Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 40 (1999), No. 2, 363-372.

A Locally Symmetric Kaehler Einstein Structure on the Tangent Bundle of a Space Form

Vasile Oproiu*

Faculty of Mathematics, University "Al.I.Cuza",
6600, Iaşi, România.
e-mail: voproiu@uaic.ro

Abstract. On the tangent bundle of a Riemannian manifold of constant negative sectional curvature there exists a structure of locally symmetric Kaehler Einstein manifold. Such a structure exists in a tube around zero section in the case of a Riemannian manifold of constant positive sectional curvature.

MSC 1991: 53C07, 53C55.
Keywords: tangent bundle, Kaehler Einstein metric, locally symmetric Riemannian manifold.

Introduction

The tangent bundle TM of a Riemannian manifold (M, g) can be organized as an almost Kaehlerian manifold (see [3], [9], [17]) by using the Sasaki metric and an almost complex structure defined by the splitting of the tangent bundle to TM into the vertical and horizontal distributions VTM, HTM (the last one being determined by the Levi Civita connection on M) (see also [15], [16]). However, this structure is Kaehler only in the case where the base manifold is locally Euclidean. The Sasaki metric is not a "good" metric in the sense of [1] since its Ricci curvature is not constant, i.e. the Sasaki metric is not, generally, Einstein.

In [14] the author and N. Papaghiuc, inspired by an idea of Calabi ([2], [1]) to define a hyper-Kaehler structure on the cotangent bundle of a Kaehler manifold of positive constant

*Partially supported by the Grant 644/1997, Ministerul Invăţământului, România

holomorphic sectional curvature, have considered a Lagrangian on a Riemannian manifold (M, g), defined by a real valued smooth function depending on the energy density (kinetic energy) only. They have shown that the nonlinear connection defined by the Euler-Lagrange equations associated to the considered Lagrangian does coincide with the nonlinear connection defined by the Levi Civita connection of g and have obtained a Riemannian metric G and an almost complex structure J such that (TM, G, J) is an almost Kaehler manifold (like in the case of the Sasaki metric). Further, if (M, g) has positive constant sectional curvature, then there exists a Lagrangian defined on the bundle of nonzero tangent vectors T_0M, such that (T_0M, G, J) is Kaehlerian. However, nor G is a "good" metric on T_0M.

In the present paper we are interested in finding a Kaehler Einstein structure on the tangent bundle of a space form. We have changed the metric G on the tangent bundle (so that it is no longer obtained from a Lagrangian) by using an M-tensor field on TM obtained by adding to the Riemannian metric g on the base manifold (thought of as an M-tensor field on TM), a term of the form $v(t)g_{0i}g_{0j}$ where v is a smooth real valued function depending on the energy density $t = \frac{1}{2}g_{kl}y^ky^l$. Here $g_{0i} = g_{hi}y^h$ and y^h are the components of the current tangent vector in a local chart on the base manifold M. Our main result is given by:

Theorem A. *If (M, g) is a Riemannian manifold of negative constant sectional curvature then its tangent bundle has a structure of Kaehler Einstein manifold. If (M, g) has positive constant sectional curvature then a tube around the zero section in TM has a structure of Kaehler Einstein manifold.*

The proof will be given in the last section of the paper.

Calculating the covariant derivative of the curvature tensor field of the Levi Civita connection of the considered Riemannian metric on TM we found the quite interesting result that the considered Riemannian strucure is locally symmetric. Then our theorem A can be obtained as a consequence of this result. However, the direct proof has been more convenient.

Remark that the Kaehlerian structures studied in [14] and in this paper are, in a certain sense, exceptional cases arising in the study of a Kaehler Einstein structure on the tangent bundle defined by an M-tensor field given by the components

$$u(t)g_{ij} + v(t)g_{0i}g_{0j}$$

where u, v are real valued functions defined on $[0, \infty)$, satisfying the conditions $u > 0,\ u + 2tv > 0$.

The manifolds, tensor fields and geometric objects we consider in this paper, are assumed to be differentiable of class C^∞ (i.e. smooth). We use the computations in local coordinates in a fixed local chart, but many results from this paper may be expressed in an invariant form. The well-known summation convention is used throughout this paper, the range for the indices i, j, k, l, h, s, r being always$\{1, \dots, n\}$ (see [5], [4], [12], [13]). We shall denote by $\Gamma(TM)$ the module of smooth vector fields on M.

