this condition is used by many authors (see [9], [14], [15] for instance). When X is p.n. and $H^1(X, \Lambda^2 M \otimes L^{q-2}) = 0$ for $q \geq 3$ then I_X is generated in degree 2, when X is not p.n. the condition yields only some information about the generators of R.

3. Syzygies of scrolls

In this section we consider the vanishing of $H^1(X, \Lambda^2 M \otimes L^{q-2})$, $q \geq 2$, for r-dimensional scrolls $X = \mathbb{P}(E)$ over smooth curves C, $r \geq 2$, where E is a very ample rank r vector bundle over C. In this case L is the tautological bundle T, $p_* T = E$ and we have the exact sequence $0 \to M_E \to H^0(C, E) \otimes \mathcal{O}_C \to E \to 0$, where $H^0(C, E) \otimes \mathcal{O}_C \to E$ is the natural evaluation map. Our strategy will be to calculate $h^1(X, \Lambda^2 M \otimes T^{q-2})$ by using $h^1(C, p_*(\Lambda^2 M \otimes T^{q-2}))$. It is well-known that the two numbers are equal if $R^i p_*(\Lambda^2 M \otimes T^{q-2}) = 0$, $\forall i \geq 1$ (see [12], p. 253) and this is true if $h^j(F, (\Lambda^2 M \otimes T^{q-2})_{|F}) = 0$, $\forall j \geq 1$. We have the following

Lemma 3.1. *With the above notations $h^j(F, (\Lambda^2 M \otimes T^{q-2})_{|F}) = 0$, $\forall j \geq 1$.*

Proof. Recall that $F \cong \mathbb{P}^{r-1}$, $(T^{q-2})_{|F} = \mathcal{O}_F(q-2)$, so that $h^j(F, (\Lambda^2 M \otimes T^{q-2})_{|F}) = h^j(F, (\Lambda^2 M_{|F} \otimes \mathcal{O}_F(q-2))$.

Now let us consider $0 \to \mathcal{O}_X(T - F) \to \mathcal{O}_X(T) \to \mathcal{O}_F(T_{|F}) \to 0$ and the long exact sequence $0 \to H^0(X, T - F) \to H^0(X, T) \to H^0(F, \mathcal{O}_F(1)) \to H^1(X, T - F) \to \cdots$. As $p_* T = E$ is generated by global sections we have that $H^0(X, T) = H^0(X, T - F) \oplus H^0(X, \mathcal{O}_F(1))$.

By considering the restriction to F of $0 \to M \to H^0(X, T) \otimes \mathcal{O}_X \to T \to 0$ we get $0 \to M_{|F} \to H^0(X, T) \otimes \mathcal{O}_F \to \mathcal{O}_F(1) \to 0$.

By using the Euler sequence for F we get the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & H^0(X, T-F) \otimes \mathcal{O}_F \arrow[d] & H^0(X, T-F) \otimes \mathcal{O}_F \arrow[d] & & \\
0 \arrow[r] & M_{|F} \arrow[r] \arrow[d] & H^0(X, T-F) \oplus H^0(X, \mathcal{O}_F(1)) \otimes \mathcal{O}_F \arrow[r] \arrow[d] & \mathcal{O}_F(1) \arrow[r] & 0 \\
0 \arrow[r] & \Omega_F(1) \arrow[r] \arrow[d] & H^0(X, \mathcal{O}_F(1)) \otimes \mathcal{O}_F \arrow[r] \arrow[d] & \mathcal{O}_F(1) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

The left column splits so that $M_{|F} \cong H^0(X, T-F) \otimes \mathcal{O}_F \oplus \Omega_F(1)$,
$\Lambda^2(M_{|F}) = \Omega_F^2(2) \oplus \Omega_F(1) \otimes H^0(X, T-F) \otimes \mathcal{O}_F \oplus \Lambda^2(H^0(X, T-F)) \otimes \mathcal{O}_F$, and
$\Lambda^2(M_{|F}) \otimes \mathcal{O}_F(q-2) = \Omega_F^2(q) \oplus \Omega_F(q-1) \otimes H^0(X, T-F) \otimes \mathcal{O}_F \oplus \Lambda^2(H^0(X, T-F)) \otimes \mathcal{O}_F(q-2)$,
now it is very easy to see that $h^j(F, (\Lambda^2 M \otimes T^{q-2})_{|F}) = 0$, $\forall j \geq 1$. □

