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On the Cohen-Macaulayness of the Coordinate Ring of Certain Projective Monomial Curves

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Abstract. Let K be a field and let $n_1, \ldots, n_e$ be a sequence of positive integers with $\gcd(n_1, \ldots, n_e) = 1$ and $n_1 < n_2 < \cdots < n_e$. Let A' be the coordinate ring of the projective monomial curve in the projective e-space $\mathbb{P}^e_K$ defined parametrically by $Z_0 = X^{n_e}, \ldots, Z_i = X^{n_e - n_i} Y^{n_i}, \ldots, Z_e = Y^{n_e}$ where $n_0 := 0$. In this paper under some assumptions, we discuss when exactly the graded ring A' is Cohen-Macaulay and we give numerical criterion for this in terms of the standard basis of the semigroup generated by $n_1, \ldots, n_e$ in the case when some $e-1$ terms of $n_1, \ldots, n_e$ form an arithmetic sequence. Our special assumptions are satisfied in the case $e = 3$, in particular, for the class of monomial projective space curves, we get a criterion for arithmetically Cohen-Macaulayness.

1. Introduction

In this article we assume that A' is the coordinate ring of a projective monomial curve in the projective e-space $\mathbb{P}^e_K$ defined parametrically by $Z_0 = X^{n_e}, \ldots, Z_i = X^{n_e - n_i} Y^{n_i}, \ldots, Z_e = Y^{n_e}$

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where $n_0 := 0$ and $n_1 < \cdots < n_e$ is a sequence of positive integers with $\gcd(n_1, \ldots, n_e) = 1$. In special case when the sequence $n_1, \ldots, n_e$ is an arithmetic sequence then A' is always Cohen-Macaulay (see [3; Proposition (1.2)]). Cavaliere and Niesi (see [1; Lemma (3.4)] and [2; Lemma (4.3)]) proved that A' is Cohen-Macaulay if and only if Z_0, Z_e is an A'-regular sequence. However, in concrete examples to check that Z_0, Z_e is an A'-regular sequence is tedious and requires long computation.

Our aim of this paper is to give simple necessary and sufficient conditions for A' to be Cohen-Macaulay and which are easy to check. We consider the case when the sequence $m_0 := n_1, \ldots, m_{e-1} := n_e$ is an almost arithmetic sequence, that is, $m_0 < \cdots < m_{e-2}$ is an arithmetic sequence and $n := m_{e-1}$ is arbitrary. We prove that (in most cases) A' is Cohen-Macaulay if and only if exactly one inequality between two positive integers holds. This inequality involves the non-negative integers $\lambda, \mu, \nu, u, v, z, w$ which were defined in [6], using the explicit description of the standard basis S_{m_0} of the semigroup $\Gamma := \sum_{i=0}^{e-2} \mathbb{N}m_i + \mathbb{N}n$. The explicit description of S_{m_0} and the properties of the non-negative integers $\lambda, \mu, \nu, u, v, z, w$ were used to find explicit minimal sets of generators for the derivation module and the relation ideal of A' (see [5] and [6]). In addition to these non-negative integers we define four more non-negative integers z', μ', λ' and w'. Given integers $m_0, \ldots, m_{e-2}, n$, it is easy to find the non-negative integers $\lambda, \mu, \nu, u, v, z, w$ z', μ', λ', w' and then we only have to check essentially only one inequality for A' to be Cohen-Macaulay.

For an integer $t \in \mathbb{Z}^+$, let $q_t \in \mathbb{Z}$, $r_t \in \{1, 2, \ldots, p\}$ be the unique integers defined by the equation $t = q_t p + r_t$. Then, in 4.8, 5.8 and 6.6 we prove the following results:

- In the case $n < m_0$:
  (1) If $w = 0$ then A' is Cohen-Macaulay.
  (2) If $w \neq 0$ then A' is Cohen-Macaulay if and only if $\nu \geq q_{u-z} + 1 + (v - w)$.
- In the case $m_0 < n < m_p$:
  (1) If $z' \geq p$ then A' is Cohen-Macaulay if and only if $\nu \geq q_{u-z} + 1 + (v - w)$ and $v \geq \mu' + q_{z'} + 1$.
  (2) If $1 \leq z' < p$ then :
    (a) If A' is Cohen-Macaulay then $v + 1 \geq \operatorname{mindeg}((\mu' + 1)m_0) + 1$.
    (b) If $v \geq \mu' + 1$ then A' is Cohen-Macaulay.
  (3) If $z' = 0$ then A' is Cohen-Macaulay if and only if $v + w' \geq \mu - \lambda' + q_u + 1$.
- In the case $m_p < n$:
  (1) If $w' \neq 0$ then A' is Cohen-Macaulay if and only if
  $q_{u+k} + 1 \geq \operatorname{mindeg}((\lambda' - \varepsilon_k)m_0 + g_k) + w'$ for every $k \in [0, p - r_u]$.
  (2) If $w' = 0$ then the following statements are equivalent :
    (i) A' is Cohen-Macaulay.
    (ii) $\operatorname{mindeg}(h) = \deg(\delta(h))$ for every $h \in \Gamma$.

(iii) For every $j \in \mathbb{N}$ with $a(g_j) \geq 1$ and $b(g_j) \geq 1$, there exist unique integers $s, t \in \mathbb{N}^+$ such that $j = i(g_j) + tz + su$ and $s\lambda'' \leq tv''$, where $\lambda'' := \lambda' - q_u - 1$ and $v'' := \mu + q_z + 1 - v$,

where $\delta(h)$, $a(g_j)$, $i(g_j)$ and $b(g_j)$ are as in the Definition 6.2.

Since any three integers are in almost arithmetic sequence with $n < m_0 < m_1$, the Theorem 4.8 determines all monomial projective space curves which are arithmetically Cohen-Macaulay. We also give many examples to illustrate.

For many values of $m_0, \ldots, m_p, n$, we used the Computer algebra programme CoCoA to test whether or not the graded ring A' is Cohen-Macaulay. This helped us to find the right inequalities.

2. Numerical criterion for Cohen-Macaulayness

In this section we give a numerical criterion for the Cohen-Macaulayness of the coordinate ring of a projective monomial curve.

2.1. Notations. Let $\mathbb{Z}$ (respectively $\mathbb{N}$, $\mathbb{N}^+$) denote the set of all (respectively non-negative, positive) integers. For $a, b \in \mathbb{Z}$, let $[a, b] := \{i \in \mathbb{Z} \mid a \leq i \leq b\}$. Unless mentioned otherwise, the symbols $a, b, c, d, e, i, j, k, l, m, n, p, q, r, s, t, u, v, w, z$ denote integers.

Let $n_1, \ldots, n_e$ be a sequence of positive integers with $n_1 < \cdots < n_e$ and $\gcd(n_1, \ldots, n_e) = 1$. We assume that $n_1, \ldots, n_e$ is a minimal set of generators for the semigroup $\Gamma := \sum_{i=1}^{e} \mathbb{N}n_i$.

- Let $A' := K[Y^{n_e}, X^{n_1}Y^{n_e-n_1}, \ldots, X^{n_i}Y^{n_e-n_i}, \ldots, X^{n_e}]$. Let $n_0 := 0$ and for $0 \leq i \leq e$, let $Z_i := X^{n_i}Y^{n_e-n_i}$.
- Let $\mathcal{E} := (\mathbb{N})^e$.
- For $i \in [1, e]$, we put $\mathbf{e}_i := (\delta_{ij})_{1 \leq j \leq e}$, where δ_{ij} denote the Kronecker delta.
- For $\alpha = \sum_{i=1}^{e} a_i \mathbf{e}_i$, let $\partial(\alpha) := \sum_{i=1}^{e} a_i n_i$.
- For $h \in \Gamma$, let $\mathcal{E}(h) := \{\alpha \in \mathcal{E} \mid \partial(\alpha) = h\}$.
- For $\alpha = \sum_{i=1}^{e} a_i \mathbf{e}_i \in \mathcal{E}(h)$, we put $\deg(\alpha) := \sum_{i=1}^{e} a_i$.
- For $\alpha, \beta \in \mathcal{E}(h)$, we write $\alpha \leq_{\deg} \beta$ if $\deg(\alpha) \leq \deg(\beta)$.

Then $\leq_{\deg}$ is an order on $\mathcal{E}(h)$ and since $\mathcal{E}(h)$ is a finite set, $\mathcal{E}(h)$ has minimal elements with respect to the order $\leq_{\deg}$. Let $\min(\mathcal{E}(h))$ denote the set of all minimal elements in $\mathcal{E}(h)$. Note that all elements of $\min(\mathcal{E}(h))$ have the same degree, therefore this degree we shall denote by $\operatorname{mindeg}(h)$.

2.2. Lemma. *With the notations as above,*

(1) *Let* $(a_0, a_1, \ldots, a_e), (b_0, b_1, \ldots, b_e) \in (\mathbb{N})^{e+1}$. *Then* $Z_0^{a_0} Z_1^{a_1} \cdots Z_e^{a_e} = Z_0^{b_0} Z_1^{b_1} \cdots Z_e^{b_e}$ *in* A' *if and only if* $\sum_{i=0}^{e} a_i = \sum_{i=0}^{e} b_i$ *and* $\sum_{i=0}^{e} a_i n_i = \sum_{i=0}^{e} b_i n_i$.

(2) *If* $Z_1^{a_1} \cdots Z_e^{a_e} = Z_0^{b_0} Z_1^{b_1} \cdots Z_e^{b_e}$ *with* $b_0 > 0$ *then* $\sum_{i=1}^{e} a_i > \sum_{i=1}^{e} b_i$.

(3) $Z_1^{a_1} \cdots Z_e^{a_e} \notin Z_0 A'$ *if and only if* $\operatorname{mindeg}(\sum_{i=1}^{e} n_i a_i) = \sum_{i=1}^{e} a_i$.

Proof. (1) is clear and (2) follows from (1).

(3) A monomial $Z_1^{a_1}\cdots Z_e^{a_e} \in Z_0A'$ if and only if there exists $(b_0, b_1, \ldots, b_e) \in (\mathbb{N})^{e+1}$ such that $b_0 \neq 0$ and $Z_1^{a_1}\cdots Z_e^{a_e} = Z_0^{b_0}Z_1^{b_1}\cdots Z_e^{b_e}$. Therefore the assertion follows from (1) and (2). $\square$

2.3. Proposition. *The following statements are equivalent:*

(i) *A' is Cohen-Macaulay.*

(ii) $\operatorname{mindeg}(h + tn_e) = \operatorname{mindeg}(h) + t$ *for every $h \in \Gamma$ and $t \in \mathbb{N}$.*

Proof. Since A' is Cohen-Macaulay if and only if Z_e^t is a non-zero divisor in A'/Z_0A' for every $t \geq 1$ by [2; Lemma (4.3)], the equivalence follows from 2.2 (3). $\square$

3. Key-Lemma and its consequences

We continue using the notations introduced in 2.1, except that the sequence $n_1, \ldots, n_e$ will be denoted by $m_0, \ldots, m_{e-1}$. This is convenient for using the same notations and results from the papers [4], [5] and [6].

