Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 40 (1999), No. 2, 475-477.

The Maximum Number of Unit Distances Among n Points in Dimension Four

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Abstract. We complete P. Brass' determination of the maximum number of unit distances that can occur among n points in dimension 4.

1. Introduction

The problem of finding the maximal number of unit distances that can occur among n points in various dimensions is a well-known problem in combinatorial geometry. In [2] P. Brass gives an intricate argument proving a formula for the exact maximum number of unit distances that can occur in $\mathbb{R}^4$. He showed that for $n \geq 10$ the extremal sets consist of n points equally distributed on two circles in orthogonal planes with a common center such that each point on one circle has distance one to each point on the other (so $r_1^2 + r_2^2 = 1$), and such that the number of unit distances within each circle is maximal. This maximum, for m points, occurs if the graph of unit distances is a union of cycles (giving m unit distances) or a path with $m - 1$ unit distances. In order to fit cycles on both circles the following equation must have a solution,

$$\left(\sin \frac{\nu_1 \pi}{\left\lfloor \frac{1}{2} n \right\rfloor}\right)^{-2} + \left(\sin \frac{\nu_2 \pi}{\left\lceil \frac{1}{2} n \right\rceil}\right)^{-2} = 4, \tag{1}$$

with $\nu_1, \nu_2 \in \mathbb{N}$ and $\nu_1, \nu_2 < \frac{n}{2}$. In this paper we determine exactly those n for which this equation has solutions. Combining this with Brass' result leads to a very simple formula for the maximum number of unit distances among n points in dimension 4.

See Brass' article [2] for more background and references to other results in this field.

I would like to thank K. J. Swanepoel for bringing this problem to my attention.

2. The theorems

Lemma 1. *The only solutions to the equation*

$$\frac{1}{\sin^2(p\pi)} + \frac{1}{\sin^2(q\pi)} = 4, \text{ with } p, q \in \mathbb{Q} \cap [0, \frac{1}{2}] \text{ and } p \leq q$$

are

$$p = \frac{1}{4} \text{ and } q = \frac{1}{4},$$

or

$$p = \frac{1}{5} \text{ and } q = \frac{2}{5}.$$

Note that, as $\sin^2(p\pi) = \sin^2((1-p)\pi)$, all solutions with $p, q \in \mathbb{Q}$ can be generated from those given.

Proof. In [1] D. H. Lehmer shows that for an integer $i > 2$ and an integer s relatively prime to i, $\cos(\frac{s}{i}2\pi)$ is an algebraic number of degree $\phi(i)/2$ (where ϕ is the Euler Totient function) and that its conjugates are the $\cos(\frac{s'}{i}2\pi)$ with s' relatively prime to i. Using the identity $\cos(\frac{s}{i}2\pi) = 1 - 2\sin^2(\frac{s}{i}\pi)$ we immediately see that

$$\frac{1}{\sin^2(s\pi/i)}$$

is an algebraic integer of degree $\phi(i)/2$ and it's conjugates are all $\frac{1}{\sin^2(s'\pi/i)}$ with s' prime to i.

Now suppose

$$\frac{1}{\sin^2(s\pi/i)} + \frac{1}{\sin^2(r\pi/j)} = 4 \tag{2}$$

with $(i, s) = 1$, $(j, r) = 1$ and $i \geq j$. Let $p(x)$ be the minimal polynomial for $\frac{1}{\sin^2(s\pi/i)}$ and $q(x)$ the minimal polynomial for $\frac{1}{\sin^2(r\pi/j)}$. Then equation (2) expresses the fact that $p(x)$ and $q(4-x)$ have a root in common. As they are both irreducible they must have all their roots in common. But $\frac{1}{\sin^2(\pi/i)}$ is a root of $p(x)$ and so we see that $4 - \frac{1}{\sin^2(\pi/i)}$ is a root of $q(x)$. Therefore there must exist an r' (relatively prime to j) such that $\frac{1}{\sin^2(r'\pi/j)} = 4 - \frac{1}{\sin^2(\pi/i)}$. But if $i \geq 6$, then the right hand side of this equation is ≤ 0 while the left hand side is positive. We conclude that for equation (2) to hold we must have $i, j \leq 5$. This leaves only a finite number of cases to check and we find that essentially the only solutions are

$$\frac{1}{\sin^2(\pi/4)} + \frac{1}{\sin^2(\pi/4)} = 4$$

and

$$\frac{1}{\sin^2(\pi/5)} + \frac{1}{\sin^2(2\pi/5)} = 4. \qquad \square$$

Now we apply this result to Brass' formula.

Theorem 1. *The maximum number of unit distances among $n \geq 5$ points in $\mathbb{R}^4$ is*

$$\begin{cases} \left\lfloor \frac{1}{4}n^2 \right\rfloor + n & \textit{if } 8 \mid n \textit{ or } 10 \mid n, \\ \left\lfloor \frac{1}{4}n^2 \right\rfloor + n - 1 & \textit{otherwise.} \end{cases}$$

Proof. In [2] Brass shows that this number is $\left\lfloor \frac{1}{4}n^2 \right\rfloor + n$ if (1) has a solution and $\left\lfloor \frac{1}{4}n^2 \right\rfloor + n - 1$ otherwise. From Lemma 1 we see that (1) only has a solution if $\frac{\nu_1}{\lfloor \frac{1}{2}n \rfloor}$ and $\frac{\nu_2}{\lceil \frac{1}{2}n \rceil}$, in reduced form, have the same denominators. As these denominators must be either both 4 or both 5, we see that $\left\lfloor \frac{1}{2}n \right\rfloor - \left\lceil \frac{1}{2}n \right\rceil$ must be divisible by 4 or 5. This can only happen if $\left\lfloor \frac{1}{2}n \right\rfloor = \left\lceil \frac{1}{2}n \right\rceil$. So n must be even. Furthermore $\left\lfloor \frac{1}{2}n \right\rfloor$ must be divisible by 4 or 5. So (1) can only have solutions if $8 \mid n$ or $10 \mid n$ (and then (1) will have solutions). This proves the theorem. $\square$

So we see that the extremal sets have the following structure. If $8 \mid n$, two orthogonal circles of radius $\frac{1}{\sqrt{2}}$, with the $\frac{n}{2}$ points arranged on each in $\frac{n}{8}$ inscribed squares of sidelength one; and for $10 \mid n$ two orthogonal circles of radii $\frac{1}{2 \sin \frac{\pi}{5}}$ and $\frac{1}{2 \sin \frac{2\pi}{5}}$, with the $\frac{n}{2}$ points arranged on each in $\frac{n}{10}$ inscribed regular pentagons of sidelength one and diagonal length one, respectively. For the other n, one radius is chosen such that $\lfloor \frac{n}{2} \rfloor$ (or $\lceil \frac{n}{2} \rceil$) points can form unit distance cycles on one circle and the remaining points on the other circle form a unit distance path.

References

[1] Lehmer, D. H.: *A Note on Trigonometric Algebraic Numbers.* Amer. Math. Monthly **40** (1933), 165–166.

[2] Brass, P.: *On the maximum number of unit distances among n points in dimension four.* Intuitive Geometry (Budapest 1995), Bolyai Soc. Math. Stud. **6** (1997), 277–290.

Received June 24, 1998