1. An almost Kaehler structure on the tangent bundle

Let (M, g) be a smooth n-dimensional Riemannian manifold and denote its tangent bundle by $\tau : TM \longrightarrow M$. Recall that TM has a structure of $2n$-dimensional smooth manifold induced from the smooth manifold structure of M. A local chart $(U,\varphi)=(U,x^1,\dots,x^n)$ on M induces a local chart $(\tau^{-1}(U),\Phi)=(\tau^{-1}(U),x^1,\dots,x^n,\ y^1,\dots,y^n)$ on TM where the local coordinates $x^i,y^i;\ i=1,\dots,n$ are defined as follows. The first n local coordinates $x^i=x^i\circ\tau;\ i=1,\dots,n$ on TM are the local coordinates in the local chart (U,φ) of the base point of a tangent vector from $\tau^{-1}(U)$. The last n local coordinates $y^i; i=1,\dots,n$ are the vector space coordinates of the same tangent vector, with respect to the natural local basis in the corresponding tangent space, defined by the local chart (U,φ).

This special structure of TM allows us to introduce the notion of M-tensor field on it (see [7]). An M-tensor field of type (p,q) on TM is defined, in a classical manner, by sets of functions

$$T^{i_1\dots i_p}_{j_1\dots j_q};\ i_1,\dots,i_p,j_1,\dots,j_q=1,\dots,n\ ,$$

assigned to any induced local charts $(\tau^{-1}(U),\Phi)$ on TM, such that the change rule is that of the components of a tensor field of type (p,q) on the base manifold, when a change of local charts on the base manifold is performed. Remark that any M-tensor field on TM may be thought of as an ordinary tensor field T with the expression

$$T=T^{i_1\dots i_p}_{j_1\dots j_q}\frac{\partial}{\partial y^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial y^{i_p}}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_q}$$

However, we should prefer to work with M-tensor fields on TM. From such M-tensor fields we are able to obtain some other authentic tensor fields on TM. Remark that any ordinary tensor field on the base manifold may be thought of as an M-tensor field on TM, having the same type and with the components in the induced local chart on TM, equal to the local coordinate components of the given tensor field in the chosen local chart on the base manifold. The M-tensor field associated to a covariant tensor field on the base manifold M may be thought of as the pull back of this tensor field by the smooth submersion $\tau: TM\longrightarrow M$.

The tangent bundle TM of a Riemannian manifold (M,g) can be organized as a Riemannian or a pseudo-Riemannian manifold in many ways. The most known such structures are given by the Sasaki metric on TM defined by g (see [15], [3]) and the complete lift type pseudo-Riemannian metric defined by g (see [16], [17], [10], [11]). Recall that the Levi Civita connection of g defines a direct sum decomposition

$$TTM=VTM\oplus HTM \tag{1}$$

of the tangent bundle to TM into the vertical distribution $VTM=\operatorname{Ker}\tau_*$ and the horizontal distribution HTM defined by the Levi Civita connection $\dot\nabla$ of g. The vector fields $(\frac{\partial}{\partial y^1},\ \dots,\frac{\partial}{\partial y^n})$ define a local frame field for VTM and for HTM we have the local frame field $(\frac{\delta}{\delta x^1},\dots,\ \frac{\delta}{\delta x^n})$ where

$$\frac{\delta}{\delta x^i}=\frac{\partial}{\partial x^i}-\Gamma^h_{i0}\frac{\partial}{\partial y^h};\quad \Gamma^h_{i0}=\Gamma^h_{ik}y^k$$

and $\Gamma^h_{ik}(x)$ are the Christoffel symbols defined by the Riemannian metric g.