Now we can prove

Theorem 3.2. *Let (X, T) be a scroll as above over a genus g curve, $r \geq 3$, $q \geq 4$, then $H^1(X, \Lambda^2 M \otimes L^{q-2}) = 0$ if $\mu^-(M_E) + \mu^-(E) > g-1$.*

Proof. By Lemma 3.1 we have to show that $h^1(C, p_*(\Lambda^2 M \otimes T^{q-2})) = 0$. As $p_*(M \otimes M \otimes T^{q-2}) = p_*[\Lambda^2 M \otimes T^{q-2}] \oplus p_*[S^2 M \otimes T^{q-2}]$ it suffices to show that $h^1(C, p_*(M \otimes M \otimes T^{q-2})) = 0$. This is true if $\mu^-[p_*(M \otimes M \otimes T^{q-2})] > 2g-2$, (see [9]).

We can use Prop. 4.2 of [9], in fact $p_*T = E$ is generated by global sections, T is 0 p-regular and $M \otimes T^{q-2}$ is -1 p-regular, hence we have that $\mu^-[p_*(M \otimes M \otimes T^{q-2})] \geq \mu^-(M_E) + \mu^-[p_*(M \otimes T^{q-2})]$; (note that, in this case, the inequality given by Prop. 4.2 of [9] is very simple because for $i = r-2$ we have that $R^i p_*(T^{-i}) = 0$, so that $\min\{\mu^-(M_E) + \mu^-[p_*(M \otimes T^{q-2})], +\infty\} = \mu^-(M_E) + \mu^-[p_*(M \otimes T^{q-2})]$; see [9] for the notion of k p-regular vector bundles).

Now we can use the same proposition for $\mu^-[p_*(M \otimes T^{q-2})]$, in fact T is 0 p-regular and T^{q-2} is -1 p-regular. By the same previous reason we get: $\mu^-[p_*(M \otimes T^{q-2})] \geq \mu^-(M_E) + \mu^-[S^{q-2}(E)]$ and $\mu^-[S^{q-2}(E)] = (q-2)\mu^-(E)$.

Hence we have $\mu^-[p_*(M \otimes M \otimes T^{q-2})] \geq 2\mu^-(M_E) + (q-2)\mu^-(E)$, but as T is very ample, this inequality is satisfied for $q \geq 4$ if it is true for $q = 4$, so that the condition is simply $\mu^-(M_E) + \mu^-(E) > g-1$. □

Remark 3.3. Assume that E is semistable, $g \geq 2$ and $\mu(E) < 2g$, then we have $\mu^-(M_E) > r(\mu^-(E) - 2g) - 2 + 2h^1(E)$ (see [9], Prop. 1.5), so the condition in Theorem 3.2 becomes: $(2+r)\mu(E) + 2h^1(E) > (2r+1)g + 1$.

Remark 3.4. Although Theorem 3.2 is stated for E very ample, exactly the same proof works also when E is ample and generated by global sections.

For the rest of this section we are concerned with the vanishing of $H^1(X, \Lambda^2 M \otimes L^{q-2})$, $q \geq 2$, when X is a divisor of $W = \mathbb{P}(E)$ where E is an ample, globally generated vector bundle over a smooth, genus g, curve C. We assume that L, the restriction to X of the tautological divisor T of W, is very ample. X is fibered over C and the generic fibre is a smooth hypersurface of $\mathbb{P}^{r-1}$ whose degree is fixed. We can prove the following

Proposition 3.5. *Let (X, L) be as above, assume that $X \equiv aT + bF$ with $a \geq 2$, then $H^1(X, \Lambda^2 M \otimes L^{q-2}) = 0$ if $r \geq 4$, $q \geq 4$, $\mu^-(M_E) + \mu^-(E) > g-1$ except, possibly, for $q = a$ and $q = a+1$.*

Proof. As E is generated by global sections we can consider the usual exact sequence

$$0 \to M_T \to H^0(W, T) \otimes \mathcal{O}_W \to T \to 0.$$

By restricting it to X we get: $0 \to (M_T)_{|X} \to H^0(W, T) \otimes \mathcal{O}_X \to L \to 0$ as $L = T_{|X}$. On the other hand we have $0 \to M \to H^0(X, L) \otimes \mathcal{O}_X \to L \to 0$, but it is easy to see that

