Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 40 (1999), No. 2, 373-383.

On Induced Modules over Strongly Group-graded Algebras

Th. Theohari-Apostolidi **H. Vavatsoulas**

Department of Mathematics
Aristotle University of Thessaloniki
Thessaloniki, 54006 Greece

1. Introduction

Let G be a group, Λ a strongly G-graded R-algebra for a commutative ring R with unity and H a subgroup of G of finite index. The aim of this paper is to examine the induced representations of the R-algebra Λ, whenever Λ is an artinian ring or an R-order and R is a complete discrete valuation ring, that is in such conditions that the Krull-Schmidt theorem holds.

In the second section Λ is a strongly G-graded ring without other assumptions. We firstly examine the relative H-projectivity on Λ under a condition on H which generalizes the condition that $[G : H]$ is a unit in Λ (Theorem 2.2). So we generalize the relevant theorems ([15], Theorem 1.1), ([17], Proposition 7), ([12], Theorem 7.26) and ([2], Lemma 2.5) on strongly group graded rings or crossed products.

In the third section we assume that G is finite and H is a normal subgroup of G. Moreover Λ is an Artin R-algebra or an R-order with R a complete discrete valuation ring and modΛ is the category of finitely generated left Λ-modules or the category of left Λ-lattices. So the Krull-Schmidt theorem holds in modΛ. Let $\Lambda = \bigoplus_{\sigma\in G}\Lambda_\sigma$ and $\Lambda_H = \bigoplus_{\sigma\in H}\Lambda_\sigma$. In this section given an indecomposable module V in modΛ_H and an indecomposable module W in modΛ we examine the decomposition of $(V^G)_H$ and $(W_H)^G$ into indecomposables. The Clifford theorem for the decomposition of induced modules from a normal subgroup H over group algebras ([3], Theorem 11.1) has been generalized by Tucker, Conlon and Ward for Clifford Systems and for $H = \{1\}$ ([3], Proposition 11.16). F. Van Oystaeyen ([18], Proposition 2.5) and Z.B. Rong ([16], Proposition 1.3) considering strongly group-graded rings generalized the Clifford theorem for $H = \{1\}$. Here we generalize this theorem for strongly group graded rings and for a non-trivial normal subgroup H correcting the result of S. Hussein and F. Van Oystayen ([11], Theorem 2.7).

In the fourth section we prove the corresponding theorem for invariant modules of group rings ([3], Proposition 10.31) for crossed-products.

0138-4821/93 $ 2.50

We cite [1] and [3] for the representation theory, [4], [5], [6], [7], [13] for the theory of group-graded rings, and [8], [9] for the theory of graded Artin algebras.

2. Basic concepts and relative projectivity

By a ring Λ we mean an associative ring with identity $1 = 1_\Lambda$. All Λ-modules are left unitary modules unless otherwise stated. Let G be a group with identity $1 = 1_G$. A G-graded ring Λ is a ring together with a direct sum decomposition

$$\Lambda = \bigoplus_{\sigma \in G} \Lambda_\sigma$$

as additive groups, where the additive subgroups Λ_σ of Λ satisfy

$$\Lambda_\sigma \Lambda_\tau \subseteq \Lambda_{\sigma\tau} \text{ for all } \sigma, \tau \in G$$

and $\Lambda_\sigma \Lambda_\tau$ is the usual module structure of additive subgroups of Λ. If the equality holds, that is

$$\Lambda_\sigma \Lambda_\tau = \Lambda_{\sigma\tau} \text{ for all } \sigma, \tau \in G$$

then Λ is called strongly (or fully) G-graded ring. A consequence of the definition of a G-graded ring is that $1 \in \Lambda_1$.

A special type of strongly G-graded ring is the crossed product in which every component contains at least one unit of the ring Λ. In this case there exist units $\overline{\sigma}$ in Λ_σ, for $\sigma \in G$, a map

$$\alpha : G \times G \to U(\Lambda_1),$$

where by $U(\Lambda_1)$ we mean the group of units of a ring Λ_1, and an action

$$\varphi : G \to \mathrm{Aut}(\Lambda_1), \quad \varphi(\sigma)(r) = {}^\sigma r$$

for all $\sigma \in G$ and $r \in \Lambda_1$, such that

$$\Lambda = \bigoplus_{\sigma \in G} \Lambda_1 \overline{\sigma}$$

and

$$\overline{\sigma} r = {}^\sigma r \, \overline{\sigma}, \quad \overline{\sigma} \, \overline{\tau} = \alpha(\sigma, \tau) \overline{\sigma\tau}$$

for σ, τ in G and $r \in \Lambda_1$, that is α is a factor set of G on $U(\Lambda_1)$.