3.1. Standard basis. The set $S_{m_0} := \{z \in \Gamma \mid z - m_0 \notin \Gamma\}$ is called the *standard basis* or *Apéry set of* Γ *with respect to* m_0. It is clear that S_{m_0} depends on Γ and m_0, but for simplicity we put $S := S_{m_0}$. It is easy to see that $S = \{s_0 := 0, s_1, \ldots, s_{m_0-1}\}$, where $s_1, \ldots, s_{m_0-1} \in \Gamma$ are positive integers with the following properties:

(i) $s_i \equiv i (\mathrm{mod}\ m_0)$ for every $i \in [0, m_0 - 1]$.

(ii) If $z \in \Gamma$ then $z \equiv i (\mathrm{mod}\ m_0)$ for a unique $i \in [0, m_0 - 1]$ and $z \geq s_i$.

3.2. Explicit description of the standard basis of a semigroup generated by almost arithmetic sequence.

Put $p := e - 2$ and assume that $m_0, \ldots, m_p$ is an arithmetic sequence, that is, there is a positive integer d with $m_i = m_0 + id$ for all $0 \leq i \leq p$. Let n be an arbitrary positive integer with $\gcd(m_0, d, n) = 1$. Let $\Gamma' := \sum_{i=0}^{p} \mathbb{N}m_i$ and $\Gamma = \Gamma' + \mathbb{N}n$. Let $S := S_{m_0}$ be the standard basis of Γ with respect to m_0. For $t \in \mathbb{N}$, let $q_t \in \mathbb{Z}$, $r_t \in [1, p]$, $g_t \in \Gamma'$ and $\gamma_t \in \mathcal{E}(g_t)$ be defined by the equations $t = q_tp + r_t$, $g_t = q_tm_p + m_{r_t}$ and $\gamma_t = q_t\mathbf{e}_p + \mathbf{e}_{r_t}$ respectively.

The Key-Lemma from [6; §3] (see also [4; §2]) is the following:

3.3. Key-Lemma. *With the notations as in 3.2,*

(1) $g_s + g_t = \varepsilon m_0 + g_{s+t}$ *with $\varepsilon = 1$ or 0 according as $r_s + r_t \leq p$ or $r_s + r_t > p$.*

(2) *Let $u := \min\{t \in \mathbb{N} \mid g_t \notin S\}$ and $v := \min\{b \geq 1 \mid bn \in \Gamma'\}$. Then there exist unique integers $w \in [0, v-1]$, $z \in [0, u-1]$, $\lambda \geq 1$, $\mu \geq 0$ such that*

  (i) $g_u = \lambda m_0 + wn$,

  (ii) $vn = \mu m_0 + g_z$,

(iii) $g_{u-z} + (v-w)n = \nu m_0$. *Moreover,* $\nu = \lambda + \mu + \varepsilon$ *where* $\varepsilon = 1$ *or* 0 *according as* $r_{u-z} < r_u$ *or* $r_{u-z} \geq r_u$.

(3) *Let* $V := [0, u-1] \times [0, v-1]$, $W := [u-z, u-1] \times [v-w, v-1]$ *and* $U := V \setminus W$. *Then* $S = \{g_s + bn \mid (s,b) \in U\}$. *In particular, if* $(s,b), (t,c) \in U$ *with* $g_s + bn \equiv g_t + cn (\mathrm{mod}\ m_0)$ *then* $(s,b) = (t,c)$.

(4) *Every element of* Γ *can be expressed uniquely in the form* $am_0 + g_s + bn$ *with* $a \in \mathbb{N}$ *and* $(s,b) \in U$.

(5) *The map* $\mathcal{E} \to (\mathbb{N})^2$ *defined by* $\sum_{i=0}^{p+1} a_i \mathbf{e}_i \mapsto (\sum_{i=0}^{p} a_i m_i, a_{p+1} n)$ *defines a bijection between* S *and* U.

Proof. See [6 ; §3]. □

3.4. Definition. *Let* $z' := \max\{i \in \mathbb{N} \mid vn = am_0 + g_i \text{ for some } a \in \mathbb{N}\}$ *and let* $\lambda' := \min\{a \in \mathbb{N}^+ \mid g_u = am_0 + bn \text{ for some } b \in \mathbb{N}\}$.

3.5. Lemma.

(1) $z' \geq z$ *and there is a unique integer* $\mu' \in \mathbb{N}$ *such that* $vn = \mu' m_0 + g_{z'}$. *In particular, if* $z' = 0$ *then* $z = 0$ *and* $\mu' = \mu$.

(2) $\lambda' \leq \lambda$ *and there is a unique integer* $w' \in \mathbb{N}$ *such that* $g_u = \lambda' m_0 + w' n$.

(3) *If* $z' > z$ *then* $z' \geq u$. *Moreover, if* $w \neq 0$ *then* $z' = z$ *and* $\mu' = \mu$ *and if* $w = 0$ *then* $z' = su + z$ *for some* $s \in \mathbb{N}$.

(4) *If* $1 \leq z' < p$ *then* $z = z'$.

(5) *If* $w' > w$ *then* $w' \geq v$. *Moreover, if* $z \neq 0$ *then* $w' = w$ *and if* $z = 0$ *then* $w' = tv + w$ *for some* $t \in \mathbb{N}$.

Proof. (1) and (2) clear from the definitions.

(3) If $z' \neq z$ then from $g_z \equiv g_{z'} (\mathrm{mod}\ m_0)$ and the Key-Lemma, we have $z' \geq u$. Now, from the equations $g_u = \lambda m_0 + wn$ and $\varepsilon m_0 + g_{z'} = g_u + g_{z'-u}$ with $\varepsilon \in \{0,1\}$, we get $\mu m_0 + g_z = vn = \mu' m_0 + g_{z'} = (\mu' + \lambda - \varepsilon) m_0 + g_{z'-u} + wn$. If $w \neq 0$ this contradicts the definition of v. Let $s \in \mathbb{Z}^+$, $j \in [0, u-1]$ be the unique integers defined by the equation $z' = su + j$. If $w = 0$ then $g_u \equiv 0 (\mathrm{mod}\ m_0)$ and so $g_z \equiv g_{z'} \equiv g_j (\mathrm{mod}\ m_0)$. Therefore $z = j$ by the Key-Lemma.

(4) follows from $z' = su + z$ and $u > p$.

(5) If $w' \neq w$ then from $wn \equiv w'n (\mathrm{mod}\ m_0)$ and the Key-Lemma, we have $w' \geq v$. Now, from the equations $vn = \mu m_0 + g_z$, $g_u = \lambda' m_0 + w' n$ and $\varepsilon m_0 + g_u = g_z + g_{u-z}$ with $\varepsilon \in \{0,1\}$, we get $g_{u-z} = (\lambda' + \mu + \varepsilon) m_0 + (w' - v) n \notin S$. If $z \neq 0$ this contradicts the definition of u (see Key-Lemma). Let $t \in \mathbb{Z}^+$, $b \in [0, v-1]$ be the unique integers defined by the equation $w' = tv + b$. If $z = 0$ then $wn \equiv g_u = (\lambda' + t\mu) m_0 + bn \equiv bn (\mathrm{mod}\ m_0)$ and so $w = b$ by the Key-Lemma. □

In the sequel the symbols $u, v, w, z, \lambda, \mu, \nu$, U, W and z', μ', λ', w' have the meaning assigned to them by the above Key-Lemma, 3.4 and 3.5. Sometimes giving reference of the Key-Lemma 3.3 is omitted.

3.6. Lemma. *Suppose that $z' = 0$. Then*

(1) $\mu \geq \lambda'$.

(2) *If $w = 0$ and $\lambda > \lambda'$ then $\lambda \geq \lambda' + \mu$.*

(3) *If $n < m_p$ then $\lambda' + w' \geq q_u + 1$.*

(4) $(v + w')\mathbf{e}_{p+1} \in \mathcal{E}\left((\mu - \lambda')m_0 + g_u\right)$.

Proof. Since $z' = 0$, $z = 0$ and $\mu' = \mu$ by 3.5 (1).

(1) From $vn = \mu m_0$ and $v \geq 1$, we have $\mu > 0$. If $\lambda' = \mu + a$ with $a \in \mathbb{N}^+$ then $a < \lambda'$. Therefore from the equations $vn = \mu m_0$ and $g_u = \lambda' m_0 + w'n$, we get $g_u = am_0 + (v + w')n$ a contradiction to the definition of λ'.

(2) Since $w = 0$ and $\lambda > \lambda'$, $w' > 0$ and $\lambda m_0 = g_u = \lambda' m_0 + w'n$ and so $w' \geq v$ by the definition of v. Therefore $(\lambda - \lambda')m_0 = w'n \geq vn = \mu m_0$ and so $\lambda - \lambda' \geq \mu$.

(3) If $n < m_p$ and $\lambda' + w' < q_u + 1$ then $q_u m_p + m_{r_u} = g_u = \lambda' m_0 + w'n < (\lambda' + w')m_p \leq q_u m_p < g_u$, a contradiction.

(4) Follows from (1) and the equations $vn = \mu m_0$ and $g_u = \lambda' m_0 + w'n$. $\square$

3.7. Proposition.

(1) $\operatorname{mindeg}(\mu' m_0) = \min\{v, \mu'\}$.

(2) *If $z' \neq 0$ then* $\operatorname{mindeg}(\mu' m_0 + m_{r_{z'}}) = \min\{v, \mu' + 1\}$.

(3) *Suppose that $z' = 0$. Then*

 (a) $\operatorname{mindeg}(\lambda' m_0) \in \{v, \lambda', q_u + 1\}$.

 (b) $\operatorname{mindeg}\left((\mu - \lambda')m_0 + m_{r_u}\right) = (\mu - \lambda') + 1$.

Proof. Let $j \in \{0, r_{z'}\}$ and let α be a minimal element in $\mathcal{E}(\mu' m_0 + g_j)$. Then, in view of [4; Lemma 2.6 (3)] we may assume that $\alpha = a\mathbf{e}_0 + \gamma_i + b\mathbf{e}_{p+1}$. From $vn = \mu' m_0 + g_{z'}$ and $am_0 + g_i + bn = \mu' m_0 + g_j$, we have

$$\text{(3.7.a)} \qquad vn = \begin{cases} (a + \varepsilon)m_0 + g_{i+z'} + bn, & \text{if } j = 0, \\ am_0 + g_{z'-j+i} + bn & \text{if } j = r_{z'}. \end{cases}$$

with $\varepsilon \in \{0, 1\}$.

(1) For $j = 0$ from the equation (3.7.a) and the definitions of v and z', it follows that either $b = v$ and $a = 0 = i$, or $b = 0$, $i = 0$ and $a = \mu'$.

(2) Since $z' \neq 0$, $z' - r_{z'} \geq 0$. Therefore for $j = r_{z'}$ from the equation (3.7.a) and the definitions of v and z', it follows that either $b = v$, $a = 0$, $i = 0$, $z' = r_{z'}$ or $b = 0$, $i = r_{z'}$ and $a = \mu'$.

