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On Full Partial Quasigroups of Finite Order and Local Cardinal Maximum Codes

Jörn Quistorff

Speckenreye 48, D-22119 Hamburg, Germany

Abstract. Stojaković/Ušan's result on the minimal size of a full partial quasigroup of a given finite order is improved and generalized to full partial n-quasigroups: An implicite lower bound is proved, some upper bounds are given. The minimal size of a full partial quasigroup is determined. Some cases for full partial n-quasigroups are also solved. The correspondence between full partial n-quasigroups of finite order and local cardinal maximum codes is used.

1. Introduction

Stojaković/Ušan [10] consider partial quasigroups of finite order, they introduce the notion of a full partial quasigroup and prove a first lower bound on the minimal size of a full partial quasigroup of a given order.

Dörnte [4] generalizes the notion of a group to an n-group. The notion of a quasigroup can be generalized to an n-quasigroup in the same way.

A correspondence between quasigroups of finite order respectively Latin squares and codes is developed especially by Golomb/Posner [6], Dénes/Gergely [1] and Ušan [11]. Golomb/Posner [6] show for the first time the relation between systems of $n-2$ mutually orthogonal Latin squares and $(n, 2)$-MDS-codes.

Because of these well-known connections, confer also Dénes/Keedwell [2], [3] and Heise/Quattrocchi [7], each n-quasigroup corresponds to a code of length $n+1$. Every full partial n-quasigroup of finite order corresponds to a local cardinal maximum code.

In the present work, Stojaković/Ušan's [10] mentioned result is considerably improved and generalized to the case of full partial n-quasigroups: An implicite lower bound on the minimal size of a full partial n-quasigroup is proved. Some upper bounds are also given.

Thus, it is possible to determine the minimal size of a full partial quasigroup. Some special cases for full partial n-quasigroups can also be solved.

Because of the correspondence between full partial n-quasigroups and local cardinal maximum codes, this note shows some results in the language of coding theory. MDS-codes will be used in order to construct local cardinal maximum codes and, thereby, also full partial n-quasigroups.

2. Foundations

Let Q be a non-empty set and $A : Q \times Q \to Q$ a mapping. (Q, A) is called quasigroup iff for all $x, z \in Q$ there exists exactly one $y \in Q$ and exactly one $y' \in Q$ with $A(x, y) = z$ and $A(y', x) = z$. If Q is finite, "exactly" can be substituted by "at least" or "at most" and $|Q|$ is called the order of the quasigroup.

If $\emptyset \neq D \subseteq Q \times Q$ and $A : D \to Q$ is a mapping, (Q, D, A) is called partial quasigroup iff for all $x, z \in Q$ there exists at most one $y \in Q$ and at most one $y' \in Q$ with $A(x, y) = z$ and $A(y', x) = z$. In the finite case, $|D|$ is called the size of the partial quasigroup.

A partial quasigroup (Q, D, A) is called full iff for all partial quasigroups $(Q, \overline{D}, \overline{A})$ with $D \subseteq \overline{D}$ and with the condition that the restriction $\mathrm{res}(\overline{A}, D)$ is equal to A, neccessarily $D = \overline{D}$ holds true.

Stojaković/Ušan [10] prove the following statement: If (Q, D, A) is a full partial quasigroup of order $q \in \mathbf{N} \setminus \{1, 2\}$ then

$$|D| \geq 2q - 1. \tag{1}$$

Let Q be a set, $\emptyset \neq D \subseteq Q^n$ with $n \in \mathbf{N}$ and $A : D \to Q$. Preparing for the n-dimensional case, let $\mathbf{Z}_n := \{i \in \mathbf{N} | i \leq n\}$ as well as $x^{(i)}\#y := (x_1, ..., x_{i-1}, y, x_{i+1}, ..., x_n) \in Q^n$ for $x^{(i)} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) \in Q^{n-1}$ and $y \in Q$. The triple (Q, D, A) is called a partial n-quasigroup iff for all $i \in \mathbf{Z}_n$, all $x^{(i)} \in Q^{n-1}$ and all $z \in Q$ there exists at most one $y \in Q$ with $x^{(i)}\#y \in D$ and $A(x^{(i)}\#y) = z$. Analogous to the twodimensional case, (Q, D, A) is called a full partial n-quasigroup iff for every partial n-quasigroup $(Q, \overline{D}, \overline{A})$ with $D \subseteq \overline{D}$ and with the conditon that the restriction $\mathrm{res}(\overline{A}, D)$ is equal to A, neccessarily $D = \overline{D}$ holds true.