$H^0(W,T)\otimes \mathcal{O}_X = H^0(X,L)\otimes \mathcal{O}_X$ so that $(M_T)_{|X} = M$. Hence we can tensorize the exact sequence $0\to \mathcal{O}_W(-X)\to \mathcal{O}_W \to \mathcal{O}_X \to 0$ with $\Lambda^2 M_T \otimes T^{q-2}$ and in cohomology we have

$$\cdots \to H^1(W,\Lambda^2 M_T\otimes T^{q-2}) \to H^1(X,\Lambda^2 M\otimes L^{q-2}) \to H^2(W,\Lambda^2 M_T\otimes \mathcal{O}_W(-X+(q-2)T) \to \cdots$$

as $H^1(X,\Lambda^2 M\otimes L^{q-2}) = H^1(X,\Lambda^2(M_T)_{|X}\otimes (T_{|X})^{q-2})$. By arguing as in 3.2 and by recalling that E ample implies $\mu^-(E) > 0$, we have $H^1(W,\Lambda^2 M_T\otimes T^{q-2}) = 0$. To deal with $H^2(W,\Lambda^2 M_T\otimes \mathcal{O}_W(-X+(q-2)T))$ recalling Remark 3.4, we proceed as in the proof of 3.2. Thus that group vanishes if $h^j(F,(\Lambda^2 M_T\otimes \mathcal{O}_W(-X+(q-2)T))_{|F}) = 0$, $\forall j\geq 1$.

As in the proof of 3.1 we have that $(\Lambda^2 M_T\otimes \mathcal{O}_W(-X+(q-2)T))_{|F}$ is the direct sum of some copies of $\Omega^2_F(q-a)$, $\Omega^1_F(q-a-1)$ and $\mathcal{O}_F(q-a-2)$ so that we get the vanishing for $q\geq a+2$. If $q\leq a-1$ (if necessary, recall that $a\geq 1$ in any case) we can consider $H^{r-2}(W,[\Lambda^2 M_T\otimes \mathcal{O}_W(-X+(q-2)T)]^*\otimes K_W)$ and we can proceed analogously as $r-2\geq 2$. □

Remark 3.6. If $a=2$ or $a=3$ (i.e. the fibres are hypersurfaces of degree 2 or 3) Proposition 3.5 shows that under the same assumptions of 3.2 for $\mathbb{P}(E)$, with $r\geq 4$, $b_{0,q}(X)=0$ if $q\geq 3$ and therefore X is p.n. if and only if it is 2-normal. If $r=3$ the previous proof works only for $q\geq a+2$.

Proposition 3.7. *Let (X,L) as above, with the same assumptions of* 3.5*, then* $b_{1,a}(X)=0$ *if* $b\geq 1$*.*

Proof. We have only to show that $H^2(W,\Lambda^2 M_T\otimes \mathcal{O}_W(-X+(a-2)T)) = 0$. By using $0\to \Lambda^2 M_T \to \Lambda^2(H^0(W,T))\otimes \mathcal{O}_W \to H^0(W,T)\otimes T \to S^2(T) = T\otimes T\to 0$ tensorized by $\mathcal{O}_W(-X+(a-2)T)$ we see that

$$H^2(W,\Lambda^2 M_T\otimes \mathcal{O}_W(-X+(a-2)T)) = H^1(W,M_T\otimes \mathcal{O}_W(-X+(a-1)T))$$

as $M_T\otimes T$ is the kernel of $H^0(W,T)\otimes T\to T\otimes T$.

Let B be a degree b divisor on C such that $X = aT+p^*B$, then $-X+(a-1)T = -T+p^*(-B)$ and $H^1(W,M_T\otimes \mathcal{O}_W(-T+p^*(-B))) = H^0(C,-B)$ by using Leray's spectral sequence as usual. If $b\geq 1$ we have the required vanishing. □

Remark 3.8. By arguing as in the previous proof we can show that

$$H^2(W,\Lambda^2 M_T\otimes \mathcal{O}_W(-X+(a-1)T)) = H^1(C,M_E\otimes B)$$

which does not vanish for $b\geq 1$, so that it is not possible to get conditions under which $b_{1,a+1}(X) = b_{1,a}(X) = 0$ by using this method.