(3) Since $z' = 0$, $z = 0$, $\mu' = \mu$ and $\mu \geq \lambda'$ by 3.6 (1).

(a) Let α be a minimal element in $\mathcal{E}(\lambda' m_0)$. Then, in view of [4; Lemma 2.6 (3)] we may assume that $\alpha = a\mathbf{e}_0 + \gamma_i + b\mathbf{e}_{p+1}$. If $b \neq 0$ then $b = v$ by definition of v and $a = 0 = i$. Since $g_u \geq \lambda' m_0 = am_0 + g_i + bn$, $i \leq u$. Therefore, if $b = 0$ then either $i = 0$ and $a = \lambda'$, or $i = u$ and $a = 0$ by the Key-Lemma.

(b) Let α be a minimal element in $\mathcal{E}\left((\mu - \lambda')m_0 + m_{r_u}\right)$. Then, in view of [4; Lemma 2.6 (3)] we may assume that $\alpha = a\mathbf{e}_0 + \gamma_i + b\mathbf{e}_{p+1}$. From $vn = \mu m_0$ and $(\mu - \lambda')m_0 + m_{r_u} = am_0 + g_i + bn$, it follows that $b < v$ and $\mu - \lambda' \geq a$. Suppose that $\mu - \lambda' > a$. Then $(\mu - \lambda' - a)m_0 + m_{r_u} = g_i + bn$ and so $i \geq u$, since $(r_u, 0) \in U$ and $z = 0$. Therefore $(v - b)n = (\lambda' + a - \varepsilon)m_0 + g_{i-r_u}$ where $\varepsilon \in \{0, 1\}$ be such that $\varepsilon m_0 + g_i = g_{r_u} + g_{i-r_u}$. Now, from the definition of v, it follows that $b = 0$ and $(\mu - \lambda' - a + \varepsilon)m_0 = g_{i-r_u}$. Therefore $vn = \mu m_0 = (\lambda' + a - \varepsilon)m_0 + g_{i-r_u}$. Now, from the definition of z', it follows that $i = r_u$, a contradiction, since $i \geq u > r_u$. Therefore $\mu - \lambda' = a$. $\square$

3.8. Lemma. *Let $s, i \in \mathbb{N}$. Then there exists a unique integer $b_s \in [0, s-1]$ such that $sg_i = b_s m_0 + g_{si}$ and $q_{si} = sq_i + (s - 1 - b_s)$.*

Proof. Use [4; Lemma 2.6] and induction on s. $\square$

3.9. Lemma. *Suppose that $w = 0$. Then for every $i \in \mathbb{N}$ there exists $j \in [0, u-1]$ such that $g_i \equiv g_j (\mathrm{mod}\ m_0)$.*

Proof. Let $i \in \mathbb{N}$ and let $k \in \mathbb{N}, j \in [0, u-1]$ be the unique integers defined by the equation $i = ku + j$. Now, since $w = 0$ and $g_u \equiv 0 (\mathrm{mod}\ m_0)$, we have $g_i = g_{ku+j} \equiv kg_u + g_j \equiv g_j (\mathrm{mod}\ m_0)$. $\square$

3.10. Lemma. *Let $a, a' \in \mathbb{N}$ and $b, b' \in [0, v-1]$ be such that $a'\mathbf{e}_0 + b'\mathbf{e}_{p+1} \in \mathcal{E}(am_0 + bn)$. Then $a = a'$ and $b = b'$.*

Proof. Since $bn \equiv am_0 + bn = a'm_0 + b'n \equiv b'n (\mathrm{mod}\ m_0)$ and $(0, b), (0, b') \in U$, we have $b = b'$ by the Key-Lemma. $\square$

3.11. Lemma. *Let $a, a' \in [0, \mu - 1]$ and $b, b' \in \mathbb{N}$ be such that $am_0 + bn = a'm_0 + b'n$. Then $a = a'$ and $b = b'$.*

Proof. If $b < b'$ then $a > a'$ and $(a - a')m_0 = (b' - b)n$. Therefore $b' - b \geq v$ by the definition of v and so $(a - a')m_0 \geq vn \geq \mu m_0$, a contradiction. $\square$

3.12. Lemma. *Suppose that $m_0 < n < m_p$ and $z' = 0$. Let $c, c', b, b' \in \mathbb{N}$ and $i \in [1, u-1]$ be such that $c'm_0 + b'n = cm_0 + g_i + bn$. Then $c < c'$, $b' < b$ and $c' - c > q_i + 1 + b - b'$.*

Proof. Since $z' = 0$, $z = 0$ and $\mu' = \mu$ by 3.5 (1). Suppose that $c \geq c'$. Then from $vn = \mu m_0$ and $c'm_0 + b'n = cm_0 + g_i + bn$, we get $b'_0 n \equiv g_i + b_0 n (\mathrm{mod}\ m_0)$ for some $b'_0, b_0 \in [0, v-1]$. Now, since $z = 0$, both $(0, b'_0)$, $(i, b_0) \in U$ and so $(0, b'_0) = (i, b_0)$, a contradiction. Therefore $c < c'$ and so $b' < b$, since $(i, 0) \in U$. Now, from $m_0 < n < m_p$ and $(c' - c)m_0 = g_i + (b - b')n$, we have $c' - c > q_i + 1 + b - b'$. $\square$

4. The case $n < m_p$

Let $m_0, \dots, m_p, n$ be an almost arithmetic sequence of positive integers with $\gcd(m_0, \dots, m_p, n) = 1$ and $m_0 < \cdots < m_p$. Let Γ be the semigroup generated by $m_0, \dots, m_p, n$ and let $A' := K[Y^{n_e}, \dots, X^{n_i}Y^{n_e - n_i}, \dots, X^{n_e}]$. We assume that $m_0, \dots, m_p, n$ is a minimal set of generators for the semigroup Γ.

We continue using the same notations as in Section 3. In particular, remember the Key-Lemma 3.3 and sometimes giving reference of the Key-Lemma is omitted.

In this section we assume that $n < m_p$.

4.1. Lemma. *Suppose that either $w = 0$ or $\nu \geq q_{u-z} + 1 + (v - w)$. Let $a, i \in \mathbb{N}$, $b \in [0, v-1]$ be such that $\gamma_i + b\mathbf{e}_{p+1} \in \mathcal{E}(am_0)$. Then $a \geq q_i + 1 + b$.*

Proof. If either $n > m_0$ or $b = 0$ then the assertion is clear.

First suppose that $w = 0$. Then by 3.9 there exist $j \in [0, u-1]$ and $a' \in \mathbb{N}$ such that $g_i = a'm_0 + g_j$. Now, from the equation $am_0 = a'm_0 + g_j + bn$ and $(j, b) \in U$ (since $w = 0$), we get $j = 0 = b$ and $a = a'$ by the Key-Lemma. Now, from the equation $am_0 = g_i$, we have $q_i + 1 \leq a$.

Now, assume that $n < m_0$, $b \neq 0$ and $\nu \geq q_{u-z} + 1 + (v - w)$.

Since $g_i + bn = am_0 \equiv 0 (\bmod\ m_0)$, we have $(i, b) \notin U$ and so $i \geq u - z$, since $b < v$. We consider the two cases $b < v - w$ and $b \geq v - w$ separately.

Case $b < v - w$: In this case from the equations $\nu m_0 = g_{u-z} + (v - w)n$, $am_0 = g_i + bn$ and $b \neq 0$, we get $(a + \varepsilon - \nu)m_0 + (v - w - b)n = g_{i-u+z}$ where $\varepsilon \in \{0, 1\}$ be such that $\varepsilon m_0 + g_i = g_{u-z} + g_{i-u+z}$. Therefore $a + \varepsilon - \nu + (v - w - b) \geq q_{i-u+z} + 1$, since $n, m_0 < m_p$.

Case $b \geq v - w$: In this case we have $(a + \varepsilon - \nu)m_0 = g_{i-u+z} + (b - v + w)n$ where $\varepsilon \in \{0, 1\}$ be such that $\varepsilon m_0 + g_i = g_{u-z} + g_{i-u+z}$. Now, since $b - v + w < b$, it follows by the case $b < v - w$ and induction on b that $a + \varepsilon - \nu \geq q_{i-u+z} + 1 + (b - v + w)$.
In any case, since $\nu \geq q_{u-z} + 1 + (v - w)$ by assumption and $\varepsilon + q_i = q_{i-u+z} + q_{u-z} + 1$ by [4; Lemma 2.6 (2)], we have $a \geq b + q_i + 1$. $\square$

4.2. Lemma. *Suppose that $v \geq \mu' + q_{z'} + 1$. Then for every $h \in \Gamma$ and $\alpha \in \mathcal{E}(h)$, there exist $a, i \in \mathbb{N}$ and $b \in [0, v-1]$ such that $a\mathbf{e}_0 + \gamma_i + b\mathbf{e}_{p+1} \in \mathcal{E}(h)$ and $a + q_i + 1 + b \leq \deg(\alpha)$.*

Proof. By [4; Lemma 2.6 (3)] there exist $a', b', j \in \mathbb{N}$ such that $\deg(\alpha) = a' + q_j + 1 + b'$. If $b' < v$ then there is nothing to prove. Suppose that $b' \geq v$. Then, since $\varepsilon\mathbf{e}_0 + \gamma_{j+z'} = \gamma_j + \gamma_{z'}$ with $\varepsilon \in \{0, 1\}$, we have $\deg(\beta) \leq \deg(\alpha)$, where $\beta := (a' + \mu' + \varepsilon)\mathbf{e}_0 + \gamma_{j+z'} + (b' - v)\mathbf{e}_{p+1} \in \mathcal{E}(h)$. If $b' - v \geq v$ then repeating we get $b \in [0, v-1]$ such that $a\mathbf{e}_0 + \gamma_i + b\mathbf{e}_{p+1} \in \mathcal{E}(h)$ and $a + q_i + 1 + b = \deg(a\mathbf{e}_0 + \gamma_i + b\mathbf{e}_{p+1}) \leq \deg(\alpha)$ for some $a, i \in \mathbb{N}$. $\square$

4.3. Lemma. *Suppose that $m_0 < n < m_p$ and $z' = 0$. Then for every $h \in \Gamma$ and $\alpha \in \mathcal{E}(h)$, there exist $a \in [0, \mu - 1]$ and $i, b \in \mathbb{N}$ such that $a + q_i + 1 + b \leq \deg(\alpha)$.*

Proof. Since $z' = 0$, $z = 0$ and $\mu' = \mu$. By [4; Lemma 2.6 (3)] there exist $a', b', j \in \mathbb{N}$ such that $\deg(\alpha) = a' + q_j + 1 + b'$. Let $a_1 \in \mathbb{N}$ and $a_0 \in [0, \mu - 1]$ be the unique integers defind by the equation $a' = a_1\mu + a_0$. Since $n < m_p$ and $vn = \mu m_0$, we have $v < \mu$. Now, the assertion is clear by taking $a := a_0$, $b := (a_1 v + b')$ and $i := j$. □