The multiplication table of a (2-)quasigroup of order $q \in \mathbf{N}$ is a Latin square of order q and vice versa. If (Q, A) is a quasigroup of order q then

$$C := \{(x, y, A(x, y)) \in Q^3 | x, y \in Q\}$$

is a code of length 3 and order q consisting of $u(C) = q^2$ codewords. In case $q \geq 2$, this code has minimal distance $d(C) = 2$. Herewith,

$$d(C) = \min\{d(w, w') | w, w' \in C \text{ with } w \neq w'\}$$

is the minimal Hamming distance $d(w, w') = |\{i \in \mathbf{Z}_3 | w_i \neq w'_i\}|$ between different codewords. Conversely, a code with the mentioned parameters induces again a quasigroup.

The code C is a $(3, 2)$-MDS-code, a notion which will be used in the last part of the paper. It is a special case of the construction given by Golomb/Posner [6] and Singleton [9] who show the more general connection between a system of $n-2$ mutually orthogonal Latin squares (or mutually orthogonal quasigroups) and an $(n,2)$-MDS-code.

If (Q,D,A) is a partial quasigroup of order q and size $|D|\geq 2$ then

$$C=\{(x,y,A(x,y))\in Q^3|(x,y)\in D\}$$

is a code of length 3 and order q with $u(C)=|D|$ and $d(C)\geq 2$. Also in this situation, a code with the mentioned parameters induces conversely again a partial quasigroup.

In case of a partial n-quasigroup of order q and size $|D|\geq 2$, a code

$$C=\{(x,A(x))\in Q^{n+1}|x\in D\}$$

of length $n+1$ and order q is induced with $u(C)=|D|$ and $d(C)\geq 2$. Also in this situation, the converse statement is true.

The three mentioned codes might be seen as check character systems, confer Ecker/Poch [5] and Schulz [8].

Let $q,n,d\in\mathbf{N}$ with $2\leq q$ and $d\leq n$ as well as Q a set with $|Q|=q$ and $C\subseteq Q^n$. A code C is called a (q,n,d)-cardinal maximum code iff $d(C)\geq d$ holds true and for every code $C'\subseteq Q^n$ with $d(C')\geq d$, the statement $u(C')\leq u(C)$ follows.

Weakening this notion, C is called a local (q,n,d)-cardinal maximum code iff $d(C)\geq d$ holds true and for every proper including set $C'\supset C$ the statement $d(C')<d$ follows. The minimal number of codewords of a local (q,n,d)-cardinal maximum code is called $v(q,n,d)$.

Every full partial n-quasigroup of order $q\in\mathbf{N}\setminus\{1\}$ induces a local $(q,n+1,2)$-cardinal maximum code and vice versa. So, $v(q,n+1,2)$ is also equal to the minimal size of a full partial n-quasigroup of order q.

3. Results

Trivially, a partial 1-quasigroup (Q,D,A) is full iff $D=Q$ is valid.

In the following, the announced improvement and generalisation of the inequality (1) is treated. In order to prove an implicite lower bound on the minimal size of a full partial n-quasigroup, a helping statement will be used:

Let $p\in\mathbf{N}\setminus\{1\}$ and $u\in\mathbf{N}_0$. Let $u=ap+b=b(a+1)+(p-b)a$ with $a,b\in\mathbf{N}_0$ and $b<p$. Let $M:=\{(s_1,...,s_p)\in(\mathbf{N}_0)^p|\sum_{l=1}^p s_l=u\}$. Then

$$\sum_{l=1}^{p}(s_l)^2\geq b(a+1)^2+(p-b)a^2 \tag{2}$$

is valid for all $(s_1,...,s_p)\in M$.