The distributions VTM and HTM are isomorphic each other and it is possible to derive an almost complex structure on TM which, together with the Sasaki metric, determines a structure of almost Kaehlerian manifold on TM (see [3]). Consider now the energy density (kinetic energy or "forza viva" according to the terminology used by Levi Civita):

$$t = \frac{1}{2} g_{ik}(x) y^i y^k \ , \tag{2}$$

defined on TM by the Riemannian metric g of M, where g_{ik} are the components of g in the local chart (U, φ). Let $v : [0, \infty) \longrightarrow \mathbf{R}$ be a real smooth function. We shall assume that v has the following property:

$$1 + 2tv > 0$$

Remark that this condition is automatically fulfilled by a positive valued function v. Later on, we shall study the conditions under which this property of v is fulfilled. Then we may consider the following symmetric M-tensor field of type (0,2) on TM, defined by the components (see [14], [13]):

$$G_{ij} = g_{ij} + v(t) g_{0i} \ g_{0j} \ , \tag{3}$$

where $g_{0i} = g_{hi} y^h$. It follows that the matrix (G_{ij}) is symmetric and positive and has the inverse with the entries

$$H^{kl} = g^{kl} + w y^k y^l \ . \tag{4}$$

where g^{kl} are the components of the inverse of the matrix (g_{ij}) and:

$$w = w(t) = -\frac{v}{1 + 2tv}.$$

The components $H^{kl}(x, y)$ define a symmetric M-tensor field of type $(2, 0)$ on TM. We shall use also the components $H_{ij}(x, y)$ of a symmetric M-tensor field of type $(0, 2)$ obtained from the components H^{kl} by "lowering" the indices:

$$H_{ij} = g_{ik} H^{kl} g_{lj} = g_{ij} + w \ g_{0i} g_{0j} \ . \tag{5}$$

Throughout this paper we shall use also the following M-tensor fields on TM:

$$\left.\begin{aligned} G^{kl} &= g^{ki} G_{ij} g^{jl} = g^{kl} + v y^k y^l, \\ G^i_k &= G^{ih} g_{hk} = G_{kh} g^{hi} = \delta^i_k + v \ y^i g_{0k}, \\ H^i_k &= H^{ih} g_{hk} = H_{kh} g^{hi} = \delta^i_k + w \ y^i g_{0k}. \end{aligned}\right\} \tag{6}$$

Remark that the matrix (H^i_k) is the inverse of the matrix (G^i_k). The following Riemannian metric may be considered on TM:

$$G = G_{ij} dx^i dx^j + H_{ij} \dot{\nabla} y^i \dot{\nabla} y^j \ , \tag{7}$$

where $\dot{\nabla} y^i = dy^i + \Gamma^i_{j0} dx^j$ is the absolute differential of y^i with respect to the Levi Civita connection $\dot{\nabla}$ of g. Equivalently, we have

$$G(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = G_{ij}, \quad G(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = H_{ij}, \quad G(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}) = \ G(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i}) = 0.$$

Remark that HTM, VTM are orthogonal each other with respect to G but HTM and VTM, with the Riemannian metrics induced from G, are not isometric vector bundles on TM, so the considered metric G on TM is no longer a metric of Sasaki type. Remark also that the system of 1-forms $(dx^1,\ldots,dx^n,\dot\nabla y^1,\ldots,\dot\nabla y^n)$ defines a local frame of T^*TM, dual to the local frame $(\frac{\delta}{\delta x^1},\ldots,\frac{\delta}{\delta x^n},\frac{\partial}{\partial y^1},\ldots,\frac{\partial}{\partial y^n})$ adapted to the direct sum decomposition (1).

An almost complex structure J may be defined on TM by:

$$J\frac{\delta}{\delta x^i}=G_i^k\frac{\partial}{\partial y^k};\qquad J\frac{\partial}{\partial y^i}=-H_i^k\frac{\delta}{\delta x^k}\ . \tag{8}$$

Then we obtain:

Theorem 1. *(TM,G,J) is an almost Kaehlerian manifold.*

Proof. First of all we may check easily that

$$J^2\frac{\delta}{\delta x^i}=-\frac{\delta}{\delta x^i};\qquad J^2\frac{\partial}{\partial y^i}=-\frac{\partial}{\partial y^i},$$

due to the property of the matrix (H_k^i) to be the inverse of the matrix (G_k^i), thus J really defines an almost complex structure on TM. Then we have

$$G(J\frac{\delta}{\delta x^i},J\frac{\delta}{\delta x^j})=G_{ik}g^{ka}G_{jh}g^{hb}G(\frac{\partial}{\partial y^a},\frac{\partial}{\partial y^b})=$$

$$=G_{ik}g^{ka}G_{jh}g^{hb}g_{ac}G^{cd}g_{db}=G_{ij}=G(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}).$$