4.4. Definition. *For $h \in \Gamma$, let*

$$\mathcal{F}(h) := \begin{cases} \{a\mathbf{e}_0 + \gamma_i + b\mathbf{e}_{p+1} \in \mathcal{E}(h) \mid b \in [0, v-1]\}, & \text{if either } n < m_0 \text{ or } z' \neq 0, \\[2ex] \{a\mathbf{e}_0 + \gamma_i + b\mathbf{e}_{p+1} \in \mathcal{E}(h) \mid a \in [0, \mu-1]\}, & \text{if } m_0 < n < m_p \text{ and } z' = 0. \end{cases}$$

and let

$$i(h) := \max\{i \in \mathbb{N} \mid a\mathbf{e}_0 + \gamma_i + b\mathbf{e}_{p+1} \in \mathcal{F}(h)\}.$$

4.5. Proposition. *Let $h \in \Gamma$. Then*

(1) *The set $\mathcal{F}(h)$ is finite and non-empty. In particular, $i(h)$ exists.*

(2) *There exists a unique pair $(a(h), b(h))$ such that*

$$a(h)\mathbf{e}_0 + \gamma_{i(h)} + b(h)\mathbf{e}_{p+1} \in \mathcal{F}(h)\,.$$

(3) *Let $\delta(h) := a(h)\mathbf{e}_0 + \gamma_{i(h)} + b(h)\mathbf{e}_{p+1} \in \mathcal{F}(h)$. Then $i(h+m_p) = i(h) + p$ and $\delta(h+m_p) = \delta(h) + \gamma_p$.*

Proof. (1) Since $\mathcal{F}(h) \subseteq \mathcal{E}(h)$ and $\mathcal{E}(h)$ is finite, $\mathcal{F}(h)$ is finite. Further, $\mathcal{F}(h) \neq \emptyset$ by

$$\begin{cases} 3.3, & \text{if either } n < m_0 \text{ or } z' \neq 0, \\ 4.3, & \text{if } m_0 < n < m_p \text{ and } z' = 0. \end{cases}$$

(2) is immediate from $\begin{cases} 3.10, & \text{if either } n < m_0 \text{ or } z' \neq 0, \\ 3.11, & \text{if } m_0 < n < m_p \text{ and } z' = 0. \end{cases}$

(3) Let $\alpha = a\mathbf{e}_0 + \gamma_{i(h)} + b\mathbf{e}_{p+1} \in \mathcal{F}(h)$. Then $\alpha + \gamma_p \in \mathcal{F}(h + m_p)$ and so $i(h+m_p) \geq i(h) + p$. Let $\beta = a'\mathbf{e}_0 + \gamma_{i'} + b\mathbf{e}_{p+1} \in \mathcal{F}(h+m_p)$. If $i' \geq i(h) + p$ then $\alpha' := \beta - \gamma_p = a'\mathbf{e}_0 + \gamma_{i'-p} + b'\mathbf{e}_{p+1} \in \mathcal{F}(h)$ and so $i' - p \leq i(h)$ by definition of $i(h)$. □

4.6. Proposition. *Suppose that $v \geq \mu' + q_{z'} + 1$ and either $w = 0$ or $\nu \geq q_{u-z} + 1 + v - w$. Let $h \in \Gamma$ and $\alpha' := a'\mathbf{e}_0 + \gamma_{i'} + b'\mathbf{e}_{p+1} \in \mathcal{E}(h)$. Then*

(1) $\deg(\delta(h)) \leq \deg(\alpha')$,

(2) $\operatorname{mindeg}(h) = \deg(\delta(h))$,

(3) $\operatorname{mindeg}(h + tm_p) = \operatorname{mindeg}(h) + t$ *for every* $t \in \mathbb{N}$.

Proof. (1) Since $v \geq \mu' + q_{z'} + 1$, we have $z' \neq 0$ if $m_0 < n$. In view of 4.2 we may assume that $b' \in [0, v-1]$. Then $\alpha' \in \mathcal{F}(h)$. Put $i := i(h), a := a(h)$ and $b := b(h)$. Then $i \geq i'$ by definition of $i(h)$. If $i = i'$ then $a' = a$ and $b' = b$ by 4.5 (2) and there is nothing to prove. Therefore assume that $i > i'$. Then $am_0 + g_{i-i'} + bn = (a' + \varepsilon)m_0 + b'n$ where $\varepsilon \in \{0, 1\}$ be such that $\varepsilon m_0 + g_i = g_{i'} + g_{i-i'}$. Now, since $(0, b') \in U$, $a < a' + \varepsilon$ and so

$$g_{i-i'} + bn = (a' + \varepsilon - a)m_0 + b'n\,. \tag{4.6.a}$$

We consider the two cases $b < b'$ and $b \geq b'$ separately.

Case $b < b'$: In this case from the equation (4.6.a), we get

$$(a' + \varepsilon - a)m_0 + (b' - b)n = g_{i-i'}$$

with $a' + \varepsilon - a > 0$ and so $i - i' \geq u > p$. Therefore, since $n < m_p$, we have

$$q_{i-i'} + 1 \leq (a' + \varepsilon - a) + (b' - b)\,. \tag{4.6.b}$$

Case $b \geq b'$: In this case from the equation (4.6.a), we get $(a' + \varepsilon - a)m_0 = g_{i-i'} + (b - b')n$. Therefore by 4.1 we have

$$q_{i-i'} + 1 + (b - b') \leq (a' + \varepsilon - a)\,. \tag{4.6.c}$$

Now, in any case, use [4; Lemma 2.6 (2)] to conclude that $\deg(\delta(h)) = a + q_i + 1 + b \leq a' + q_{i'} + 1 + b' = \deg(\alpha')$.

(2) Follows from [4; Lemma 2.6 (3)] and (1).

(3) Since $\delta(h + m_p) = \delta(h) + \gamma_p$, the assertion follows from (2) by induction on t. □

4.7. Proposition. *Suppose that* $m_0 < n < m_p$ *and* $z' = 0$. *Let* $\alpha' := a'\mathbf{e}_0 + \gamma_{i'} + b'\mathbf{e}_{p+1} \in \mathcal{E}(h)$, $h \in \Gamma$. *Then:*

(1) *If* $i(h) - i' \notin \mathbb{N}u$ *then* $\deg(\delta(h)) < \deg(\alpha')$.

(2) *If* $i(h) - i' \in \mathbb{N}u$ *and* $v + w' \geq \mu - \lambda' + q_u + 1$ *then* $\deg(\delta(h)) \leq \deg(\alpha')$.

(3) *If* $v + w' \geq \mu - \lambda' + q_u + 1$ *then* $\operatorname{mindeg}(h) = \deg(\delta(h))$.

(4) *If* $v + w' \geq \mu - \lambda' + q_u + 1$ *then* $\operatorname{mindeg}(h + tm_p) = \operatorname{mindeg}(h) + t$ *for every* $t \in \mathbb{N}$.

Proof. Since $z' = 0$, $z = 0$ and $\mu' = \mu$ by 3.5 (1). In view of 4.3 we may assume that $a' < \mu$. Then $\alpha' \in \mathcal{F}(h)$. Put $i := i(h), a := a(h)$ and $b := b(h)$. Then $i \geq i'$ by definition of $i(h)$. If $i = i'$ then $a' = a$ and $b' = b$ by 4.5 (2) and there is nothing to prove. Therefore assume that $i > i'$. Then $a'm_0 + g_{i'} + b'n = h = am_0 + g_i + bn$ and so

$$(a' + \varepsilon)m_0 + b'n = am_0 + g_{i-i'} + bn \tag{4.7.a}$$

where $\varepsilon \in \{0,1\}$ be such that

$$\varepsilon m_0 + g_i = g_{i'} + g_{i-i'}\,. \tag{4.7.b}$$

Let $b_1, b_1' \in \mathbb{N}$ and $b_0, b_0' \in [0, v-1]$ be the unique integers defined by the equations $b = b_1 v + b_0$ and $b' = b_1' v + b_0'$. Then from the equations $vn = \mu m_0$ and (4.7.a), we get $b_0' n \equiv g_{i-i'} + b_0 n (\text{mod } m_0)$. Now, since $(0, b_0') \in U$ and $z = 0$, it follows from the Key-Lemma that $i - i' \geq u$.

Put $j := i - i'$ and let $j_1 \in \mathbb{N}$ and $j_0 \in [0, u-1]$ be the unique integers defined by the equation $j = j_1 u + j_0$. Then $j_1 g_u + g_{j_0} = (b_{j_1} + \varepsilon') m_0 + g_j$ where $b_{j_1} \in [0, j_1 - 1]$ and $\varepsilon' \in \{0,1\}$ with

$$\varepsilon' m_0 + g_j = g_{j_0} + g_{j_1 u}\,. \tag{4.7.c}$$

With this from the equations (4.7.a) and $g_u = \lambda' m_0 + w' n$, we get

$$(a' + \varepsilon + \varepsilon' + b_{j_1}) m_0 + b' n = (a + j_1 \lambda') m_0 + g_{j_0} + (b + j_1 w') n\,. \tag{4.7.d}$$

(1) Since $i - i' \notin \mathbb{N}u$, we have $j_0 \neq 0$. Therefore from the equation (4.7.d) and 3.12, we get $(a' + \varepsilon + \varepsilon' + b_{j_1}) - (a + j_1 \lambda') > q_{j_0} + 1 + (b + j_1 w') - b'$. Now, by using the equations (4.7.b), (4.7.c), 3.6 (3), 3.8 (applied to $s = j_1$ and $i = u$) and [4; Lemma 2.6 (2)], it is easy to see that $\deg(\alpha') = a' + b' + q_{i'} + 1 > a + b + q_i + 1 = \deg(\delta(h))$.

(2) Since $i - i' \in \mathbb{N}u$, we have $j_0 = 0$, $\varepsilon' = 0$ and the equation (4.7.d) is

$$c' m_0 + b' n = c m_0 + (b + j_1 w') n \tag{4.7.e}$$

where

$$c' := (a' + \varepsilon + b_{j_1}) \text{ and } c := (a + j_1 \lambda')\,. \tag{4.7.f}$$

We consider two cases $c' \geq c$ and $c' < c$ separately.

Case $c' \geq c$: From the equation (4.7.e), we have $c' - c \geq (b + j_1 w') - b'$ and so $c' + b' \geq c + b + j_1 w'$. Therefore by 3.6 (3), [4; Lemma 2.6 (2)] and 3.8 (applied to $s = j_1$ and $i = u$), we have $\deg(\delta(h)) = a + q_i + 1 + b \leq a' + q_{i'} + 1 + b' = \deg(\alpha')$.