4. Koszul groups for $X = \mathbb{P}(E)$

For any smooth n-dimensional X, embedded in $\mathbb{P}^N$ by a very ample line bundle L and for any vector bundle $\mathcal{E}$ over X we can consider the Koszul $\mathbb{C}$-vector spaces $K_{p,q}(X,\mathcal{E},L)$ (see [4]). Let $k_{p,q}(X,\mathcal{E},L)$ be the dimension of $K_{p,q}(X,\mathcal{E},L)$. It is $k_{p,q}(X,\mathcal{O}_X,L) = b_{p,p+q}$ for any p, q, so that the computation of these Koszul groups is related to the minimal resolutions of R.

For the convenience of the reader we recall the following basic results, due to M. Green, which will be used in the sequel:

Theorem 4.1. ([4], Th. 3.a.1) *$K_{p,q}(X,\mathcal{E},L)=0$ if $h^0(X,\mathcal{E}\otimes L^q)\le p$.*

Theorem 4.2. ([4], Th. 2.c.6)
$K_{p,q}(X,\mathcal{E},L)^*\cong K_{N-n-p,n+1-q}(X,\mathcal{E}^*\otimes K_X,L)$ if $h^i(X,\mathcal{E}\otimes L^{q-i})=0$ and $h^i(X,\mathcal{E}\otimes L^{q-i-1})=0$ for $i=1,2,\dots,n-1$.

In this section E is a rank r vector bundle over a smooth genus g curve C, $r\ge 2$, $X=\mathbb{P}(E)$ is embedded in $\mathbb{P}^N$ by a very ample line bundle $L\sim aT+p^*B$, where B is a divisor of C, $\deg(B)=b$. X is linearly normal, $N=h^0(L)-1$, $n=r$, $L\equiv aT+bF$, $\delta=c_1(E)$. We want to compute $k_{p,q}$ by using 4.1 and 4.2 when $\mathcal{E}=\mathcal{O}_X$.

First of all we consider $h^i(X,L^{q-i})$ for $i=1,2,\dots,r-1$ and $q\ge 2$. Recall that $L^{q-i}\sim(q-i)aT+p^*[(q-i)B]$ and that $a\ge 1$ and $a\mu^-(E)+b>0$ as L is very ample; moreover, by using Leray's spectral sequence and Kodaira's vanishing we have that all cohomology groups vanish but for $i=1$, in this case we have $h^1(X,L^{q-1})=h^1(C,S^{(q-1)a}(E)\otimes(q-1)B)=0$ if $(q-1)(a\mu^-(E)+b)>2g-2$.

Now we consider $h^i(X,L^{q-i-1})$ for $i=1,2,\dots,r-1$ and $q\ge 3$. Reasoning as in the previous case we get that all groups vanish if $(q-2)(a\mu^-(E)+b)>2g-2$. Note that if $q=2$, $i=1$ the corresponding group does not vanish unless $g=0$.

We have proved the following

Lemma 4.3. *With the notation as in this section let $q\ge 3$, $g\ge 1$, then $K_{p,q}(X,\mathcal{O}_X,L)^*\cong K_{N-r-p,r+1-q}(X,K_X,L)$ if $(q-2)(a\mu^-(E)+b)>2g-2$.*

Lemma 4.3 and Theorem 4.2 tell us that, under some conditions, for our varieties $k_{p,q}(X,\mathcal{O}_X,L)=0$ if $N-r-p<0$ and $k_{p,q}(X,\mathcal{O}_X,L)=0$ if $N-r-p\ge 0$ and $h^0(X,K_X+(r+1-q)L)\le N-r-p$. If $g\ge 1$ it is well-known that $N\ge 2r$, hence $N-r-p\ge r-p$, so that for $p=0,1,\dots,r$ to get $k_{p,q}(X,\mathcal{O}_X,L)=0$ it suffices that $h^0(X,K_X+(r+1-q)L)=0$.