Proof of the helping statement. Let $(s_1^{(0)},...,s_p^{(0)})\in M$ and, without loss of generality, $s_l^{(0)}\geq s_m^{(0)}$ for $l<m$. In case of $s_1^{(0)}-s_p^{(0)}\leq 1$, it follows that $(s_1^{(0)},...,s_p^{(0)})=(a+1,..,a+1,a,...,a)$ and, hence,

$$\sum_{l=1}^{p}(s_l^{(0)})^2=\sum_{l=1}^{b}(a+1)^2+\sum_{l=b+1}^{p}a^2=b(a+1)^2+(p-b)a^2.$$

In case of $s_1^{(0)} - s_p^{(0)} \geq 2$, put

$$s_l^{(1)} := \begin{cases} s_1^{(0)} - 1 & \text{if } l = 1, \\ s_l^{(0)} & \text{if } 1 < l < p, \\ s_p^{(0)} + 1 & \text{if } l = p. \end{cases}$$

It holds true that $(s_1^{(1)}, ..., s_p^{(1)}) \in M$ and $\sum_{l=1}^{p}(s_l^{(0)})^2 \geq \sum_{l=1}^{p}(s_l^{(1)})^2$. One also can get $s_l^{(1)} \geq s_m^{(1)}$ for $l < m$ by renumbering. After a finite number k of such steps, $(s_1^{(k)}, ..., s_p^{(k)}) = (a+1, .., a+1, a, ..., a)$ and

$$\sum_{l=1}^{p}(s_l^{(0)})^2 \geq \sum_{l=1}^{p}(s_l^{(k)})^2 = b(a+1)^2 + (p-b)a^2$$

are valid. □

Theorem 1. *Let (Q, D, A) be a partial n-quasigroup of order q as well as $u := |D| = aq^{n-1}+b$ with $a, b \in \mathbf{N}_0$ and $b < q^{n-1}$. If (Q, D, A) is full then*

$$n(qu - b(a+1)^2 - (q^{n-1} - b)a^2) \geq (q^n - u)q \tag{3}$$

and, equivalently,

$$(n+1)qu \geq q^{n+1} + n(au + ab + b) \tag{4}$$

are valid.

Proof. The case $n = 1$ is trivial. Let $n \geq 2$ and $n(qu - b(a+1)^2 - (q^{n-1} - b)a^2) < (q^n - u)q$. For $i \in \mathbf{Z}_n$ and $x^{(i)} \in Q^{n-1}$, put $\varphi_i(x^{(i)}) := |\{y \in Q | x^{(i)}\#y \in D\}|$. It holds true that

$$\sum_{x^{(i)} \in Q^{n-1}} \varphi_i(x^{(i)}) = u = b(a+1) + (q^{n-1} - b)a. \tag{5}$$

Remark that the left hand side of this chain of equations is a sum of q^{n-1} items. On the right hand side one can find b times the item $(a+1)$ and $(q^{n-1} - b)$ times the item a. Hence, also this side is a sum of q^{n-1} items. Put

$$E_i := \{(x, r) \in Q^n \times Q | x^{(i)}\#r \in D\}$$

as well as

$$E_i' := \{(x, r) \in E_i | x \in D\}.$$

It holds true that $|E_i| = q|D| = qu$ and

$$\begin{aligned} |E_i'| &= \sum_{x \in D} \varphi_i(x^{(i)}) \\ &= \sum_{x^{(i)} \in Q^{n-1}} \quad \sum_{x_i \in \{y \in Q | x^{(i)}\#y \in D\}} \varphi_i(x^{(i)}) \\ &= \sum_{x^{(i)} \in Q^{n-1}} \varphi_i(x^{(i)}) \cdot |\{y \in Q | x^{(i)}\#y \in D\}| \\ &= \sum_{x^{(i)} \in Q^{n-1}} (\varphi_i(x^{(i)}))^2. \end{aligned}$$