Case $c' < c$: Let $c_1, c_1' \in \mathbb{N}$ and $c_0, c_0' \in [0, \mu - 1]$ be the unique integers defined by the equations

$$c = c_1 \mu + c_0 \text{ and } c' = c_1' \mu + c_0'\,. \tag{4.7.g}$$

Then from the equations $vn = \mu m_0$ and (4.7.e), we get $c_0' m_0 + (c_1' v + b') n = c_0 m_0 + (c_1 v + b + j_1 w') n$. Therefore by 3.11 we have

$$c_0' = c_0 \text{ and } c_1' v + b' = c_1 v + b + j_1 w' \tag{4.7.h}$$

and so from (4.7.f), (4.7.g) and (4.7.h), we have

$$a = (c_1 - c_1')\mu + a' + \varepsilon + b_{j_1} - j_1 \lambda' \text{ and } b = b' - (c_1 - c_1') v - j_1 w'\,. \tag{4.7.i}$$

Further, since $\lambda' \leq \mu$ by 3.6 (1) and $a \leq \mu$, we have $(c_1 - c_1')\mu = c - c' \leq c = a + j_1\lambda' \leq a + j_1\mu < (j_1+1)\mu$ and so

$$(c_1 - c_1') \leq j_1 \,. \tag{4.7.j}$$

Now, from the equations (4.7.i), (4.7.j), (4.7.b), (4.7.c), [4; Lemma 2.6 (2)], 3.8 (applied to $s = j_1$ and $i = u$) and the assumption $v + w' \geq \mu - \lambda' + q_u + 1$, it is easy to see that $\deg(\alpha') = a' + b' + q_{i'} + 1 \geq a + b + q_i + 1 = \deg(\delta(h))$.

(3) is immediate from (1) and (2).

(4) Since $\delta(h + m_p) = \delta(h) + \gamma_p$, the assertion follows from (3) by induction on t. □

4.8. Theorem.

(1) *If $n < m_0$ and $w = 0$ then A' is Cohen-Macaulay.*

(2) *In each of the following cases*

(a) $n < m_0$ *and* $w \neq 0$,

(b) $m_0 < n < m_p$ *and* $z' \geq p$,

A' is Cohen-Macaulay if and only if $\nu \geq q_{u-z} + 1 + (v - w)$ and $v \geq \mu' + q_{z'} + 1$.

(3) *Suppose that $m_0 < n < m_p$ and $1 \leq z' < p$. Then*

(a) *If A' is Cohen-Macaulay then $v + 1 \geq \operatorname{mindeg}((\mu'+1)m_0) + 1$.*

(b) *If $v \geq \mu' + 1$ then A' is Cohen-Macaulay.*

(4) *If $m_0 < n < m_p$ and $z' = 0$ then A' is Cohen-Macaulay if and only if $v + w' \geq \mu - \lambda' + q_u + 1$.*

Proof. (1) Since $n < m_0$ and $vn = \mu' m_0 + g_{z'}$, $v \geq \mu' + q_{z'} + 1$. Therefore the assertion follows from 4.6 (3) and 2.3.

(2) First assume that $n < m_0$ and $w \neq 0$. Then $v \geq \mu' + q_{z'} + 1$. If $\nu \geq q_{u-z} + 1 + (v-w)$ then A' is Cohen-Macaulay by 4.6 (3) and 2.3. Conversely, suppose that A' is Cohen-Macaulay. Then, since $(q_{u-z}+1)\mathbf{e}_p + (v-w)\mathbf{e}_{p+1} \in \mathcal{E}\left((\nu-1)m_0 + m_{p-r_{u-z}}\right)$ and $\operatorname{mindeg}((v-w)n) = v - w$ (by definition of v), by 2.3 we have

$$\begin{aligned}(\nu - 1) + 1 &\geq \operatorname{mindeg}((q_{u-z}+1)m_p + (v-w)n) \\ &= \operatorname{mindeg}((v-w)n) + q_{u-z} + 1 = (v-w) + q_{u-z} + 1.\end{aligned}$$

Now, assume that $m_0 < n < m_p$ and $z' \geq p$. Then $\nu \geq q_{u-z} + 1 + (v-w)$. If $v \geq \mu' + q_{z'} + 1$ then A' is Cohen-Macaulay by 4.6 (3) and 2.3. Conversely, suppose that A' is Cohen-Macaulay. Then from the equation $vn = \mu' m_0 + g_{z'} = \mu' m_0 + m_{r_{z'}} + q_{z'} m_p$ and 2.3, we get

$$v \geq \begin{cases} \operatorname{mindeg}(\mu' m_0 + m_{r_{z'}}) + q_{z'} & \text{if } z' > p, \\ \operatorname{mindeg}(\mu' m_0) + 1 & \text{if } z' = p. \end{cases}$$

Therefore $v \geq \mu' + q_{z'} + 1$ by 3.7 (1) and 3.7 (2).

(3) (a) Follows from $vn + m_{p-z'} = \mu' m_0 + m_{r_{z'}} + m_{p-z'} = (\mu' + 1)m_0 + m_p$ and 2.3.

(b) Since $1 \leq z' < p$, $q_{z'} = 0$ and so $v \geq \mu' + 1 = \mu' + q_{z'} + 1$ by assumption. Moreover, $\nu \geq q_{u-z} + 1 + (v - w)$, since $m_0 < n < m_p$. Therefore A' is Cohen-Macaulay by 4.6 (3) and 2.3.

(4) Since $m_0 < n < m_p$ and $\nu m_0 = g_{u-z} + (v - w)n$, $\nu \geq q_{u-z} + 1 + (v - w)$. Further, since $z' = 0$, $z = 0$ and $\mu' = \mu$ by 3.5 (1). If $v + w' \geq \mu - \lambda' + q_u + 1$ then A' is Cohen-Macaulay by 4.7 (3) and 2.3. Conversely, suppose that A' is Cohen-Macaulay. Then from the equation $(v + w')n = (\mu - \lambda')m_0 + g_u$ and 2.3, we get $v + w' \geq \operatorname{mindeg}((\mu - \lambda')m_0 + m_{r_u}) + q_u = \mu - \lambda' + q_u + 1$ by 3.7 (3)(b). □

4.9. Remark. The Theorem 4.8 gives a numerical criterion for the Cohen-Macaulayness of A' in the following cases:

(1) $n < m_0$ (see 4.8 (1) and 4.8 (2)).

(2) $m_0 < n < m_p$ and either $z' \geq p$ or $z' = 0$ (see 4.8 (2) and 4.8 (4)).

In the case $m_0 < n < m_p$ and $1 \leq z' < p$ partial results are given in 4.8 (3). These results are not complete, the necessary condition given in 4.8 (3)(a) is not sufficient and the sufficient condition given in 4.8 (3)(b) is not necessary, as is demonstrated in Section 7 by examples (see 7.3 (3) and 7.3 (4)).

4.10. Lemma. *Suppose that $m_0 < n < m_p$. Let $\varepsilon \in \{0, 1\}$ be such that $\varepsilon m_0 + g_{u+z'} = g_u + g_{z'}$. Then:*

(1) $(v + w')n = (\mu' - \lambda' + \varepsilon)m_0 + g_{u+z'}$.

(2) *If $(\mu' - \lambda' + \varepsilon) \leq 0$ then $(v + w') - (\mu' - \lambda' + \varepsilon) \geq q_{u+z'} + 1$.*

(3) *If $0 \leq (\mu' - \lambda' + \varepsilon) \leq \mu'$ then* $\operatorname{mindeg}((\mu' - \lambda' + \varepsilon)m_0) = (\mu' - \lambda' + \varepsilon)$.

(4) *If $(\mu' - \lambda' + \varepsilon) \geq 0$ and A' is Cohen-Macaulay then*

$$(v + w') \geq \operatorname{mindeg}\Big((\mu' - \lambda' + \varepsilon + m_{r_{u+z'}})\Big) + q_{u+z'} .$$

Proof. (1) Follows from the equations $vn = \mu' m_0 + g_{z'}$ and $g_u = \lambda' m_0 + w'n$.

(2) Since $n < m_p$ and $u + z' \geq u > p$, the assertion follows from (1).

(3) Let α be a minimal element in $\mathcal{E}((\mu' - \lambda' + \varepsilon)m_0)$. By [4; Lemma 2.6 (3)] we may assume that $\alpha = a\mathbf{e}_0 + \gamma_i + b\mathbf{e}_{p+1}$. From the equations $vn = \mu' m_0 + g_{z'}$, $g_u = \lambda' m_0 + w'n$ and $am_0 + g_i + bn = (\mu' - \lambda' + \varepsilon)m_0$, we have

$$vn = (\lambda' - \varepsilon + a + \varepsilon')m_0 + g_{i+z'} + bn \qquad (4.10.a)$$

where $\varepsilon' \in \{0, 1\}$ be such that $\varepsilon' m_0 + g_{i+z'} = g_i + g_{z'}$. Now, since $(\mu' - \lambda' + \varepsilon) < \mu'$, from the definitions of v, z' and the equation (4.10.a), it is easy to see that $b = 0$, $i = 0$, $\varepsilon' = 0$ and $a = (\mu' - \lambda' + \varepsilon)$. Therefore $\operatorname{mindeg}((\mu' - \lambda' + \varepsilon)m_0) = \deg(\alpha) = a = (\mu' - \lambda' + \varepsilon)$.

(4) In view of (2) we may assume that $(\mu' - \lambda' + \varepsilon) \geq 1$. Now, since $m_0 < n < m_p$, the assertion follows from the equation $(v + w')n = (\mu' - \lambda' + \varepsilon)m_0 + g_{u+z'}$, 2.3 and (3). $\square$

4.11. Question. Suppose that $m_0 < n < m_p$ and $z' \in [1, p-1]$. Let $\varepsilon \in \{0, 1\}$ be such that $\varepsilon m_0 + g_{u+z'} = g_u + g_{z'}$. Then:

(a) If $\mu' - \lambda' + \varepsilon \leq 0$ is then A' always Cohen-Macaulay?

(b) If $\mu' - \lambda' + \varepsilon \geq 0$ and $(v + w') \geq \operatorname{mindeg}\Big((\mu' - \lambda' + \varepsilon + m_{r_{u+z'}})\Big) + q_{u+z'}$ is then A' always Cohen-Macaulay?

5. The case $m_p < n$ and $w' \neq 0$

In this section we continue using the same notations as in Section 3 and assume that $m_p < n$.

5.1. Notations. We fix the following notations:

For $k \in [0, p - r_u]$, let $\varepsilon_k := \begin{cases} 0 & \text{if } k = 0, \\ 1 & \text{if } k \neq 0, \end{cases}$ and $\delta_k := a_k\mathbf{e}_0 + \gamma_{u_k} + b_k\mathbf{e}_{p+1}$ be a fixed minimal element in $\mathcal{E}\left((\lambda' - \varepsilon_k)m_0 + g_k\right)$. In particular, $\deg(\delta_k) = \operatorname{mindeg}\left((\lambda' - \varepsilon_k)m_0 + g_k\right))$.