A special type of crossed product is the skew group ring in which the factor set is trivial.

If the G-graded ring Λ is an algebra over a commutative ring R and if each Λ_σ, $\sigma \in G$, is an R-submodule of Λ, then we call Λ a G-graded R-algebra. So if C is the center of Λ_1, then Λ is a G-graded C-algebra.

If H is a subset of G and Λ is a G-graded ring, we define

$$\Lambda_H = \bigoplus_{\sigma \in H} \Lambda_\sigma.$$

Thus if H is a subgroup of G, Λ_H is an H-graded ring and if Λ is strongly graded, then so is Λ_H.

If V is a left Λ_H-module we denote by V^G the induced module of V in G, that is

$$V^G = \Lambda \bigotimes_{\Lambda_H} V$$

and for the left Λ-module W we write W_H to indicate that W is regarded as left Λ_H-module, W_H is called the restriction of W on H.

In the sequel by Λ we will mean a strongly G-graded ring for a group G and H a subgroup of G of finite index in G. Moreover by T we will indicate a left transversal of H in G containing $1 \in G$.

Given the left Λ_H-module V and an element $\sigma \in G$, we define the conjugate module

$$\Lambda_\sigma \bigotimes_{\Lambda_H} V \cong \Lambda_{\sigma H} \bigotimes_{\Lambda_H} V,$$

which is a left $\Lambda_{\sigma H \sigma^{-1}}$-module, and we denote it by ${}^\sigma V$. Hence, if $H \trianglelefteq G$, $(V^G)_H = \bigoplus_{t \in T} {}^t V$. I. Reiten and C. Riedtmann defined the conjugate module for crossed products ([15], p. 235) and they showed for $H \trianglelefteq G$ that ${}^\sigma V$ is Λ_H-isomorphic to the left Λ_H-module whose underlying set and additive structure is the same as for V and where $r * v = {}^\sigma r v$ and $\overline{h} * v = \overline{\sigma}^{-1} \overline{h} \overline{\sigma} v$ for $r \in \Lambda_1$, $v \in V$ and $h \in H$.

Since $\Lambda_\sigma \Lambda_{\sigma^{-1}} = \Lambda_1$ for all $\sigma \in G$, we can find elements $\alpha_\sigma^{(i)} \in \Lambda_\sigma$ and $\alpha_{\sigma^{-1}}^{(i)} \in \Lambda_{\sigma^{-1}}$ such that

$$\sum_{i=1}^{n_\sigma} \alpha_\sigma^{(i)} \alpha_{\sigma^{-1}}^{(i)} = 1 \tag{2.1}$$

for some natural number n_σ.

The next proposition describes the elementary properties of the conjugate modules. The proof is analogue of that in group ring case. See also ([2], Lemma 2.1).

Proposition 2.1. *Let V, Z be left Λ_H-modules and $\sigma \in T$.*

i) ${}^g({}^\sigma V) \cong {}^{g\sigma} V$ *as $\Lambda_{g\sigma H (g\sigma)^{-1}}$-modules.*

ii) *Let $H \trianglelefteq G$. $({}^\sigma V)^G \cong V^G$ as left Λ-modules.*

iii) *Let $H \trianglelefteq G$. ${}^\sigma V$ is a simple (resp. indecomposable) Λ_H-module if and only if V is a simple (resp. indecomposable) Λ_H-module.*

We recall that a left Λ-module W is called H-projective if the exact sequence of left Λ-modules $0 \to X \to Y \to W \to 0$, for which the associated sequence of Λ_H-modules $0 \to X_H \to Y_H \to W_H \to 0$ splits, is also a splitting sequence of Λ-modules. Equivalently, W is H-projective if and only if $W \,|\, (W_H)^G$, where the notation $X \,|\, Y$ means that the module X is isomorphic to a direct summand of the module Y.

In this paper by (s) we indicate the following

Condition. *There exists an element $r \in C_{\Lambda_1}(\Lambda_H) = \{\lambda \in \Lambda_1 \colon \lambda x = x\lambda,\ \forall x \in \Lambda_H\}$, the centralizer of Λ_H in Λ_1, such that*

$$\sum_{t \in T} \sum_{i=1}^{n_t} \alpha_t^{(i)} r \alpha_{t^{-1}}^{(i)} = 1,$$

where $\alpha_\sigma^{(i)}$ *for* $\sigma \in G$ *are as in the relation* (2.1) *and for some integer* n_t.