Hence, the number of elements of E_i' can be written as a sum of q^{n-1} items as above. By equation (5) and helping statement (2) with $p = q^{n-1}$ and $\{s_1, ..., s_p\} = \{\varphi_i(x^{(i)}) | x^{(i)} \in Q^{n-1}\}$, it follows that

$$|E_i'| \geq b(a+1)^2 + (q^{n-1} - b)a^2.$$

Moreover, $\sum_{x \in Q^n \setminus D} \varphi_i(x^{(i)}) = |E_i \setminus E_i'|$ is valid. Hence, it follows that

$$\begin{aligned}\sum_{x \in Q^n \setminus D} \sum_{i \in \mathbf{Z}_n} \varphi_i(x^{(i)}) &= \sum_{i \in \mathbf{Z}_n} (|E_i| - |E_i'|)| \\ &\leq n(qu - b(a+1)^2 - (q^{n-1} - b)a^2) \\ &< (q^n - u)q\end{aligned}$$

by the hypothesis. Because of $|Q^n \setminus D| = q^n - u$, it exists a $w \in Q^n \setminus D$ with $\sum_{i \in \mathbf{Z}_n} \varphi_i(w^{(i)}) < q$. Put

$$Q_i := \{z \in Q | \text{ it exists a } y \in Q \text{ with } w^{(i)} \# y \in D \text{ and } A(w^{(i)} \# y) = z\}.$$

Because of

$$|\bigcup_{i \in \mathbf{Z}_n} Q_i| \leq \sum_{i \in \mathbf{Z}_n} |Q_i| = \sum_{i \in \mathbf{Z}_n} \varphi_i(w^{(i)}) < q,$$

it exists a $\overline{z} \in Q \setminus (\bigcup_{i \in \mathbf{Z}_n} Q_i)$. Put $\overline{D} := D \cup \{w\}$ and

$$\overline{A} : \overline{D} \to Q, \quad x \mapsto \begin{cases} A(x) & \text{if } x \in D, \\ \overline{z} & \text{if } x = w. \end{cases}$$

Hence, $(Q, \overline{D}, \overline{A})$ is a partial n-quasigroup and, through construction, (Q, D, A) is not full. So for every partial n-quasigroup of order q, the inequality (3) and its equivalent (4) are valid. □

Corollary 1. *Let $k, n \in \mathbf{N}$ and (Q, D, A) be a full partial n-quasigroup of order $q := kn$. Then $|D| \geq n^{n-1}k^n = kq^{n-1}$.*

Proof. Let (Q, D, A) be a partial n-quasigroup of order q. Let

$$\begin{aligned} u := |D| &= kq^{n-1} - (lq^{n-1} + m) \\ &= (k - l - 1)q^{n-1} + (q^{n-1} - m) \\ &= (q^{n-1} - m)(k - l) + m(k - l - 1)\end{aligned}$$

with $l, m \in \mathbf{N}_0$ and $0 < m \leq q^{n-1}$ as well as $lq^{n-1} + m < kq^{n-1}$. Hence,

$$\begin{aligned} & 0 < q^{n-1}l(q - k + l) + m(q - k) + 2lm + m, \qquad \text{since } q \geq k, \\ \Rightarrow\ & -q^n l + q^{n-1}l(k - l) - qm + km - 2lm - m < 0 \\ \Rightarrow\ & n(q^n(k - l) - qm - q^{n-1}(k - l)^2 + 2m(k - l) - m) < (q^n k - q^{n-1}k(k - l) + km)n \\ \Rightarrow\ & n(qu - (q^{n-1} - m)(k - l)^2 - m(k - l)^2 + 2m(k - l) - m) < (q^n - u)kn \\ \Rightarrow\ & n(qu - (q^{n-1} - m)(k - l)^2 - m(k - l - 1)^2) < (q^n - u)q \end{aligned}$$

follows. By Theorem 1 with $a = k - l - 1$ and $b = q^{n-1} - m$, the partial n-quasigroup (Q, D, A) is not full. □

Corollary 2. *Let $k \in \mathbf{N}$ and (Q, D, A) a full partial quasigroup of order $2k+1$. Then $|D| \geq k(2k+1)+(k+1) = k^2+(k+1)^2$.*