5.2. Lemma. *$\delta_k + w'\mathbf{e}_{p+1} \in \mathcal{E}(g_{u+k})$ for every $k \in [0, p - r_u]$. Moreover, if $w' \neq 0$ then $u_k < u + k$ for every $k \in [0, p - r_u]$.*

Proof. Clear from the equations $g_{u+k} = (\lambda' - \varepsilon_k)m_0 + g_k + w'n$, $\partial(\delta_k) = (\lambda' - \varepsilon_k)m_0 + g_k$ and $\delta_k = a_k\mathbf{e}_0 + \gamma_{u_k} + b_k\mathbf{e}_{p+1}$. $\square$

5.3. $q_{u+k} + 1 \geq \operatorname{mindeg}((\lambda' - \varepsilon_k)m_0 + g_k) + w'$ for every $k \in [0, p - r_u]$.

5.4. Lemma. *Suppose that $w' \neq 0$ and 5.3 holds. Then for every $h \in \Gamma$ and $\alpha \in \mathcal{E}(h)$, there exist $a, b \in \mathbb{N}$ and $i \in [0, u-1]$ such that $a\mathbf{e}_0 + \gamma_i + b\mathbf{e}_{p+1} \in \mathcal{E}(h)$ and $a + q_i + 1 + b \leq \deg(\alpha)$.*

Proof. By [4; Lemma 2.6 (3)] there exist $a', b', j \in \mathbb{N}$ such that $\deg(\alpha) = a' + q_j + 1 + b'$. If $j < u$ there is nothing to prove. Therefore assume that $j \geq u$. Let $k := \max\{0, r_j - r_u\}$. Then $k \in [0, p - r_u]$ and it is easy (by using Key-Lemma) to see that $g_j = g_{u+k} + g_{j-u-k}$. Let

$$\beta := (a' + a_k + \varepsilon)\mathbf{e}_0 + \gamma_{j-u-k+u_k} + (b' + b_k + w')\mathbf{e}_{p+1}$$

where $\varepsilon \in \{0, 1\}$ be such that $\varepsilon\mathbf{e}_0 + \gamma_{j-u-k+u_k} = \gamma_{u_k} + \gamma_{j-u-k}$. Using 5.2 and the assumption 5.3, it is easy to see that $\beta \in \mathcal{E}(h)$, $j-u-k+u_k < j$ and $\deg(\beta) \leq \deg(\alpha)$. If $j-u-k+u_k < u$ there is nothing to prove. If $j - u - k + u_k \geq u$ then repeating we get $i \in [0, u-1]$ such that $a\mathbf{e}_0 + \gamma_i + b\mathbf{e}_{p+1} \in \mathcal{E}(h)$ and $a + q_i + 1 + b = \deg(a\mathbf{e}_0 + \gamma_i + b\mathbf{e}_{p+1}) \leq \deg(\alpha)$. $\square$

5.5. Definition. *For $h \in \Gamma$, let*

$$\mathcal{G}(h) := \{a\mathbf{e}_0 + \gamma_i + b\mathbf{e}_{p+1} \in \mathcal{E}(h) \mid i \in [0, u-1]\}$$

and let

$$b(h) := \max\{b \in \mathbb{N} \mid a\mathbf{e}_0 + \gamma_i + b\mathbf{e}_{p+1} \in \mathcal{G}(h)\}\,.$$

5.6. Proposition. *Let $h \in \Gamma$. Then:*

(1) *The set $\mathcal{G}(h)$ is a finite and non-empty. In particular, $b(h)$ exists.*

(2) *There exists a unique pair $(a(h), i(h))$ such that*

$$a(h)\mathbf{e}_0 + \gamma_{i(h)} + b(h)\mathbf{e}_{p+1} \in \mathcal{G}(h)\,.$$

(3) *Let $\delta(h) := a(h)\mathbf{e}_0 + \gamma_{i(h)} + b(h)\mathbf{e}_{p+1} \in \mathcal{G}(h)$. Then $b(h+n) = b(h)+1$ and $\delta(h+n) = \delta(h) + \mathbf{e}_{p+1}$.*

Proof. (1) Since $\mathcal{G}(h) \subseteq \mathcal{E}(h)$ and $\mathcal{E}(h)$ is finite, $\mathcal{G}(h)$ is finite. Further, $\mathcal{G}(h) \neq \emptyset$ by the Key-Lemma.

(2) is immediate from the Key-Lemma.

(3) Let $\alpha = a\mathbf{e}_0 + \gamma_i + b(h)\mathbf{e}_{p+1} \in \mathcal{G}(h)$. Then $\alpha + \mathbf{e}_{p+1} \in \mathcal{G}(h+n)$ and so $b(h+n) \geq b(h)+1$. Let $\beta = a'\mathbf{e}_0 + \gamma_{i'} + b'\mathbf{e}_{p+1} \in \mathcal{G}(h+n)$. If $b' \geq b(h)+1$ then $\alpha' := \beta - \mathbf{e}_{p+1} = a'\mathbf{e}_0 + \gamma_{i'} + (b'-1)\mathbf{e}_{p+1} \in \mathcal{G}(h)$ and so $b'-1 \leq b(h)$ by definition of $b(h)$. □

5.7. Proposition. *Suppose that $w' \neq 0$ and 5.3 holds. Let $\alpha' := a'\mathbf{e}_0 + \gamma_{i'} + b'\mathbf{e}_{p+1} \in \mathcal{E}(h)$, $h \in \Gamma$. Then*

(1) $\deg(\delta(h)) \leq \deg(\alpha')$,

(2) $\mathrm{mindeg}(h) = \deg(\delta(h))$,

(3) $\mathrm{mindeg}(h+tn) = \mathrm{mindeg}(h) + t$ *for every* $t \in \mathbb{N}$.

Proof. (1) In view of 5.4 we may assume that $i' \in [0, u-1]$. Then $\alpha' \in \mathcal{G}(h)$. Put $a := a(h), i := i(h)$ and $b := b(h)$. Then $b \geq b'$ by definition of $b(h)$. If $b = b'$ then $i = i'$ and $a = a'$ by 5.6 (2) and there is nothing to prove. Therefore assume that $b > b'$. Then $a'm_0 + g_{i'} = am_0 + g_i + (b-b')n$. We consider the cases $i' \leq i$ and $i' > i$ separately.

Case $i \leq i'$: In this case $(a'-a-\varepsilon)m_0 + g_{i'-i} = (b-b')n$ where $\varepsilon \in \{0,1\}$ be such that $\varepsilon m_0 + g_{i'} = g_i + g_{i'-i}$. Therefore $a'-a-\varepsilon+q_{i'-i}+1 \geq (b-b')$, since $n > m_p > m_0$.

Case $i > i'$: In this case $(a'-a+\varepsilon)m_0 = g_{i-i'} + (b-b')n$ where $\varepsilon \in \{0,1\}$ be such that $\varepsilon m_0 + g_i = g_{i'} + g_{i-i'}$. Therefore $a'-a+\varepsilon \geq q_{i-i'} + 1 + (b-b')$, since $n > m_p > m_0$.

Now, in any case, use [4; Lemma 2.6 (2)] to conclude that $\deg(\delta(h)) = a + q_i + 1 + b \leq a' + q_{i'} + 1 + b' = \deg(\alpha')$.

(2) Follows from [4; Lemma 2.6 (3)] and (1).

(3) Since $\delta(h+n) = \delta(h) + \mathbf{e}_{p+1}$, the assertion follows from (2) by induction on t. □

5.8. Theorem. *Suppose that $m_p < n$ and $w' \neq 0$. Then A' is Cohen-Macaulay if and only if 5.3 holds.*

Proof. If 5.3 holds then A' is Cohen-Macaulay by 5.7 (3) and 2.3. Conversely, suppose that A' is Cohen-Macaulay. Then by 2.3, we have

$$\begin{aligned} q_{u+k}+1=\operatorname{mindeg}(g_{u+k}) &= \operatorname{mindeg}\left((\lambda'-\varepsilon_k)m_0+g_k+w'n\right) \\ &= \operatorname{mindeg}\left((\lambda'-\varepsilon_k)m_0+g_k\right)+w' \end{aligned}$$

for every $k \in [0, p-r_u]$. □

6. The case $m_p < n$ and $w' = 0$

In this section we continue using the same notations as in Section 3 and assume that $m_p < n$ and $w' = 0$. Therefore $g_u = \lambda' m_0$.

6.1. Lemma. *For every $h \in \Gamma$ and $\alpha \in \mathcal{E}(h)$, there exist $a \in [0, \lambda'-1]$ and $i, b \in \mathbb{N}$ such that $a\mathbf{e}_0 + \gamma_i + b\mathbf{e}_{p+1} \in \mathcal{E}(h)$ and $a + q_i + 1 + b \le \deg(\alpha)$.*

Proof. Let $h \in \Gamma$ and let $\alpha \in \mathcal{E}(h)$. Then by [4; Lemma 2.6 (3)] there exist $a', j, b' \in \mathbb{N}$ such that $a'\mathbf{e}_0 + \gamma_j + b'\mathbf{e}_{p+1} \in \mathcal{E}(h)$ and $\deg(\alpha) = a' + q_j + 1 + b'$. If $a' < \lambda'$ then there is nothing to prove. Suppose that $a' \ge \lambda'$. Put $\beta := (a' - \lambda' + \varepsilon)\mathbf{e}_0 + \gamma_{u+j} + b'\mathbf{e}_{p+1}$ where $\varepsilon \in \{0,1\}$ be such that $\varepsilon m_0 + g_{u+j} = g_u + g_j$. Then $\beta \in \mathcal{E}(h)$. Further, since $g_u = \lambda' m_0$, we have $\lambda' \ge q_u + 1 \ge 2$ and so $\deg(\beta) \le \deg(\alpha)$ and $a' - \lambda' + \varepsilon < a'$. If $a' - \lambda' + \varepsilon \ge \lambda'$ then repeating we get $a \in [0, \lambda'-1]$ such that $a\mathbf{e}_0 + \gamma_i + b\mathbf{e}_{p+1} \in \mathcal{E}(h)$ and $a + q_i + 1 + b \le \deg(\alpha)$ for some $i, b \in \mathbb{N}$. □

6.2. Definition. *For $h \in \Gamma$, let*

$$\mathcal{G}(h) := \{(a,i,b) \in \mathbb{N}^3 \mid a \in [0, \lambda'-1], a\mathbf{e}_0 + \gamma_i + b\mathbf{e}_{p+1} \in \mathcal{E}(h)\}.$$

Then $\mathcal{G}(h) \ne \emptyset$ by 6.1. Order $\mathcal{G}(h)$ by the lexicographic order from right to left and let $\delta(h) := a(h)\mathbf{e}_0 + \gamma_{i(h)} + b(h)\mathbf{e}_{p+1}$ denote the maximal element in $\mathcal{G}(h)$.