The condition (s) holds if the order of G is a unit in Λ_1, since we can take in this case $r = [G : H]^{-1}$. Moreover in the crossed product case the condition (s) means the existence of an element $r \in C_{\Lambda_1}(\Lambda_H)$ such that

$$\sum_{t \in T} \bar{t} r \bar{t}^{-1} = 1.$$

Now if we consider the local classical crossed product case with L/K a Galois extension of local fields and G the corresponding finite Galois group, then the condition (s) means that the field F corresponding to the normal subgroup H is a tamely ramified extension over K. In this case Λ_1 is a Dedekind domain with quotient field L.

The next theorem generalizes ([15] , Theorem 1.1) in which $|G|$ is a unit in Λ and Λ is a crossed product, ([17], Proposition 7) in which Λ is a local classical crossed product, and ([2], Lemma 2.5) in which H contains a Sylow p-subgroup of G and Λ is a finitely dimensional strongly G-graded algebra over a field of characteristic p.

Theorem 2.2. *Let* H *be a subgroup of* G *of finite index and suppose that condition* (s) *holds. Then every left* Λ*-module* M *is relative* H*-projective.*

Proof. We shall prove under the conditions of the theorem that $M \,|(M_H)^G$. For this we define the maps

$$f : (M_H)^G \to M, \quad \lambda \otimes m \mapsto \lambda m,$$

for $\lambda \in \Lambda$ and $m \in M$, which is obviously a Λ-homomorphism, and

$$\Psi : M \to (M_H)^G, \quad m \mapsto \sum_{t \in T} \sum_{i=1}^{n_t} \alpha_t^{(i)} \otimes r \alpha_{t^{-1}}^{(i)} m$$

for $m \in M$, and r, $\alpha_t^{(i)}$, $\alpha_{t^{-1}}^{(i)}$, n_t as in the statement of the condition (s). We shall prove that Ψ is also a Λ-homomorphism and $f \circ \Psi = 1_M$.

At first we shall prove that Ψ does not depend on the chosen decomposition of 1. For this let $\beta_t^{(j)}$ and $\beta_{t^{-1}}^{(j)}$ be elements of Λ_t and $\Lambda_{t^{-1}}$ respectively, $1 \leq j \leq m_t$, such that

$$1 = \sum_{j=1}^{m_t} \beta_t^{(j)} \beta_{t-1}^{(j)}.$$

Then

$$\sum_{i=1}^{n_t} \alpha_t^{(i)} \otimes r\alpha_{t^{-1}}^{(i)} m = \sum_{i=1}^{n_t} \left(\sum_{j=1}^{m_t} \beta_t^{(j)} \beta_{t^{-1}}^{(j)} \right) \alpha_t^{(i)} \otimes r\alpha_{t^{-1}}^{(i)} m =$$
$$= \sum_{i=1}^{n_t} \sum_{j=1}^{m_t} \left(\beta_t^{(j)} \otimes r\beta_{t^{-1}}^{(j)} \alpha_t^{(i)} \alpha_{t^{-1}}^{(i)} m \right) =$$
$$= \sum_{j=1}^{m_t} \beta_t^{(j)} \otimes r\beta_{t^{-1}}^{(j)} \left(\sum_{i=1}^{n_t} \alpha_t^{(i)} \alpha_{t^{-1}}^{(i)} \right) m =$$
$$= \sum_{j=1}^{m_t} \beta_t^{(j)} \otimes r\beta_{t^{-1}}^{(j)} m.$$

Moreover Ψ is independent of the choice of T. Indeed, let S be another left transversal of H in G. Then for every $t \in T$ there exists exactly one $h_t \in H$ such that $th_t = \sigma_t$. Therefore if we choose elements $\alpha_{h_t}^{(j)}$ and $\alpha_{h_t^{-1}}^{(j)}$ in Λ_{h_t} and $\Lambda_{h_t^{-1}}$ respectively such that

$$1 = \sum_{j \in J} \alpha_{h_t}^{(j)} \alpha_{h_t^{-1}}^{(j)}$$

for some finite index set J, then we define

$$\alpha_{\sigma_t}^{(i)} = \sum_{j \in J} \alpha_t^{(i)} \alpha_{h_t}^{(j)} \quad \text{and} \quad \alpha_{\sigma_t^{-1}}^{(i)} = \sum_{j \in J} \alpha_{h_t^{-1}}^{(j)} \alpha_{t^{-1}}^{(i)}, \quad 1 \le i \le n_t,$$