The relations

$$G(J\frac{\partial}{\partial y^i},J\frac{\partial}{\partial y^j})=G(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j});\quad G(J\frac{\partial}{\partial y^i},J\frac{\delta}{\delta x^j})=G(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j})=0$$

may be obtained in a similar way, thus G is almost Hermitian with respect to J. The associated 2-form Ω is given by

$$\Omega(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j})=G(\frac{\delta}{\delta x^i},J\frac{\delta}{\delta x^j})=G(\frac{\delta}{\delta x^i},G_j^k\frac{\partial}{\partial y^k})=0,$$

$$\Omega(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j})=G(\frac{\partial}{\partial y^i},J\frac{\partial}{\partial y^j})=G(\frac{\partial}{\partial y^i},-H_j^k\frac{\delta}{\delta x^k})=0,$$

$$\Omega(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j})=G(\frac{\partial}{\partial y^i},J\frac{\delta}{\delta x^j})=G_j^kG(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^k})=G_j^kH_{ik}=g_{ij}.$$

Hence we have

$$\Omega=g_{ij}\dot\nabla y^i\wedge dx^j \tag{9}$$

and Ω is closed since it does coincide with the 2-form associated to the Sasaki metric on TM (see [3]).

2. A Kaehler structure on *TM*

In this section we shall study the integrability of the almost complex structure defined by J on TM. To do this we need the following well known formulas for the brackets of the vector fields $\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^i}$; $i = 1, \ldots, n$:

$$[\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}] = 0; \quad [\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}] = -\Gamma^h_{ij}\frac{\partial}{\partial y^h}; \quad [\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}] = -R^h_{0ij}\frac{\partial}{\partial y^h}\ , \tag{10}$$

where $R^h_{0ij} = R^h_{kij}y^k$ and R^h_{kij} are the local coordinate components of the curvature tensor field of $\dot\nabla$ on M.

Theorem 2. *The almost complex structure J on TM is integrable if and only if (M, g) has constant sectional curvature c and*

$$v = -c. \tag{11}$$

If $c < 0$ we obtain a Kaehler structure on whole TM. In the case where the constant c is positive we have a Kaehler structure in the tube around the zero section in TM defined by

$$t < \frac{1}{2c}. \tag{12}$$

Proof. Consider the M-tensor field of type $(0, 3)$ on TM, defined by the components $\dot\nabla_i G_{jk}$, as well as the M-tensor field of type $(1, 2)$ on TM defined by the components $\dot\nabla_i G^j_k$, where:

$$\dot\nabla_i G_{jk} = \frac{\delta G_{jk}}{\delta x^i} - \Gamma^h_{ij}G_{hk} - \ \Gamma^h_{ik}G_{jh},$$

$$\dot\nabla_i G^j_k = \frac{\delta G^j_k}{\delta x^i} + \Gamma^j_{ih}G^h_k - \ \Gamma^h_{ik}G^j_h.$$

Then the formulas $\dot\nabla_i G_{jk} = 0$, $\dot\nabla_i G^j_k = 0$ can be checked by straightforward computation. In the same way we can obtain the relations $\dot\nabla_i H_{jk} = 0$, $\dot\nabla_i H^j_k = 0$. Then, by using the definition of the Nijenhuis tensor field N_J of J we have

$$N_J(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) =$$

$$= [J\frac{\delta}{\delta x^i}, J\frac{\delta}{\delta x^j}] - J[\frac{\delta}{\delta x^i}, J\frac{\delta}{\delta x^j}] - J[J\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}] - [\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}] =$$

$$= (G^h_i\frac{\partial G^k_j}{\partial y^h} - G^h_j\frac{\partial G^k_i}{\partial y^h} + R^k_{0ij})\frac{\partial}{\partial y^k} - (\frac{\delta G^k_i}{\delta x^j} - \frac{\delta G^k_j}{\delta x^i} + G^h_i\Gamma^k_{hj} - G^h_j\Gamma^k_{ih})H^l_k\frac{\delta}{\delta x^l}.$$

The coefficient of $H^l_k\frac{\delta}{\delta x^l}$ is just $-\dot\nabla_j G^k_i + \ \dot\nabla_i G^k_j = 0$, so we have to study the vanishing of the coefficient of $\frac{\partial}{\partial y^k}$. By using the expression (6) of G^k_i we get:

$$R^k_{0ij} + v\ (\delta^k_i\ g_{0j} - \delta^k_j\ g_{0i}) = 0.$$

It follows that the curvature tensor field of $\dot{\nabla}$ must have the expression

$$R^k_{hij} = c(\delta^k_i g_{hj} - \delta^k_j g_{hi}) \ ,$$

where c is a constant and $v(t) = -c$.