We have $K_X+(r+1-q)L\equiv[(r+1-q)a-r]T+[d+2g-2+(r+1-q)b]F$, and such a line bundle has no sections if $(r+1-q)a-r<0$ or (see [9], Lemma 1.12) if $[(r+1-q)a-r]\mu^+(E)+\delta+2g-2+(r+1-q)b<0$ and $(r+1-q)a-r\ge 0$. Then we have proved the following:

Lemma 4.4. *With the notation as in this section let $q\ge 3$, $r\ge p\ge 0$, $g\ge 1$, then $K_{p,q}(X,\mathcal{O}_X,L)=0$ if $(q-2)(a\mu^-(E)+b)>2g-2$ and $(r+1-q)a-r<0$ or if $(r+1-q)a-r\ge 0$ and $[(r+1-q)a-r]\mu^+(E)+\delta+2g-2+(r+1-q)b<0$.*

Corollary 4.5. *Let $X=\mathbb{P}(E)$ as above with $r=2$, $g\ge 1$, then $b_{0,q}(X)=0$ for $q\ge 3$ if $a\mu^-(E)+b>2g-2$.*

Corollary 4.6. *Let $X=\mathbb{P}(E)$ be a scroll over a curve of genus $g\ge 1$ (hence $a=1$, $b=0$) then $b_{0,q}(X)=0$ for $q\ge 3$ if $\mu^-(E)>2g-2$, therefore X is p.n. if and only if it is 2-normal.*

Corollary 4.7. *Let $X=\mathbb{P}(E)$ be a scroll over a curve of genus $g=2$ then X is p.n. if and only if it is 2-normal; in fact in this case $\mu^-(E)>3$, (see [2]); moreover, $b_{p,p+q}(X)=0$ for $q\ge 3$, $r\ge p\ge 0$.*

Corollary 4.8. *Let $X = \mathbb{P}(E)$ be a scroll over a curve of genus $g \geq 1$ with $\mu^-(E) > 2g$, then X is projectively normal* (*see* [9]) *and I_X is generated in degree two.*

Remark 4.9. The previous results hold also if X is embedded by a linear subspace W of $H^0(X, L)$, i.e. if X is not linearly normal. To see this it is enough to use Green's Theorems 4.1 and 4.2, being careful to use $N' = \dim W$ instead of N in the previous formulas.

5. Long exact sequences for Koszul groups and applications

Let X be a smooth variety in $\mathbb{P}^N$ as usual and let Y be a smooth, one codimensional subvariety of X. Let L be a very ample divisor of X. Then from the natural exact sequence given by Y: $0 \to \mathcal{O}_X(-Y) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$ we get $0 \to H^0(X, qL - Y) \to H^0(X, qL) \to H^0(Y, qL_{|Y}) \cdots$ for any $q \geq 0$.

Assume that the previous sequence is exact for any $q \geq 0$, then if we put

$$\mathcal{A} = \oplus_{q\geq 0} H^0(X, qL - Y),\ \mathcal{B} = \oplus_{q\geq 0} H^0(X, qL) \text{ and } \mathcal{C} = \oplus_{q\geq 0} H^0(Y, qL_{|Y})$$

we get an exact sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ of S-modules from which we deduce some long exact sequences for Koszul groups (see [4], Cor. 1.d.4):

$$\cdots \to K_{p,q}(\mathcal{B}) \to K_{p,q}(\mathcal{C}) \to K_{p-1,q+1}(\mathcal{A}) \to K_{p-1,q+1}(\mathcal{B}) \to \cdots,$$

where $K_{p,q}(\mathcal{A}) = K_{p,q}(X, \mathcal{O}_X(-Y), L)$, $K_{p,q}(\mathcal{B}) = K_{p,q}(X, \mathcal{O}_X, L)$, $K_{p,q}(\mathcal{C}) = K_{p,q}(Y, \mathcal{O}_Y, L_{|Y})$.