and we get

$$\sum_{i=1}^{n_t} \alpha_{\sigma_t}^{(i)} \alpha_{\sigma_t^{-1}}^{(i)} = \sum_{i=1}^{n_t} \sum_{j \in J} \alpha_t^{(i)} \alpha_{h_{t^{-1}}}^{(j)} \alpha_{h_t^{-1}}^{(j)} \alpha_{t^{-1}}^{(i)} = \sum_{i=1}^{n_t} \alpha_t^{(i)} \alpha_{t^{-1}}^{(i)} = 1.$$

Now we remark that

$$\sum_{\sigma_t \in S} \sum_{i=1}^{n_t} \alpha_{\sigma_t}^{(i)} \otimes r\alpha_{\sigma_t^{-1}}^{(i)} m = \sum_{\sigma_t \in S} \sum_{i=1}^{n_t} \sum_{j \in J} \alpha_t^{(i)} \alpha_{h_t}^{(j)} \otimes r\alpha_{h_t^{-1}}^{(j)} \alpha_{t^{-1}}^{(i)} m =$$
$$= \sum_{t \in T} \sum_{i=1}^{n_t} \sum_{j \in J} \alpha_t^{(i)} \otimes r\alpha_{h_t}^{(j)} \alpha_{h_t^{-1}}^{(j)} \alpha_{t^{-1}}^{(i)} m = \sum_{t \in T} \sum_{i=1}^{n_t} \alpha_t^{(i)} \otimes r\alpha_{t^{-1}}^{(i)} m = \Psi(m),$$

so Ψ does not depend on the chosen set T. We will show that Ψ is a Λ-homomorphism. For this it is enough to show that

$$\Psi(\lambda_\sigma m) = \lambda_\sigma \Psi(m) \quad \text{for } \lambda_\sigma \in \Lambda_\sigma,\ \sigma \in G,\ m \in M.$$

Let σ be a fixed element of G, obviously $\sigma^{-1}t$ ranges over the left transversal of H in G as t ranges over the elements of T. We choose a decomposition of 1 in $\Lambda_{\sigma^{-1}t} \cdot \Lambda_{t^{-1}\sigma}$,

$$1 = \sum_{j \in J} \alpha_{\sigma^{-1}t}^{(j)} \alpha_{t^{-1}\sigma}^{(j)},$$

for a finite index set J. Then

$$\begin{aligned}\Psi(\lambda_\sigma m) &= \sum_{t\in T}\sum_{i=1}^{n_t} \alpha_t^{(i)} \otimes r\alpha_{t^{-1}}^{(i)}\lambda_\sigma m = \\ &= \sum_{t\in T}\sum_{i=1}^{n_t}\sum_{j\in J} \alpha_t^{(i)} \otimes r\alpha_{t^{-1}}^{(i)}\lambda_\sigma \alpha_{\sigma^{-1}t}^{(j)}\alpha_{t^{-1}\sigma}^{(j)} m = \\ &= \sum_{t\in T}\sum_{i=1}^{n_t}\sum_{j\in J} \alpha_t^{(i)}\alpha_{t^{-1}}^{(i)}\lambda_\sigma \alpha_{\sigma^{-1}t}^{(j)} \otimes r\alpha_{t^{-1}\sigma}^{(j)} m = \\ &= \sum_{t\in T}\sum_{j\in J} \lambda_\sigma \alpha_{\sigma^{-1}t}^{(j)} \otimes r\alpha_{t^{-1}\sigma}^{(j)} m = \\ &= \lambda_\sigma \Psi(m).\end{aligned}$$

Hence $\Psi(\lambda_\sigma m) = \lambda_\sigma\Psi(m)$ for all $\sigma \in G$ and $m \in M$. Finally for all $m \in M$ we have

$$f\Psi(m) = f\left(\sum_{t\in T}\sum_{i=1}^{n_t} \alpha_t^{(i)} \otimes r\alpha_{t^{-1}}^{(i)} m\right) = \sum_{t\in T}\sum_{i=1}^{n_t} \alpha_t^{(i)} r\alpha_{t^{-1}}^{(i)} m = m.$$

because of the condition (s), so f is a splitting epimorphism and the theorem is proved.