Next, we have

$$N_J(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}) = -H^h_i(\frac{\delta G^k_j}{\delta x^h} - \frac{\delta G^k_h}{\delta x^j} + \Gamma^k_{hl}\ G^l_j - \Gamma^k_{jl}G^l_h)\frac{\partial}{\partial y^k} +$$

$$+H^k_h H^l_i(-R^h_{0jl} + G^r_l\frac{\partial G^h_r}{\partial y^j} - \ G^r_j\frac{\partial G^h_l}{\partial y^r})\frac{\delta}{\delta x^k}$$

and

$$N_J(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = \ - H^k_i H^r_j(\frac{\delta G^l_r}{\delta x^k} - \frac{\delta G^l_k}{\delta x^r} + \ G^s_r\Gamma^l_{sk} - \Gamma^l_{rs}G^s_k)H^h_l\frac{\delta}{\delta x^h} +$$

$$+H^k_i H^l_j(-R^h_{0kl} - \frac{\partial G^h_l}{\partial y^r}G^r_k + \ \frac{\partial G^h_k}{\partial y^r}G^r_l)\frac{\partial}{\partial y^h}.$$

Hence, the condition $N_J(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = 0$ implies that $N_J(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}) = 0$ and $N_J(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = 0$.

It follows that (M, g) must have constant sectional curvature and $v = -c$. Then:

$$G_{ij} \ = \ g_{ij} \ - \ c\ g_{0i}g_{0j}, \ H_{ij} \ = \ g_{ij} \ + \ \frac{c}{1-2ct}\ g_{0i}g_{0j} \ . \tag{13}$$

From the expression of G_{ij} it follows that the symmetric matrix (G_{ij}) is positive if $c < 0$ and from the expression of H_{ij} it follows that we have a Kaehler structure even when $c > 0$ but only in the tube around the zero section of TM defined by $g_{ij}y^iy^j = 2t < \frac{1}{c}$.

Remark. In the case $c = 0$ (M, g) is a flat manifold and the metric G on TM becomes the flat Sasaki metric.

3. A Kaehler Einstein structure on *TM*

In this section we shall study the property of (TM, G) to be Einstein. We shall find the expression of the Levi Civita connection ∇ of the metric G on TM, then we shall find the expression of the curvature tensor filed of ∇. Then, computing the corresponding traces we shall find the components of the Ricci tensor field of ∇ and we find that it is proportional with G, hence (TM, G) is an Einstein manifold.

Recall that the Levi Civita connection ∇ on a Riemannian manifold (M, g) is obtained from the formula

$$2g(\nabla_X Y, Z) = X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) +$$

$$+g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X); \qquad \forall X, Y, Z \in \Gamma(TM),$$

and is characterized by the conditions

$$\nabla G = 0, \qquad T = 0 \ ,$$

where T is the torsion tensor of ∇([5], [4]).

Theorem 3.([14]) *The Levi Civita connection ∇ of G has the following expression in the local adapted frame* $(\frac{\partial}{\partial y^1},\dots,\frac{\partial}{\partial y^n},\ \frac{\delta}{\delta x^1},\dots,\frac{\delta}{\delta x^n})$:

$$\nabla_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j}=\frac{1}{2}G^{hk}(\frac{\partial H_{jk}}{\partial y^i}+\ \frac{\partial H_{ik}}{\partial y^j}-\frac{\partial H_{ij}}{\partial y^k})\ \frac{\partial}{\partial y^h}\ ,$$

$$\nabla_{\frac{\delta}{\delta x^i}}\frac{\partial}{\partial y^j}=\Gamma^h_{ij}\frac{\partial}{\partial y^h}+\ \frac{1}{2}(\frac{\partial G_{ik}}{\partial y^j}+H_{jl}R^l_{0ik})H^{kh}\ \frac{\delta}{\delta x^h}\ ,$$

$$\nabla_{\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j}=\frac{1}{2}(\frac{\partial G_{jk}}{\partial y^i}+H_{il}R^l_{0jk})H^{kh}\ \frac{\delta}{\delta x^h}\ ,$$

$$\nabla_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j}=\ \Gamma^h_{ij}\frac{\delta}{\delta x^h}+\frac{1}{2}(-R^h_{0ij}-\ \frac{\partial G_{ij}}{\partial y^k}G^{kh})\frac{\partial}{\partial y^h}.$$