6.3. Lemma. *Let $h \in \Gamma$ and let $\delta(h)$ be as in the Definition 6.2. Put $a_0 := a(h)$, $i_0 := i(h)$ and $b_0 := b(h)$.*

(1) *If $a \in [0, \lambda'-1]$ and $a'\mathbf{e}_0 + \gamma_{i'} + b'\mathbf{e}_{p+1} \in \mathcal{E}(am_0)$ with $a' \in [0, \lambda'-1]$ then either $a' = a$, $i' = 0 = b'$, or $b' > 0$.*

(2) *$a_0 \in [0, \lambda'-1]$ and $\operatorname{mindeg}(am_0) = a$ for every $a \in [0, a_0]$.*

(3) *If $a \in [0, a_0]$ then $\operatorname{mindeg}(am_0 + g_{i_0}) = a + q_{i_0} + 1$.*

Proof. (1) We need only to prove that if $b' = 0$ then $a' = a$. Suppose that $a' < a$. Then $i' > 0$ and $(a - a')m_0 = g_{i'}$. Therefore $i' \ge u$ by the Key-Lemma and so $(a-a')m_0 \ge g_u = \lambda' m_0$. Therefore $a - a' \ge \lambda'$, a contradiction to the assumption $a < \lambda'$.

(2) Follows easily from the definition of $\delta(h)$ and the part (1).

(3) It is enough to prove the assertion for $a = a_0$. Let $\alpha = a'\mathbf{e}_0 + \gamma_{i'} + b'\mathbf{e}_{p+1}$ be a minimal element in $\mathcal{E}(a_0 m_0 + g_{i_0})$. Then $a' < \lambda'$ by 6.1 and $b' = 0$, $i' \le i_0$ by the definition of $\delta(h)$.

Therefore $a_0 \le a'$ and from the equation $a_0m_0+g_{i_0} = a'm_0+g_{i'}$, we get $(a'-a_0+\varepsilon)m_0 = g_{i_0-i'}$ where $\varepsilon \in \{0,1\}$ be such that $\varepsilon m_0 + g_{i_0} = g_{i'} + g_{i_0-i'}$. Therefore, if $i_0 > i'$ then $a' - a_0 + \varepsilon > q_{i_0-i'} + 1$ and so $\deg(\alpha) > \deg(a_0\mathbf{e}_0 + g_{i_0}) \ge \deg(\alpha)$ which is absurd. This proves that $i' = i_0$ and $a' = a_0$. □

6.4. Proposition. *Let $h \in \Gamma$ and let $\delta(h)$ be as in the Definition 6.2. Put $a_0 := a(h)$, $i_0 := i(h)$ and $b_0 := b(h)$. Let $\alpha = a\mathbf{e}_0 + \gamma_i + b\mathbf{e}_{p+1} \in \mathcal{E}(h)$ with $a \in [0, \lambda' - 1]$ be such that either $b = b_0$, or $b < b_0$ and $a_0 \le a$. Then* $\deg(\delta(h)) \le \deg(\alpha)$.

Proof. We have

(6.4.a) $$a_0m_0 + g_{i_0} + b_0n = h = am_0 + g_i + bn\,.$$

First suppose that $b = b_0$. Then from the equation (6.4.a) and the definition of $\delta(h)$, we have either $i = i_0$ and $a = a_0$, or $i < i_0$. Therefore from the equation (6.4.a), we get

(6.4.b) $$g_{i_0-i} = (a - a_0 + \varepsilon)m_0$$

where $\varepsilon \in \{0,1\}$ be such that $\varepsilon m_0 + g_{i_0} = g_i + g_{i_0-i}$. Now, from $n > m_p > m_0$, the equation (6.4.b) and [4; Lemma 2.6 (2)], it is easy to see that $\deg(\delta(h)) = a_0 + q_{i_0} + 1 + b_0 \le a + q_i + 1 + b = \deg(\alpha)$.

Now, suppose that $b < b_0$ and $a \ge a_0$. Then from the equation (6.4.a) we have

(6.4.c) $$g_{i_0} + (b_0 - b)n = (a - a_0)m_0 + g_i\,.$$

We consider then two cases $i_0 \ge i$ and $i_0 < i$ separately.

Case $i_0 \ge i$: In this case from the equation (6.4.c) we have

(6.4.d) $$g_{i_0-i} + (b_0 - b)n = (a - a_0 + \varepsilon)m_0$$

where $\varepsilon \in \{0,1\}$ be such that $\varepsilon m_0 + g_{i_0} = g_i + g_{i_0-i}$. Therefore, since $n > m_p > m_0$, we get

(6.4.e) $$q_{i_0-i} + 1 + (b_0 - b) \le (a - a_0 + \varepsilon)\,.$$

Case $i_0 < i$: In this case from the equation (6.4.c) we have

(6.4.f) $$\varepsilon m_0 + (b_0 - b)n = (a - a_0)m_0 + g_{i-i_0}$$

where $\varepsilon \in \{0,1\}$ be such that $\varepsilon m_0 + g_i = g_{i_0} + g_{i-i_0}$. Therefore, since $n > m_p > m_0$, from the equation (6.4.f), it is easy to see that

(6.4.g) $$b_0 - b \le (a - a_0 - \varepsilon) + q_{i-i_0} + 1\,.$$

Now, in any case, use [4; Lemma 2.6 (2)] and equations (6.4.e), (6.4.g), to conclude that $\deg(\delta(h)) = a_0 + q_{i_0} + 1 + b_0 \le a + q_i + 1 + b = \deg(\alpha)$. □

6.5. Lemma. *Let $h \in \Gamma$ and let $\delta(h)$ be as in the Definition 6.2. Put $a_0 := a(h)$, $i_0 := i(h)$ and $b_0 := b(h)$. Let $\alpha = a\mathbf{e}_0 + \gamma_i + b\mathbf{e}_{p+1} \in \mathcal{E}(h)$ be such that $b < b_0$ and $a < a_0$. Then there exist unique integers $s, t \in \mathbb{N}^+$ such that:*

(1) *$b_0 - b = tv$ and $i - i_0 = tz + su$. In particular, $(a_0 - a)m_0 + g_{i_0} + tvn = g_i$.*

(2) *$\deg(\delta(h)) = \deg(\alpha) + (s\lambda'' - tv'')$, where $\lambda'' := \lambda' - q_u - 1$ and $v'' := \mu + q_z + 1 - v$.*

Proof. (1) From the equation $a_0 m_0 + g_{i_0} + b_0 n = h = am_0 + g_i + bn$ and $b < b_0$, $a < a_0$, it follows that $i_0 < i$ and

$$(a_0 - a + \varepsilon)m_0 + (b_0 - b)n = g_{i-i_0} \tag{6.5.a}$$

where with $\varepsilon \in \{0, 1\}$ be such that

$$\varepsilon m_0 + g_i = g_{i_0} + g_{i-i_0}\,. \tag{6.5.b}$$

Let $t \in \mathbb{N}$ and $v' \in [0, v-1]$ be the unique integers defined by the equation $b_0 - b = tv + v'$. Then from the equations $tvn = tg_z + t\mu m_0$,

$$tg_z = g_{tz} + c_t m_0 \tag{6.5.c}$$

with $c_t \in [0, t-1]$ and (6.5.a), we get

$$(a_0 - a + \varepsilon + \varepsilon' + c_t + t\mu)m_0 + v'n = g_{i-i_0-tz} \tag{6.5.d}$$

where $\varepsilon' \in \{0, 1\}$ be such that

$$\varepsilon' m_0 + g_{i-i_0} = g_{tz} + g_{i-i_0-tz} \tag{6.5.e}$$

and $a_0 - a + \varepsilon + \varepsilon' + c_t + t\mu \geq a_0 - a > 0$. Therefore from the equation (6.5.d), it follows that $i - i_0 - tz \geq u$. Let $s \in \mathbb{N}^+$ and $u' \in [0, u-1]$ be the unique integers defined by the equation $i - i_0 - tz = su + u'$. Then, since $w' = 0$, we have $g_{i-i_0-tz} \equiv sg_u + g_{u'} \equiv g_{u'}$ and so from the equation (6.5.d), it follows that $v'n \equiv g_{u'}$. Therefore, since both $(u', 0)$ and $(0, v') \in U$, we have $u' = 0 = v'$ by the Key-Lemma. This proves that $b_0 - b = tv$ and $i - i_0 - tz = su$.

(2) From $n > m_p > m_0$, $g_u = \lambda' m_0$ and $vn = \mu m_0 + g_z$, it follows that λ'', $v'' \in \mathbb{N}^+$. From the equation (6.5.d) we have

$$\begin{aligned}(a_0 - a + \varepsilon + \varepsilon' + c_t + t\mu)m_0 + v'n &= g_{i-i_0-tz}\\ &= g_{su}\\ &= sg_u - c_s m_0\\ &= (s\lambda' - c_s)m_0.\end{aligned} \tag{6.5.f}$$

where

$$c_s m_0 + g_{su} = sg_u \tag{6.5.g}$$

with $c_s \in [0, s-1]$ and $v' = 0$ by (1). Therefore

$$s\lambda' = a_0 - a + \varepsilon + \varepsilon' + t\mu + c_t + c_s\,. \tag{6.5.h}$$

Now, since

$$(6.5.i) \qquad s\lambda'' - tv'' = s\lambda' - sq_u - s - t\mu - tq_z - t + tv\,,$$

from (6.5.e), (6.5.b), (6.5.g), (6.5.c), (6.5.h), (6.5.i), [4; Lemma 2.6 (2)] and [4; Lemma 3.8], it is easy to see that $\deg(\delta(h)) = a_0 + q_{i_0} + 1 + b_0 = a + q_i + 1 + b + (s\lambda'' - tv'') = \deg(\alpha) + (s\lambda'' - tv'')$. □

6.6. Theorem. *Suppose that $m_0 < m_p < n$ and $w' = 0$. Then the following statements are equivalent :*

(i) *A' is Cohen-Macaulay.*

(ii) $\mathrm{mindeg}(h) = \deg(\delta(h))$ *for every $h \in \Gamma$.*

(iii) *For every $j \in \mathbb{N}$ with $a(g_j) \geq 1$ and $b(g_j) \geq 1$, there exist unique integers $s, t \in \mathbb{N}^+$ such that $j = i(g_j) + tz + su$ and $s\lambda'' \leq tv''$, where $\lambda'' := \lambda' - q_u - 1$ and $v'' := \mu + q_z + 1 - v$.*

Proof. (ii) ⇒ (i): Let $h \in \Gamma$ and let $c \in \mathbb{N}$. Then $\delta(h + cn) = \delta(h) + c\mathbf{e}_{p+1}$ by the definition of $\delta(h + cn)$. Therefore by (ii) we have $\mathrm{mindeg}(h + cn) = \deg(\delta(h + cn)) = \deg(\delta(h)) + c = \mathrm{mindeg}(h) + c$. Therefore A' is Cohen-Macaulay by 2.3.

(i) ⇒ (ii): Suppose that A' is Cohen-Macaulay. Let $h \in \Gamma$ and let $\delta(h)$ be as in the Definition 6.2. Put $a_0 := a(h)$, $i_0 := i(h)$ and $b_0 := b(h)$. Let $\alpha \in \mathcal{E}(h)$ be a minimal element in $\mathcal{E}(h)$. In view of 6.1, we may assume that $\alpha = a\mathbf{e}_0 + \gamma_i + b\mathbf{e}_{p+1}$ with $a \in [0, \lambda' - 1]$ and $i, b \in \mathbb{N}$. Then $b_0 \geq b$ by the definition of $\delta(h)$. We consider the following two cases separately:

- either $b = b_0$, or $b < b_0$ and $a \geq a_0$;
- $b < b_0$ and $a < a_0$.