Corollary 2.4. *Let G be a finite group and H a subgroup of G for which the condition* (s) *holds. Then Λ is of finite representation type if and only if Λ_H is of finite representation type.*

3. Induced representation modules

In this section we assume that Λ_1 is an artinian ring or a complete discrete valuation ring and G a finite group. Thus the Krull-Schmidt Theorem holds for the finitely generated left Λ-modules. We denote by R the centralizer $C_{\Lambda_1}(\Lambda)$ of Λ in Λ_1 and by modΛ the category of finitely generated left Λ-modules, if Λ_1 is artinian ring, or the category of finitely generated left Λ-modules which are R-torsion-free, that is the Λ-lattices, if Λ_1 is a complete discrete valuation ring. In this last case Λ is an R-order and by ([10], Proposition 6.4) Λ is a crossed product order. We refer to [3] for the representation theory of orders.

If $W \cong \bigoplus_{i=1}^m W_i$ is the decomposition of W in the category modΛ into indecomposable Λ-modules, then W_i, $1 \le i \le m$, are called the indecomposable components of W.

If V is an indecomposable module in modΛ_H and $V^G \cong \bigoplus_{i=1}^n Y_i$, where Y_i, $1 \le i \le n$, are the indecomposable components of V^G in modΛ, then

$$(V^G)_H \cong \bigoplus\nolimits_{t\in T} {}^tV \cong \bigoplus\nolimits_{i=1}^n (Y_i)_H.$$

In this section we shall study the decomposition of $(Y_i)_H$ into indecomposable modules in modΛ_H and for a module W in modΛ the decomposition of $(W_H)^G$ into indecomposable modules in modΛ.

In the whole of this section H will be a normal subgroup of G. For a module V in $\mathrm{mod}\Lambda_H$ we denote by N_V the set of all elements $g \in G$ such that ${}^gV \cong V$ as left Λ_H-modules. N_V is a subgroup of G and is called the inertia group of V. For convenience we include into the next proposition the elementary properties of N_V. The proof is analogous to that of the corresponding result for group rings ([13], [3]).

Proposition 3.1. *Let V be a left Λ_H-module and W a left Λ-module.*

i) *${}^gV \cong {}^{g\sigma}V$ as Λ_H-modules for $g \in G$ and $\sigma \in N_V$.*

ii) *$H \leq N_V$, that is ${}^hV \cong V$ as Λ_H-modules for $h \in H$.*

iii) *$N_{W_H} = G$. Moreover if Λ is a crossed product, then $N_{\Lambda_H} = G$.*

iv) *$N_{{}^gV} = gN_Vg^{-1}$ for $g \in G$.*

v) *$(V^G)_H \cong \bigoplus_{g\in K}({}^gV)^{([N_V:H])}$, where K is a left transversal of N_V in G.*

The next result reduces the induction on G-invariant Λ_H-modules, that is Λ_H-modules V such that ${}^gV \cong V$, $\forall g \in G$ as Λ_H-modules, and generalizes the corresponding result for group rings and for strongly graded algebras but for trivial H ([12], ch. 4, Theorem 5.2). We omit its proof since it is analogous to those of the above cases.

Proposition 3.2. *Let V be an indecomposable module in* $\mathrm{mod}\Lambda_H$ *with inertia group N and let*

$$V^N \cong U_1 \oplus \cdots \oplus U_s \tag{3.1}$$

be the decomposition of V^N into indecomposable modules in $\mathrm{mod}\Lambda_N$. *Then $s \leq [N:H]$, $(U_i)_H \cong V^{(m_i)}$ for some $m_i \leq [N:H]$, and U_i^G, $1 \leq i \leq s$, are the indecomposable components of V^G. Moreover if $U_i^G \cong U_j^G$, then $U_i \cong U_j$ for $1 \leq i,j \leq s$.*

In [2] P. R. Boisen gives the graded version of the theory of vertices and sources for group rings and among other things he proves that there exists an indecomposable Λ_H-module V such that $W \mid V^G$ for a given indecomposable H-projective Λ-module W ([2], Theorem 2.5, (2)). The next result gives more information in this direction.