Proof. Let (Q, D, A) be a partial quasigroup of order $2k+1$. Let

$$u := |D| = a(2k+1)+b < k(2k+1)+(k+1)$$

with $a, b \in \mathbf{N}_0$ and $b < 2k+1$. So, either $(a = k \wedge b \leq k)$ or $(a < k \wedge b \leq 2k)$ is valid. Hence,

$$\begin{aligned} & 0 < 4(k-a)(3k-b)+2k(2k(k-a)+k-b)+ \\ & \qquad\qquad +(k-a)(3+4k(k-a-1)+2(k-a))+3k-b+1 \\ \Rightarrow\ & 12ak^2+12ak+6bk+3a+3b < 8k^3+12k^2+6k+1+4a^2k+2a^2+4ab+2b \\ \Rightarrow\ & 3((2k+1)(2ak+a+b)) < (2k+1)^3+2(a(a(2k+1)+b)+ab+b) \\ \Rightarrow\ & 3qu < q^3+2(au+ab+b) \end{aligned}$$

follows. By Theorem 1, the partial quasigroup (Q, D, A) is not full. □

Examples. For every full partial quasigroup (Q, D, A) of order 5 respectively 6 the inequality $|D| \geq 9$ respectively $|D| \geq 11$ is valid by (1). The above corollaries show in these cases $|D| \geq 13$ respectively $|D| \geq 18$. Hence, $v(5,3,2) \geq 13$ and $v(6,3,2) \geq 18$. Furthermore, for full partial 3-quasigroups of order 5 respectively 6 the inequalities $|D| \geq 44$ respectively $|D| \geq 72$ hold true by Theorem 1 and Corollary 1. Hence, $v(5,4,2) \geq 44$ and $v(6,4,2) \geq 72$.

After these developments of lower bounds on the function $v(q, n, d)$, upper bounds will be considered.

Theorem 2. *Let $Q = Q_1 \uplus Q_2$ be a set which consists of two disjunct subsets with $|Q_1| = q_1$ and $|Q_2| = q_2$ as well as $q_1, q_2 \in \mathbf{N}$. If (Q_i, A_i) are quasigroups for $i \in \{1, 2\}$ then (Q, D, A) with $D := (Q_1)^2 \cup (Q_2)^2$ and*

$$A : D \to Q, \quad (x, y) \mapsto \begin{cases} A_1(x, y) & \text{if } x, y \in Q_1, \\ A_2(x, y) & \text{if } x, y \in Q_2, \end{cases}$$

is a full partial quasigroup. The statements

$$v(2k, 3, 2) \leq 2k^2 \tag{6}$$

and

$$v(2k+1, 3, 2) \leq k^2+(k+1)^2 \tag{7}$$

are valid for all $k \in \mathbf{N}$.

Combining the Corollaries 1 and 2 with the above theorem, the minimal size of a full partial quasigroup of a given order and hence $v(q, 3, 2)$ can be determined:

Theorem 3. *For $k \in \mathbf{N}$, the statements $v(2k, 3, 2) = 2k^2$ and $v(2k+1, 3, 2) = k^2+(k+1)^2$ hold true.*

The determination of further upper bounds on the function $v(q,n,d)$ is done in the language of coding theory. In order to do so, the following two notions are useful: Let $q,n,k \in \mathbf{N}$ with $q \geq 2$ and K be a set with $|K| = q$. Then, $B(w,k) := \{x \in K^n | d(x,w) \leq k\}$ is called the closed ball around $w \in K^n$ with radius k. Moreover, a code $C \subseteq K^n$ is called an (n,k)-MDS-code of order q iff $u(C) = q^k$ and $d(C) = n-k+1$ are valid. If $C \subseteq K^n$ is an (n,k)-MDS-code of order q, if $I \subseteq \mathbf{Z}_n$ with $|I| = k$ and if $y_i \in K$ for all $i \in I$ then it exists exactly one $w \in C$ with $w_i = y_i$ for all $i \in I$.