Consider now the case where (TM,G,J) has a structure of Kaehler manifold, i.e. M has (negative) constant sectional curvature c, $v=-c$, $w=\frac{c}{1-2ct}$ and G_{ij}, H_{ij} are given by (13). We have

Theorem 4. *The Levi Civita connection ∇ of the Kaehler manifold (TM,G,J) is given by the formulas:*

$$\left.\begin{array}{l}\nabla_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j}=Q^h_{ij}\frac{\partial}{\partial y^h}\ ;\ \nabla_{\frac{\delta}{\delta x^i}}\frac{\partial}{\partial y^j}=\Gamma^h_{ij}\frac{\partial}{\partial y^h}+P^h_{ji}\frac{\delta}{\delta x^h}\\[2ex] \nabla_{\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j}=P^h_{ij}\frac{\delta}{\delta x^h}\ ;\ \nabla_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j}=\ \Gamma^h_{ij}\frac{\delta}{\delta x^h}+S^h_{ij}\frac{\partial}{\partial y^h},\end{array}\right\}\qquad(14)$$

where the M-tensor fields P^h_{ij}, Q^h_{ij}, S^h_{ij} are given by:

$$\left.\begin{array}{l}P^h_{ij}=-c\delta^h_i g_{0j}-\frac{c^2}{1-2ct}g_{0i}g_{0j}y^h=-cg_{0j}H^h_i\ ,\\[2ex] Q^h_{ij}=cg_{ij}y^h+\frac{c^2}{1-2ct}g_{0i}g_{0j}y^h=cy^hH_{ij}\ ,\\[2ex] S^h_{ij}=c\delta^h_j g_{0i}-c^2g_{0i}g_{0j}y^h=cg_{0i}G^h_j\ .\end{array}\right\}\qquad(15)$$

Denote by K the curvature tensor field of the connection ∇. Then we obtain by a straightforward computation:

$$\left.\begin{array}{l}K(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j})\frac{\delta}{\delta x^k}=c(\delta^h_iG_{jk}-\delta^h_jG_{ik})\frac{\delta}{\delta x^h},\\[2ex] K(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j})\frac{\partial}{\partial y^k}=c(g_{jk}G^h_i-g_{ik}G^h_j)\frac{\partial}{\partial y^h},\\[2ex] K(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j})\frac{\delta}{\delta x^k}=c(g_{jk}H^h_i-g_{ik}H^h_j)\frac{\delta}{\delta x^h},\\[2ex] K(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j})\frac{\partial}{\partial y^k}=c(\delta^h_iH_{jk}-\delta^h_jH_{ik})\frac{\partial}{\partial y^h},\\[2ex] K(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j})\frac{\delta}{\delta x^k}=cg_{ij}G^h_k\frac{\partial}{\partial y^h},\ K(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j})\frac{\partial}{\partial y^k}=-cg_{ij}H^h_k\frac{\delta}{\delta x^h}.\end{array}\right\}\qquad(16)$$

Finally, we obtain the Ricci tensor field of ∇, defined as $Ric(Y,Z) = \operatorname{trace}(X \longrightarrow K(X,Y)Z)$; $X,Y,Z \in \Gamma(TM)$. It follows:

$$\left.\begin{aligned} &Ric(\tfrac{\delta}{\delta x^i},\tfrac{\delta}{\delta x^j}) = c\ n\ G_{jk}\ , \\ &Ric(\tfrac{\partial}{\partial y^i},\tfrac{\partial}{\partial y^j}) = c\ n\ H_{jk}\ . \\ &Ric(\tfrac{\partial}{\partial y^i},\tfrac{\delta}{\delta x^j}) = 0. \end{aligned}\right\} \tag{17}$$

It follows that $Ric = c\ n\ G$ and the theorem A is proved.