In [3] the authors considered the projective normality of (X, L) in the case in which L is interesting from the point of view of adjunction theory, i.e. when $L = aK_X + bA$ where A is a suitable divisor of X and a, b are integers. Notice that it is the same point of view of [8] and [14], [15], [16]. Here, by using the previous ideas we can prove the following proposition:

Proposition 5.1. *Let X be a regular surface and let L be a very ample line bundle on X. Assume that there exists a smooth curve in $|L - K_X|$. Then $b_{0,q}(X) = 0$ for $q \geq 4$.*

Proof. Firstly notice that the proposition is true for $(X, L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$. Let now $(X, L) \neq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ and let Y be a smooth curve in $|L - K_X|$. From the exact sequence $0 \to K_X \to L \to L_{|Y} \to 0$, as X is regular it is easy to see that $h^1(X, L) = h^2(X, L) = 0$ and that Y is linearly normal in the embedding given by L. Now let

$$\mathcal{A} = \oplus_{q\geq 0} H^0(X, qL - Y),\ \mathcal{B} = \oplus_{q\geq 0} H^0(X, qL),\ \mathcal{C} = \oplus_{q\geq 0} H^0(Y, qL_{|Y}).$$

The regularity of X and Kodaira's vanishing theorem give an exact sequence of S-modules $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$, which in turn gives the exact sequences

$$\cdots \to K_{0,q}(\mathcal{A}) \to K_{0,q}(\mathcal{B}) \to K_{0,q}(\mathcal{C}) \to 0$$

for any $q \geq 0$.

$K_{0,q}(\mathcal{C}) = 0$ for $q \geq 2$ because Y is p.n. in $\mathbb{P}^{N-1}$ as it is canonically embedded by $L_{|Y} = K_Y$, hence it suffices to show that $b_{0,q}(\mathcal{A}) = K_{0,q}(X, \mathcal{O}_X(-Y), L) = 0$ for $q \geq 4$. Note that here $\mathcal{C}$ is considered as an S-module, not a $\mathbb{C}[x_0, x_1, \ldots, x_{N-1}]$-module, however, Y is p.n. in $\mathbb{P}^N$ too, so that $b_{0,q}(\mathcal{C}) = k_{0,q}(\mathcal{C}) = 0$, $\forall q \geq 2$.

We can use Theorem 4.2 as $h^1(X, -Y+(q-1)L) = h^1(X, -Y+(q-2)L) = 0$ for $q \geq 4$, so we have to consider $K_{N-2,3-q}(X, \mathcal{O}_X(Y+K_X), L)$. Now we can use Theorem 4.1 because $h^0(X,(4-q)L) \leq h^0(X,L)-3$ for $q \geq 4$. □

The previous ideas can be applied in other cases, for instance when $X = \mathbb{P}(E)$ is the projectivized of a rank r vector bundle E over a surface Σ. In this case let T be the tautological bundle and $p : X \to \Sigma$ the natural projection as usual. Let C be a smooth curve on Σ, $C' = \pi^{-1}(C)$ and let us consider, for any $j \geq 0$, the exact sequences:

$$0 \to \mathcal{O}_X(jT - p^*C) \to \mathcal{O}_X(jT) \to \mathcal{O}_{C'}(jT_{|C'}) \to 0.$$

If we assume that $H^1(X, jT - p^*C) = 0$, $\forall j \geq 0$, $K_{0,q}(X, \mathcal{O}_X(-p^*C), T) = 0$, $\forall q \geq 2$, and $T_{|C'}$ embeds C' p.n. in $\mathbb{P}^{N-1}$, then we have that X is p.n. In fact by these assumptions there is an exact sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ of S-modules, and from the sequences $\cdots \to K_{0,q}(\mathcal{A}) \to K_{0,q}(\mathcal{B}) \to K_{0,q}(\mathcal{C}) \to \dots$ we have that $b_{0,q}(\mathcal{B}) = 0$, $\forall q \geq 2$, (i.e. X is p.n.) as $b_{0,q}(\mathcal{A}) = b_{0,q}(\mathcal{C}) = 0$, $\forall q \geq 2$, by assumptions. Note that here $\mathcal{C}$ is considered as an S-module, not a $\mathbb{C}[x_0, x_1, \dots, x_{N-1}]$-module, however, C' is p.n. in $\mathbb{P}^N$ too, so that $b_{0,q}(\mathcal{C}) = 0$, $\forall q \geq 2$.