Theorem 4. *Let $q_1, q_2, n, d \in \mathbf{N}$ with $2 \leq q_1, q_2$ and $d \leq n$. Let $C' \subseteq (K')^n$ be a code of order q_1 with $d(C') \geq d$ and $\bigcup_{w' \in C'} B(w', d-1) = (K')^n$. Let $C'' \subseteq (K'')^n$ be an $(n, n-d+1)$-MDS-code of order q_2. Then $v(q_1 q_2, n, d) \leq u(C')u(C'') = u(C')q_2^{n-d+1}$ holds true.*

Proof. Put $C := C' \times C'' \subseteq (K')^n \times (K'')^n$. Identify $(K')^n \times (K'')^n$ with $(K' \times K'')^n$. Then, C is a code of length n with $u(C) = u(C')u(C'')$ and $d(C) = \min\{d(C'), d(C'')\} = d$ since C'' is an $(n, n-d+1)$-MDS-code. Let

$$((x_1', x_1''), ..., (x_n', x_n'')) \in (K' \times K'')^n \setminus C.$$

It exists a $w' \in C'$ with $d(x', w') \leq d-1$. Hence, for $I' := \{i \in \mathbf{Z}_n | x_i' = w_i'\}$, it holds true that $|I'| \geq n-d+1$. Let $I'' \subseteq I'$ with $|I''| = n-d+1$. Then, it exists a $w'' \in C''$ with $x_i'' = w_i''$ for all $i \in I''$. Hence,

$$d(C \cup \{((x_1', x_1''), ..., (x_n', x_n''))\}) \leq d-1$$

is valid and C is a local $(q_1 q_2, n, d)$-cardinal maximum code. □

Corollary 3. *Let $n, q \in \mathbf{N}$ with $n \geq 3$ and $q \geq 2$. If an $(n, n-2)$-MDS-code of order $n-1$ and an $(n, n-1)$-MDS-code of order q exist then*

$$v((n-1)q, n, 2) \leq (n-1)^{n-2} q^{n-1}.$$

Proof. Let $C \subseteq K^n$ be an $(n, n-2)$-MDS-code of order $n-1$. For $w \in C$, it holds true that $|B(w,1)| = 1 + n(n-2) = (n-1)^2$. For $x, w \in C$ with $x \neq w$, it holds true that $B(x,1) \cap B(w,1) = \emptyset$ because of $d(C) = 3$. Hence,

$$(n-1)^n = u(C)(n-1)^2 = |\biguplus_{w \in C} B(w,1)| \leq |K^n| = (n-1)^n.$$

So, $\biguplus_{w \in C} B(w,1) = K^n$ follows. This means, by the way, that C is 1-perfect. Theorem 4 proves the corollary. □

Examples. The existence of a $(3,1)$-MDS-code of order 2 and of a $(3,2)$-MDS-code of order $k \in \mathbf{N} \setminus \{1\}$, confer Heise/Quattrocchi [7], shows that $v(2k, 3, 2) \leq 2k^2$. This is a proof of inequality (6) by coding theory. The existence of a $(4,2)$-MDS-code of order 3 and of a $(4,3)$-MDS-code of order $k \in \mathbf{N} \setminus \{1\}$ shows that $v(3k, 4, 2) \leq 9k^3$. As an example, $v(6,4,2) \leq 72$ is valid. If n is more general a power of a prime then the well-known existence of an $(n+1, n-1)$-MDS-code of order n and of an $(n+1, n)$-MDS-code of order $k \in \mathbf{N} \setminus \{1\}$ implies the inequality $v(nk, n+1, 2) \leq n^{n-1} k^n$.

Combining the last mentioned inequality with Corollary 1, the minimal size of a full partial n-quasigroup of given order and hence the term $v(q, n+1, 2)$ can be determined in special cases:

Theorem 5. *If n is a power of a prime then $v(nk, n+1, 2) = n^{n-1}k^n$ holds true for all $k \in \mathbf{N}$.*

The case $k = 1$ is included in this theorem because every $(n+1, n-1)$-MDS-code of order n is by the proof of Corollary 3 a local $(n, n+1, 2)$-cardinal maximum code. Hence, $v(n, n+1, 2) \leq n^{n-1}$ is valid if n is a power of a prime.

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