Theorem 3.3. *Let W be an indecomposable module in* $\mathrm{mod}\Lambda$ *and suppose that W is H-projective.*

i) *There exists an indecomposable Λ_H-module V such that*

$$W_H \cong \bigoplus_{g\in K}\left({}^gV\right)^{(m)} \quad \text{and } m \leq [N:H],$$

where K is a left transversal of the inertia group N of V in G.

ii) *If*

$$V^N \cong \bigoplus_{i=1}^{s} U_i^{(t_i)}$$

is the decomposition of V^N into indecomposable Λ_N-modules, where V is that of i), *then there exists $\rho \in \{1,2,\ldots,s\}$ such that $W \cong U_\rho^G$. Moreover if α_w is the multiplicity of W in $(W_H)^G$, then $\alpha_w = t_\rho m[G:N]$.*

Proof. i) $W\,|(W_H)^G$ since W is H-projective. Therefore $W\,|\,V^G$ for some indecomposable Λ_H-module by the Krull-Schmidt theorem. Thus $W_H\,|(V^G)_H$. Now from v) of Proposition 3.1 and the Krull-Schmidt theorem we get that W_H is a direct sum of conjugates of V. However ${}^gW_H \cong W_H$ for all $g \in G$ as follows from iii) of Proposition 3.1 and so all the multiplicities of the conjugates of V in the decomosition of W_H are equal, this proves i).

ii) We get the relation (3.1). Since $V^G \cong (V^N) \cong \bigoplus_{i=1}^{s} U_i^G$, from the Krull-Schmidt theorem and Proposition 3.2 we get that $W \cong U_i^G$ for some $i \in \{1,\ldots,s\}$. The second statement of ii) follows from i).

Corollary 3.4. *If W is an indecomposable Λ-module and $\alpha_w = 1$, then W_H is an indecomposable Λ_H-module.*

In the sequel we assume that R is an artinian ring. The next theorem generalizes the Clifford's theorem for strongly group graded algebras.

Theorem 3.5. *Let W be a simple Λ-module, $H \trianglelefteq G$ and V a simple Λ_H-module contained in W_H. Then the following holds:*

i) *W_H is a semisimple Λ_H-module, where the simple summands are isomorphic to conjugates $\{{}^\sigma V : \sigma \in G\}$ of V.*

ii) *Let N be the inertia group of V and U the homogeneous component of W_H containing V. Then $W_H \cong \bigoplus_{\sigma\in K}({}^\sigma V)^{(e)}$, where K is a left transversal of N in G and for some natural number e.*

iii) *$W \cong U^G$ as Λ-modules and U is a simple submodule of V^N.*

Proof. i) Let V be a simple Λ_H-submodule of W_H. Then the conjugates ${}^\sigma V$, $\sigma \in G$, are also simple Λ_H-modules. We remark that $\Lambda_\sigma V \cong {}^\sigma V$ as Λ_H-modules and $\sum_{\sigma\in G}\Lambda_\sigma V$ is a Λ-submodule of W. Hence $W = \sum_{\sigma\in G}\Lambda_\sigma V$, since W is simple, and so W_H is a semisimple Λ_H-module as a sum of simple Λ_H-modules. This proves i).

ii) Let U be the homogeneous component of W_H containing V. It is clear that $\Lambda_\sigma V$, $\sigma \in K$, are the non-isomorphic conjugates of V. We remark that $\Lambda_\sigma U$, $\sigma \in G$, is also a homogeneous component of W_H and since $\sum_{\sigma\in G}\Lambda_\sigma U \subset W$ we get that $W = \Sigma\Lambda_\sigma U$. Hence the group G acts transitively on the set of the homogeneous components of W_H which is in bijective correspondence with a left transversal of the stabilizer $\widetilde{N}$ of U in G. Hence $W_H = \bigoplus_{\sigma\in X}\Lambda_\sigma U$ for a left transversal X of $\widetilde{N}$ in G. Moreover since $W = \Lambda_\sigma W$, for $\sigma \in \Gamma$, we remark that the multiplicity of $\Lambda_\sigma V$ in $\Lambda_\sigma U$ is the multiplicity e of V in U, so

$$W_H = \left(\bigoplus_{\sigma\in X}\Lambda_\sigma V\right)^{(e)}.$$

It is obvious that $N \le \widetilde{N}$ and we claim that $N = \widetilde{N}$. Indeed, if $x \in \widetilde{N}$, then $\Lambda_x U = U$ and so $\bigoplus_{\sigma\in Y}\Lambda_{x\sigma}V = \bigoplus_{\sigma\in Y}\Lambda_\sigma V$, for some subset Y of a left transversal of H in N. This means that there exists $\sigma \in N$ such that $\Lambda_{x\sigma}V \cong V$ as Λ_N-modules and so $x\sigma \in N$, that is $x \in N$. Hence $N = \widetilde{N}$ and we get ii).

iii) From the proof of ii) we conclude that $W = \bigoplus_{\sigma\in K}\Lambda_\sigma V$ and hence $W \cong V^G$ as Λ-modules. If U is not a simple Λ_N-module, then there exists a non-zero proper Λ_N-module

U' of U. Thus $\sum_{\sigma\in K}\Lambda_\sigma U' \subset \sum_{\sigma\in K}\Lambda_\sigma U = W$ which is a contradiction, since W is simple. This proves iii).