Case {either $b = b_0$, or $b < b_0$ and $a \geq a_0$}: In this case by the minimality of α in $\mathcal{E}(h)$, the definition of $\delta(h)$ and 6.4, it follows that

$$(6.6.a) \qquad \deg(\delta(h)) = \deg(\alpha) = \mathrm{mindeg}(h)\,.$$

This proves (ii).

Case {$b < b_0$ and $a < a_0$}: In this case by 6.5 there exist unique integers $s, t \in \mathbb{N}^+$ such that $b_0 - b = tv$, $i - i_0 = tz + su$ and

$$(6.6.b) \qquad (a_0 - a)m_0 + g_{i_0} + tvn = g_i\,.$$

Now, since A' is Cohen-Macaulay, $t \geq 1$ and $n > m_p$ from the equation (6.6.b), 2.3 and 6.3 (3), it follows that

$$\begin{aligned} q_i + 1 = \deg(\gamma_i) \geq \mathrm{mindeg}(g_i) &= \mathrm{mindeg}((a_0 - a)m_0 + g_{i_0} + tvn) \\ &= \mathrm{mindeg}((a_0 - a)m_0 + g_{i_0}) + tv \\ &= (a_0 - a) + q_{i_0} + 1 + tv\,. \end{aligned}$$

This proves that $\deg(\delta(h)) = a_0 + q_{i_0} + 1 + b_0 \le a + q_i + 1 + b = \deg(\alpha)$ and so $\deg(\delta(h)) = \deg(\alpha) = \mathrm{mindeg}(h)$ by the minimality of α in $\mathcal{E}(h)$.

(ii) $\Rightarrow$ (iii): Let $j \in \mathbb{N}$ be such that $a(g_j) \ge 1$ and $b(g_j) \ge 1$. Put $a' := a(g_j)$, $i' := i(g_j)$ and $b' := b(g_j)$. Then applying 6.5 to $h = g_j$ and $\alpha = \gamma_j$, there exist integers $s, t \in \mathbb{N}^+$ such that $b' = tv$, $j - i' = tz + su$ and

$$\deg(\delta(g_j)) = \deg(\gamma_j) + (s\lambda'' - tv'') \tag{6.6.c}$$

where $\lambda'' := \lambda' - q_u - 1$ and $v'' := \mu + q_z + 1 - v$. Now, since $\mathrm{mindeg}(g_j) = \deg(\delta(g_j))$ by (ii) and $\gamma_j \in \mathcal{E}(g_j)$, it follows from the equation (6.6.c) that $s\lambda'' \le tv''$. This proves (iii).

(iii) $\Rightarrow$ (ii): Let $h \in \Gamma$ and let $\delta(h)$ be a s in 6.2. Put $a_0 := a(h)$, $i_0 := i(h)$ and $b_0 := b(h)$. Let $\alpha \in \mathcal{E}(h)$ be a minimal element in $\mathcal{E}(h)$. In view of 6.1, we may assume that $\alpha = a\mathbf{e}_0 + \gamma_i + b\mathbf{e}_{p+1}$ with $a \in [0, \lambda' - 1]$ and $i, b \in \mathbb{N}$. Then $b_0 \ge b$ by the definition of $\delta(h)$. If either $b = b_0$, or $b < b_0$ and $a \ge a_0$ then (ii) is proved in (6.6.a). Now, assume that $b < b_0$ and $a < a_0$. Then (6.6.b) holds and by the definitions of $\delta(g_i)$ and t, we have $b(g_i) \ge tv$. Further, from the definition of $\delta(h)$, it follows that $b(g_i) = b_0 - b = tv$ and $a(g_i) \ge a_0 - a$. Therefore by (iii) we have $s\lambda'' \le tv''$ where $\lambda'' := \lambda' - q_u - 1$ and $v'' := \mu + q_z + 1 - v$. Therefore $\deg(\delta(h)) \le \deg(\alpha)$ by 6.5 (2) and so $\deg(\delta(h)) \le \deg(\alpha) = \mathrm{mindeg}(h)$ by the minimality of α in $\mathcal{E}(h)$. □

7. Examples

7.1. Example. (see [3; Proposition (1.2)]) Let $m_0, \dots, m_p, m_{p+1}$ be an arithmetic sequence and let $n := m_{p+1}$. Then $u = p + 1$, $r_u = p$, $\lambda = \lambda' = 1$, $w = w' = 1$ and so $q_u + 1 = \lambda' + w'$. Therefore A' is Cohen-Macaulay by 5.8.

7.2. Examples. These are examples with $n < m_0$.

(1) Let $n = 7$, $m_0 = 10$ and $m_1 = 15$. Then $u = 2$, $\lambda = 3$, $w = 0$, $v = 5$, $\mu = 2$, $z = 1$ and so by 4.8 (1) A' is Cohen-Macaulay.

(2) Let $n = 7$, $m_0 = 8$ and $m_1 = 10$. Then $u = 3$, $\lambda = 2$, $w = 2$, $v = 4$, $\mu = \mu' = 1$, $z = z' = 2$ and $\nu = 3$. Therefore $\nu = 3 = q_{u-z} + 1 + (v - w)$ and so by 4.8 (2) A' is Cohen-Macaulay.

(3) Let $n = 13$, $m_0 = 16$ and $m_1 = 21$. Then $u = 2$, $\lambda = 1$, $w = 2$, $v = 9$, $\mu = 6$, $z = 1$ and $\nu = 7$. Therefore $\nu = 7 < 8 = q_{u-z} + 1 + (v - w)$ and so by 4.8 (2) A' is not Cohen-Macaulay.

7.3. Examples. These are examples with $m_0 < n < m_p$.

(1) Let $m_0 = 7$, $m_1 = 10$ and $n = 9$. Then $v = 3, z = z' = 2 > p = 1$, $\mu = \mu' = 1$ and so $v = 3 = \mu' + q_{z'} + 1$. Therefore A' is Cohen-Macaulay by 4.8 (2).

(2) Let $m_0 = 7$, $m_1 = 12$ and $n = 10$. Then $v = 4$, $z = z' = 1 = p$, $\mu = \mu' = 4$ and so $v = 4 < 5 = \mu' + q_{z'} + 1$. Therefore A' is not Cohen-Macaulay by 4.8 (2).

(3) Let $m_0 = 7$, $m_1 = 16$, $m_2 = 25$ and $n = 17$. Then $p = 2$, $v = 3$, $z = z' = 1 < p$, $\mu = \mu' = 5$ and from $(\mu' + 1)m_0 = 6m_0 = m_2 + n$, we have $\operatorname{mindeg}((\mu' + 1)m_0) = 2$ and so $v + 1 = 4 > 3 = \operatorname{mindeg}((\mu' + 1)m_0) + 1$. For $h := 4m_0 + 2m_1 = 5m_0 + m_2$, we have $\operatorname{mindeg}(h) = 6$ and from $h + m_2 = 5n$, we have $\operatorname{mindeg}(h + m_2) = 5$. Therefore $\operatorname{mindeg}(h + m_2) = 5 \neq 7 = \operatorname{mindeg}(h) + 1$ and so A' is not Cohen-Macaulay by 2.3. This example shows that the converse of 4.8 (3)(a) is not true.

(4) Let $m_0 = 7$, $m_1 = 12$, $m_2 = 17$ and $n = 11$. Then $p = 2$, $v = 3$, $z = z' = 1 < p$, $\mu = \mu' = 3$ and from $(\mu' + 1)m_0 = 4m_0 = m_2 + n$, we have $\operatorname{mindeg}((\mu' + 1)m_0) = 2$ and so $v + 1 = 4 > 3 = \operatorname{mindeg}((\mu' + 1)m_0) + 1$. In this example $v = 3 < 4 = \mu' + 1$ and by CoCoA A' is Cohen-Macaulay. Therefore the converse of 4.8 (3)(b) is not true.

(5) Let $m_0 = 10$, $m_1 = 19$, $m_2 = 28$, $m_3 = 37$ and $n = 35$. Then $p = 3$, $u = 5$, $\lambda = \lambda' = 3$, $w = w' = 1$, $q_u = 1$, $r_u = 2$, $v = 2$, $z = z' = 0$, $\mu = \mu' = 7$ and so $v + w' = 3 < 6 = \mu - \lambda' + q_u + 1$. Therefore A' is not Cohen-Macaulay by 4.8 (4).

7.4. Examples. These are examples with $m_p < n$.

(1) Let $m_0 = 10$, $m_1 = 15$ and $n = 17$. Then $u = 2$, $w = w' = 0$, $\lambda = \lambda' = 3$, $v = 5$, $z = 1$, $\mu = 7$ and A' is Cohen-Macaulay by CoCoA.

(2) Let $m_0 = 15$, $m_1 = 25$ and $n = 26$. Then $u = 3$, $w = w' = 0$, $\lambda = \lambda' = 5$, $v = 5$, $z = 1$, $\mu = 7$, $z' = 4$, $\mu' = 2$ and A' is not Cohen-Macaulay by 6.6. In fact, we have $\delta(g_7) = 3\mathbf{e}_0 + 5\mathbf{e}_{p+1}$ and so $a(g_7) \geq 1$, $b(g_7) \geq 1$, $i(g_7) = 0$ and $7 = i(g_7) + tz + su$ with $t = 1$, $s = 3$. Further, $s\lambda'' = 2(5 - q_3 - 1) = 4 > 3 = 7 + q_1 + 1 - 5 = v'' = tv''$. Therefore the condition (iii) of 6.6 does not hold for $j = 7$ and so A' is not Cohen-Macaulay.

(3) Let $m_0 = 15$, $m_1 = 18$ and $n = 20$. Then $p = 1$, $u = 5$, $r_u = 1$, $q_u = 4$, $\lambda = 6$, $w = 0$, $\lambda' = 2$, $w' = 3$ and so $q_u + 1 = 5 = \operatorname{mindeg}(\lambda' m_0) + w'$. Therefore A' is Cohen-Macaulay by 5.8.

(4) Let $m_0 = 17$, $m_1 = 22$, $m_2 = 27$ and $n = 38$. Then $p = 2$, $u = 11$, $q_u = 5$, $r_u = 1 < p$, $\lambda = \lambda' = 7$, $w = w' = 1$. Further, from $\lambda' m_0 = g_6 + n$ and $(\lambda' - 1)m_0 + g_1 = 6m_0 + m_1$, we have $\operatorname{mindeg}(\lambda' m_0) = 4$ and $\operatorname{mindeg}((\lambda' - 1) + g_1) = 7$. Therefore $q_u + 1 = 6 > 5 = \operatorname{mindeg}(\lambda' m_0) + w'$ and $q_{u+1} + 1 = 6 < 8 = \operatorname{mindeg}((\lambda' - 1) + g_1) + w'$. Therefore A' is not Cohen-Macaulay by 5.8.

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