By using the expression (16), we computed the components of the covariant derivative of the curvature tensor field K of the connection ∇ with respect to the vector fields $\frac{\delta}{\delta x^h}$ and $\frac{\partial}{\partial y^h}$, and we obtained that in all twelve cases the result is zero. Thus we may state:

Theorem 5. *If (M,g) has constant negative sectional curvature and the components of the metric G are given by (13) then (TM,G) is a locally symmetric space. If (M,g) has constant positive sectional curvature and the components of the metric G are given by (13) then the tube around the zero section in TM defined by the condition (12), endowed with the metric G, is a locally symmetric Riemannian manifold.*

Acknowledgement. The computations made in order to obtain Theorem 5 have been checked by using the Mathematica package RICCI for doing tensor calculations, elaborated by John M. Lee. The author is grateful to Eng. Victor Fecioru for supporting him with some advices, software and hardware, necessary in the computations made by using RICCI.

The author is indebted to the referee for the carefull reading of the manuscript, pointing out some unclear expressions from the initial version of the manuscript, as well as some suggestions concerning a possible continuation of the research in the direction presented here.

References

[1] Besse, A.: *Einstein Manifolds.* Ergeb. Math. Grenzgeb. (3) **10**, Springer-Verlag, Berlin 1987.

[2] Calabi, E.: *Métriques Kaehlériennes et fibrés holomorphes.* Ann. Sci. Ecole Norm. Sup. **12** (1979), 269–294.

[3] Dombrowski, P.: *On the geometry of the tangent bundle.* J. Reine Angew. Math. **210** (1962), 73–88.

[4] Gheorghiev, Gh.; Oproiu, V.: *Varietăţi diferenţiabile finit şi infinit dimensionale.* Ed. Academiei Rom. I (1976), II (1979).

[5] Kobayashi, S.; Nomizu, K.: *Foundations of Differential Geometry.* Interscience, New York, I (1963), II (1969).

[6] Kowalski, O.: *Curvature of the induced Riemannian metric on the tangent bundle of a Riemannian manifold.* J. Reine Angew. Math. **250** (1970), 124–129.

[7] Mok, K. P.; Patterson, E. M.; Wong, Y. C.: *Structure of symmetric tensors of type (0,2) and tensors of type (1,1) on the tangent bundle.* Trans. Amer. Math. Soc. **234** (1977), 253–278.

[8] Oproiu, V.: *On the differential geometry of the tangent bundles.* Rev. Roumaine Math. Pures Appl. **13** (1968), 847–856.

[9] Oproiu, V.: *Some remarkable structures and connections defined on the tangent bundle.* Rend. Mat. Roma **6** (1973), 503–540.

[10] Oproiu, V.: *A Riemannian structure in Lagrange geometry.* Rend. Sem. Fac. Sci. Univ. Cagliari **55**(2) (1985), 1–20.

[11] Oproiu, V.: *A pseudo-Riemannian structure in Lagrange geometry.* An. Ştiinţ. Univ. Al. I. Cuza Iaşi **33** (1987), 239–254.

[12] Oproiu, V.; Papaghiuc, N.: *A pseudo-Riemannian metric on the cotangent bundle.* An. Ştiinţ. Univ. Al. I. Cuza Iaşi **36** (1990), 265–276.

[13] Oproiu, V.; Papaghiuc, N.: *Some examples of almost complex manifolds with Norden metric.* Publ. Math. Debrecen **41** (1992), 199–212.

[14] Oproiu, V.; Papaghiuc, N.: *A Kaehler structure on the nonzero tangent bundle of a space form.* To appear.

[15] Sasaki, S.: *On the Differential Geometry of the Tangent Bundle of Riemannian Manifolds.* Tohoku Math. J. **10** (1958), 238–354.

[16] Yano, K.; Kobayashi, S.: *Prolongations of tensor fields and connections to tangent bundles, I. General Theory.* Jour. Math. Soc. Japan **18** (1966), 194–210.

[17] Yano, K.; Ishihara, S.: *Tangent and Cotangent Bundles.* M. Dekker Inc., New York 1973.

Received October 15, 1997 ; revised version February 16, 1998