Remark. Zhou Bo Rong proved Theorem 3.5 for $H = \{1\}$ ([16], Proposition 1.3). S. Houssein and F. Van Oystaeyen in ([11], Theorem 2.7) claimed that $W \cong U^G$, but this is not true even for group rings ([16], Counterexample).

4. *G*-invariant modules in crossed products

In this section we assume that $\Lambda = R * G = \bigoplus_{\sigma\in G} R\overline{\sigma}$ is a crossed-product, G is a finite group and H is a subgroup of G.

Let W be a left $R*G$-module. We call an element w of W G-invariant, if $uw = w$ for all graded units u of $R*G$, that is for all $u \in \operatorname{Gr} U(R*G) = \{u\overline{\sigma}\colon u \in U(R), \sigma \in G\}$. Let $\operatorname{Inv}_G(W)$ be the set of all G-invariant elements of W. It is clear that $\operatorname{Inv}_G(W)$ is a $\mathrm{C}(R*G)$-submodule of W, for $\mathrm{C}(R*G)$ the center of $R*G$.

Theorem 4.1. *Let V be a left $R*H$-module. Then*

$$\operatorname{Inv}_G(V^G) \cong \operatorname{Inv}_H(V)$$

as $\mathrm{C}_{R*H}(R*G)$-module, where $\mathrm{C}_{R*H}(R*G) = \{x \in R*H\colon\ xy = yx,\ y \in R*G\}$.

Proof. Firstly we prove that

$$\operatorname{Inv}_G(V^G) = \left\{\sum_{t\in T}\overline{t}\oplus v : v \in \operatorname{Inv}_H(V)\right\},$$

where T is a left transversal of H in G. We remark that every element of the right part of the above relation is defined independently of the set T. Indeed if S is another one and $th_t = \sigma_t$, for $t \in T$, $h_t \in H$ and $\sigma_t \in S$, then for $a_t = \alpha(t,h_t)^{-1}$ we have

$$\sum_{t\in T}\overline{\sigma}_t\otimes v = \sum_{t\in T} a_t\overline{t}\otimes\overline{h}_t v = \sum_{t\in T}\overline{t}\otimes\overline{t}^{-1}a_t\overline{t}v = \sum_{t\in T}\overline{t}\otimes v,$$

since $\overline{t}a_t\overline{t} \in U(R)$. Moreover for $g \in G$, $u \in U(R)$ and $v \in \operatorname{Inv}_H(V)$ we get that

$$\begin{aligned}\overline{g}\sum_{t\in T}\overline{t}\otimes v &= \sum_{t\in T}\alpha(g,t)\overline{gt}\otimes v = \sum\overline{gt}\otimes\overline{gt}^{-1}\alpha(g,t)\overline{gt}\,v = \\ &= \sum_{t\in T}\overline{gt}\otimes v = \sum_{t\in T}\overline{t}\otimes v,\end{aligned}$$

and

$$u\sum_{t\in T}\overline{t}\otimes v = \sum_{t\in T}\overline{t}\otimes\overline{t}^{-1}u\overline{t}v = \sum_{t\in T}\overline{t}\otimes v.$$

Hence $\sum_{t\in T}\overline{t}\otimes v\in \mathrm{Inv}_G(V^G)$.

Let now $\sum_{t\in T}\overline{t}\otimes v_t$ be an element of $\mathrm{Inv}_G(V^G)$. Then for $\sigma\in T$ we get $\overline{\sigma}^{-1}\sum_{t\in T}\overline{t}\otimes v_t=\sum_{t\in T}\overline{t}\otimes v_t$. Hence $v_\sigma=v_1$ for all $\sigma\in G$, and $\sum_{t\in T}\overline{t}\otimes v_t=\sum_{t\in T}\overline{t}\otimes v$ for some $v\in V$. It remains to prove that v belongs to $\mathrm{Inv}_H(V)$. For this we remark that if $h\in H$, then $\overline{h}\sum_{t\in T}\overline{t}\otimes v=\sum_{t\in T}\overline{t}\otimes v$. Hence $\overline{h}\cdot\overline{1}\otimes v=\overline{1}\otimes v$ so $\overline{h}v=v$. Similarly $uv=v$ for $u\in U(R)$. Therefore we have the equality mentioned above.