Now we translate our assumptions into conditions on Σ. The first one is simply $H^1(\Sigma, S^j(E) \otimes \mathcal{O}_S(-C)) = 0$, $\forall j \geq 1$, the third one is satisfied if we assume that $\mu^-(E_{|C}) > 2g(C)$ by Butler's results [9]; for the second one we use Green's Theorems 4.1 and 4.2. Let us consider $h^i(X,(q-i)T + p^*(-C))$ for $i = 1, \dots, n-1 = r$, by standard calculations they vanish if $h^i(\Sigma, S^{q-i}(E) \otimes \mathcal{O}_\Sigma(-C)) = 0$ for $i = 1, \dots, r$ and $q \geq i$, so we have only to assume further that $h^2(\Sigma, S^{q-2}(E) \otimes \mathcal{O}_\Sigma(-C)) = 0$ for $q \geq 2$. In order to have $h^i(X,(q-i-1)T + p^*(-C)) = 0$ for $i = 1, \dots, r$, it suffices to ask that $h^1(\Sigma, \mathcal{O}_\Sigma(-C)) = 0$ by similar arguments. Hence we can apply Theorem 4.2 and we consider, for $q \geq 2$, $K_{N-n,n+1-q}(X, \mathcal{O}_X(p^*C + K_X), T)$. This group vanishes if $h^0(X, p^*C + K_X + (n+1-q)T) \leq N-n$, i.e. $h^0(\Sigma, C + \det(E) + K_\Sigma) \leq h^0(E) - r - 2$, by Theorem 4.1.

Thus we have proved the following

Theorem 5.2. *Let E be a very ample, rank r, vector bundle over a smooth surface Σ, let $X = \mathbb{P}(E)$ and let T be the tautological bundle. Moreover, let C be a smooth genus g curve on S. Then (X,T) is p.n. if*

1) *$h^1(\Sigma, S^j(E) \otimes \mathcal{O}_\Sigma(-C)) = 0$ for $j \geq 0$,*
2) *$h^2(\Sigma, S^j(E) \otimes \mathcal{O}_\Sigma(-C)) = 0$ for $j \geq 0$,*
3) *$h^0(\Sigma, C + \det(E) + K_\Sigma) \leq h^0(E) - r - 2$,*
4) *$\mu^-(E_{|C}) > 2g$.*

Remark 5.3. If C is a rational curve 4) is satisfied; if C is an ample divisor 1) and 2) are satisfied for $j = 0$; if $-K_\Sigma$ is effective 3) is more easily satisfied.

Now we want to give some examples in which Theorem 5.2 can be applied.

Example 5.4. Let $\pi : \Sigma \to \mathbb{P}^2$ be the blowing up of $\mathbb{P}^2$ of k points in general position with $1 \leq k \leq 5$, let L be the generator of $\mathrm{Pic}(\mathbb{P}^2)$, let $\mathsf{E}_1, \dots, \mathsf{E}_k$ be the exceptional divisors. Σ is a well-known Del Pezzo surface and it is known that, in this range, $-K_\Sigma$ is very ample. Let E be $-K_\Sigma \oplus -K_\Sigma$ and let C be E_1. Then Theorem 5.2 proves that (X, T) is p.n.

In fact by looking at the exact sequence $0 \to \mathcal{O}_\Sigma(-\mathsf{E}_1) \to \mathcal{O}_\Sigma \to \mathcal{O}_{\mathbb{P}^1} \to 0$ we get that 1) and 2) are true for $j = 0$. 4) is true as E_1 is a rational curve. By recalling that $h^0(\Sigma, -K_\Sigma) = 10-k$ we have that $h^0(E)-r-2 = 16-2k$. Moreover, $h^0(\Sigma, C+\det(E)+K_\Sigma) = h^0(\Sigma, -K_\Sigma+\mathsf{E}_1) = h^0(\Sigma, 3\pi^*\mathsf{L}-\mathsf{E}_2, \dots, -\mathsf{E}_k) = 11-k$ as the k points are in general position, hence 3) is satisfied. Now let us consider 1) and 2) for $j \geq 1$. It suffices to show that $h^i(\Sigma, -tK_\Sigma - \mathsf{E}_1) = 0$ for $t \geq 1$, $i = 1, 2$. For $i = 2$ we can use Serre duality. For $i = 1$ we can use Kodaira vanishing because $-tK\Sigma - \mathsf{E}_1 = K_\Sigma - (t+1)K_\Sigma - \mathsf{E}_1$ and $-(t+1)K_\Sigma - \mathsf{E}_1$ is ample by Nakai-Moishezon criterion: $(-(t+1)K_\Sigma - \mathsf{E}_1)^2 > 0$ and for any curve Γ on Σ we have:

$$(-(t+1)K_\Sigma - \mathsf{E}_1)\,\Gamma = -(t+1)K_\Sigma\Gamma - \mathsf{E}_1\Gamma = -tK_\Sigma\Gamma - K_\Sigma\Gamma - \mathsf{E}_1\Gamma \geq -tK_\Sigma\Gamma > 0$$

because $\Phi_{|-K_\Sigma|}$ embeds E_1 as a line so that

$$-K_\Sigma\Gamma = \deg\left[\Phi_{|-K_\Sigma|}(\Gamma)\right] \geq \Phi_{|-K_\Sigma|}(\mathsf{E}_1)\Phi_{|-K_\Sigma|}(\Gamma) = \mathsf{E}_1\Gamma.$$

Remark 5.5. Obviously, in the previous example, when $k = 1$ Butler's criterion can be used (see [9], Theorem 5.1A), to get the projective normality of (X, T).

Example 5.6. Let Σ be $\mathbb{P}^2$, let C be a line, let E be a rank 2 very ample vector bundle on $\mathbb{P}^2$, let $\delta\mathsf{L}$ and c be, respectively, the first and second Chern classes of E (L is the generator of $\mathrm{Pic}(\mathbb{P}^2)$ as above), let $p : \mathbb{P}(E) \to \mathbb{P}^2$ be the natural projection. Under which assumptions can we apply Theorem 5.2 ?

First of all 4) is true as C is a rational curve. 1) and 2) are true for $j = 0$ as C is ample. $h^0(\Sigma, C + \det(E) + K_\Sigma) = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(\delta - 2)) = \delta(\delta - 1)/2$, so condition 3) becomes: $h^0(E) \geq 4+\delta(\delta-1)/2$. Now let us consider 1) and 2) for $j \geq 1$. Let Y be a smooth element of $|T|$, so that Y is isomorphic to the blowing up of $\mathbb{P}^2$ at c points. Let π be the blowing up and let $\mathsf{E}_1, \dots, \mathsf{E}_\mathsf{c}$ be the exceptional divisors as before. We consider $0 \to \mathcal{O}_X(-Y) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$ tensorized with $jT - p^*\mathsf{L}$ and we have:

$$0 \to \mathcal{O}_X((j-1)T - p^*\mathsf{L}) \to \mathcal{O}_X(jT - p^*\mathsf{L}) \to \mathcal{O}_Y(jT - p^*\mathsf{L})_{|Y} \to 0.$$

We can proceed by induction on $j \geq 1$ as $h^i(X, -p^*\mathsf{L}) = 0$ for $i = 1, 2$, so we have only to consider $h^i(Y, (jT - p^*\mathsf{L})_{|Y}) = 0$ for $i = 1, 2$. Recall that $T_{|Y} = \delta\pi^*\mathsf{L} - \mathsf{E}_1 \cdots - \mathsf{E}_\mathsf{c}$. Now if $i = 2$ we can use Serre duality, if $i = 1$ we can use Kodaira vanishing as in Example 5.4 when $\delta \geq 4$ and $0 \leq \mathsf{c} \leq 6$ (or $\delta \geq 2$ and $0 \leq \mathsf{c} \leq 2$).

Hence, by using Theorem 5.2, with the previously introduced notation, we get the projective normality of (X, T) if

$$h^0(E) \geq 4 + \delta(\delta - 1)/2,\ \delta \geq 4 \text{ and } 0 \leq \mathsf{c} \leq 6.$$

Example 5.7. Let Σ be any surface, let C be any rational curve on Σ, choose $E = L$, a very ample line bundle L. When $r = 1$ Theorem 5.2 is true too, moreover, condition 2) is unnecessary. So we get that (X, L) is p.n. if:

$h^1(\Sigma, jL - C) = 0$ for $j \geq 0$,
$h^0(\Sigma, C + (3-q)L + K_\Sigma) \leq h^0(L) - 3$.

Such conditions are satisfied, for example, in many cases when Σ is the blowing up of $\mathbb{P}^2$ in k points in general position and $C = \mathsf{E}_1$.

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