Now it is easy to see that the map

$$f:\mathrm{Inv}_G(V^G)\to\mathrm{Inv}_H(V),\ \sum_{t\in T}\overline{t}\otimes v\mapsto v$$

is a $\mathrm{C}_{R*H}(R*G)$-isomorphism.

Corollary 4.2. *Let W be a left $R*G$-module, then*

$$\mathrm{Tr}_H^G:\mathrm{Inv}_H(W_H)\to\mathrm{Inv}_G(W),\ w\mapsto\sum_{t\in T}\overline{t}w$$

is a $\mathrm{C}_{R*H}(R*G)$-homomorphism. Particularly Tr_H^G is a splitting epimorphism if the condition (s) *holds.*

Proof. It is clear that Tr_H^G is a $\mathrm{C}_{R*H}(R*G)$-homomorphism. If the condition (s) holds, we define

$$f:\mathrm{Inv}_G(W)\to\mathrm{Inv}_H(W_H),\ w\mapsto rw.$$

Then

$$\mathrm{Tr}_H^G\ f(w)=\sum_{t\in T}\overline{t}rw=\sum_{t\in T}\overline{t}r\overline{t}^{-1}=w.$$

So Tr_H^G is a splitting epimorphism.

Acknowledgement. The authors would like to thank the referee for his (or her) very helpful remarks.

References

[1] Auslander, M.; Reiten, I.; Smalø, S. O.: *Representation Theory of Artin Algebras.* Cambridge Stud. Adv. Math. **36**, Cambridge University Press, Cambridge 1995.

[2] Boisen, P. R.: *The Representation Theory of Fully Group-Graded Algebras.* J. Algebra **151** (1992), 160–179.

[3] Curtis, C.; Reiner, I.: *Methods of Representation Theory with Applications to Finite Groups and Orders.* Vol.I, Wiley, New York 1981.

[4] Dade, E. C.: *Group Graded Rings and Modules.* Math. Z. **174** (1980), 241–262.

[5] Dade, E. C.: *The Equivalence of Various Generalizations of Group Rings and Modules.* Math. Z. **181** (1982), 335–344.

[6] Dade, E. C.: *Clifford Theory for Group Graded Rings.* J. Reine Angew. Math. **369** (1986), 40–86.

[7] Dade, E. C.: *Clifford Theory for Group Graded Rings II.* J. Reine Angew. Math. **387** (1988), 148–181.

[8] Gordon, R.; Green, E.: *Graded Artin Algebras.* J. Algebra **76** (1982), 111–137.

[9] Gordon, R.; Green, E.: *Representation Theory of Graded Artin Algebras.* J. Algebra **76** (1982), 138–152.

[10] Haefner, J.: *Graded Equivalence Theory with Applications.* J. Algebra **172** (1995), 385–424.

[11] Hussein, S.; Van Oystaeyen, F.: *Control Subgroups for Group Graded Rings.* Comm. Algebra **20**(8) (1992), 2219–2237.

[12] Karpilovsky, G.: *The Algebraic Structure of Crossed-Products.* North-Holland Math. Stud. **142**, Amsterdam 1987.

[13] Karpilovsky, G.: *Group representations.* Vol.4, North-Holland Math. Stud. **182**, Amsterdam 1995.

[14] Năstăsescu, C.; Van Oystaeyen, F.: *Graded Ring Theory.* Noth Holland Math. Library, **28**, Amsterdam 1982.

[15] Reiten, I.; Riedtmann, C.: *Skew Group Algebras in the Representation Theory of Artin Algebras.* J. Algebra **92** (1985), 224–282.

[16] Zhou, Borong: *Two Clifford's Theorems for strongly Group Graded Rings.* J. Algebra **139** (1991), 172–189.

[17] Theohari-Apostolidi, Th.: *Local Crossed-Product Orders of Finite Representation Type.* J. Pure Appl. Algebra **41** (1986), 87–98.

[18] Van Oystaeyen, F.: *On Clifford Systems and Generalized Crossed-Products.* J. Algebra, **87**(2) (1984), 396–415.

Received April 16, 1998