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On the Radical Theory for Semirings

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Abstract. Based on an abstract concept of radical classes for semirings introduced by D. M. Olson and T. L. Jenkins in [13] we develop in this paper a Kurosh-Amitsur radical theory for semrings in a self-contained way. We define radical operators and semisimple classes for semirings independently on radical classes and exhibit among other results the bijective correspondence between each pair of these three concepts. As a special result we obtain that each radical class of semirings satisfies the A-D-S-property, which yields that each semisimple class is hereditary. We give more details in the following introduction, where we also compare Kurosh-Amitsur radical theories for different structures. Observing the remarks given there, a reader may well use this paper also to inform himself about the radical theory for associative rings.

1. Introduction

In correspondence to the Kurosh-Amitsur radical theory for associative rings, an abstract concept of radical classes and radicals for semirings has been introduced and investigated in a series of publications [12]–[15] by D. M. Olson and several co-authors. In this paper we develop the first steps of a more complete Kurosh-Amitsur radical theory for semirings. So we define, always in a fixed universal class of semirings, radical classes, radical operators and semisimple classes in an independent way, investigate their interrelation and use the latter in particular to obtain further results on these concepts.

After some preparations on semirings in the next section, we begin our radical-theoretical considerations in Section 3 with the definition of radical classes of semirings given in [13]. Then we show the equivalence of that concept to another charaterization of these classes, which corresponds directly to the most common one for radical classes of rings. The following

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statements in this section are more or less known from [13], but we organize and prove them in a somewhat different way more appropriate for our purposes. In Section 4 we introduce radical operators in a universal class of semirings and show that each radical operator determines a radical class and conversely. This allows to obtain further characterizations of radical classes, and we also mention some well-known radicals of semirings in this context.

In the following sections we include the third basic concept of a Kurosh-Amitsur radical theory in our considerations, namely semisimple classes of a universal class of semirings. An essential point is here a certain duality between semisimple classes and radical classes. It occurs in the definitions and correspondence theorems of Section 5, but also in Section 7 which deals with common characterizations of classes $\mathbb{R}$ and $\mathbb{S}$ as a pair of radical and semisimple classes corresponding to each other and with so-called complementary classes of this kind. In particular we mention that each semisimple class is inverse semi-isomorphically closed and hereditary (Theorems 5.7 and 6.3). The latter follows from the fact that each radical class of semirings satisfies the A-D-S-property, the main result of Section 6. There we also deal with hereditary radical classes and give a simple example of a non-hereditary one.

It is a peculiarity of Kurosh-Amitsur radical theories for different structures that their basic concepts as radical classes, radical operators and semisimple classes, though defined in a sometimes considerably different way, have rather similar if not even identical properties, which often can be expressed by the same formulations. Both can be seen if one compares these theories for (associative) rings (cf. e.g. [5] or [20]) and for semirings. Clearly, the former has been used as a model in [13] and here to obtain corresponding statements on semirings, but concepts and some considerations for semirings are somewhat more complicated than those for rings. This is mainly caused by the fact that semiring ideals and homomorphisms of semirings are far away of being related to each other as the corresponding ring-theoretical concepts are, and we point out those differences between rings and semirings in the following section. Conversely, however, one can read this paper as an introduction to the Kurosh-Amitsur radical theory of rings if one replaces "semirings" by "rings", interpretes each kind of semiring ideal as a ring ideal, and overlooks some complications which obviously disappear when dealing with rings.

In this context we mention a Kurosh-Amitsur radical theory established by H. J. Weinert and R. Wiegandt in [19] for proper semifields, including a radical theory for groups. Proper semifields are semirings $(T, +, \cdot)$ for which $(T, \cdot)$ is a group. This yields that, apart from the one-element case, such a semifield $(T, +, \cdot)$ has no zero, and there are also no ideals of $(T, +, \cdot)$ except T itself. In fact, instead of any kind of ideals, the considerations in [19] are based on certain invariant subgroups of $(T, \cdot)$, which correspond to the homomorphic images of a proper semifield $(T, +, \cdot)$. Therefore, proper semifields $(T, +, \cdot)$ considered in that theory and semirings $(S, +, \cdot)$ with an absorbing zero, the objects of the radical theory of semirings developed here, have just one-element semirings in common, and no statement of one of these theories can be applied to the other. Nevertheless, both theories have corresponding concepts and results, and we have also used [19] as an orientation for this paper and the closely related dissertation [11].

2. Preliminaries on semirings

A *semiring* as considered in this paper is defined as an algebra $S = (S, +, \cdot)$ such that $(S, +)$ is a commutative and $(S, \cdot)$ an arbitrary semigroup, both connected by ring-like distributivity. Apart from this section, whose general statements we want to formulate without unnecessary restrictions, we shall also assume that each semiring has a zero which is (multiplicatively) absorbing, i.e. a neutral o of $(S, +)$ which satisfies $so = os = o$ for all $s \in S$. Semirings of this kind are also called hemirings, e.g. in [12]–[15]. Note that semirings as considered here are not assumed to have an identity.

A *left ideal* of a semiring S is defined as a subsemigroup of $(S, +)$ closed under premultiplication by elements of S. *Right ideals* are defined dually and a *(two-sided) ideal* is both, a left and a right ideal. The *k-closure* $\overline{A}$ of an ideal A, defined by $\overline{A} = \{s \in S \mid s + a \in A \text{ for some } a \in A\}$, is an ideal of S as well. An ideal A of a semiring S is called a *k-ideal* if $\overline{A} = A$ is true. We denote the set of all ideals of S by $\mathcal{I}(S)$ and the set of all k-ideals by $\mathcal{K}(S)$. Note that the ideals defined in this way should be called more precisely "semiring ideals". This is of importance if (associative) rings occur in semiring-theoretical investigations, since a ring R, considered as a semiring, may have semiring ideals which are not "ring ideals" in the usual meaning. More precisely, a semiring ideal A of a ring R is a ring ideal of R iff $A = \overline{A}$ holds, i.e. iff A is a k-ideal of R (cf. [7], I.8).

Each homomorphism $\varphi : S \to T$ of semirings corresponds to a congruence κ of S and the homomorphic image $\varphi(S)$ is isomorphic to the semiring S/κ of congruence classes. In this paper we mainly use congruences that are determined by an ideal A of S according to

$$s\kappa_A s' \iff \text{there are } a_i \in A \text{ satisfying } s + a_1 = s' + a_2. \tag{2.1}$$

In this case one usually denotes S/κ_A by S/A. Moreover, $\kappa_A = \kappa_{\overline{A}}$ and thus $S/A = S/\overline{A}$ hold for all ideals A of S with the same k-closure $\overline{A}$, and S/A has always an absorbing zero, namely the congruence class $\overline{A} = [a]_A = [a]_{\overline{A}}$ determined by each $a \in A$. We also mention that a semiring has in general much more congruences than those determined by its ideals. For a last concept of this kind, let $\varphi : S \to T$ be a surjective homomorphism for semirings which have a zero. Then φ is called a *semi-isomorphism* and denoted by $\varphi : S \xrightarrow{\sim} T$ if $\varphi(o_S) = o_T$ and $\varphi^{-1}(o_T) = o_S$ are satisfied. We emphasize here that such a semi-isomorphism, despite of the misleading name, has in general very little in common with an isomorphism. For instance, there is a semi-isomorphism φ of the semiring $\mathbb{N}_0$ of natural numbers onto a semiring which consists just of two elements, an absorbing zero as the image of o and an identity as the image of all other elements of $\mathbb{N}_0$.

We further need the Homomorphism Theorem and the so-called Isomorphism Theorems for semirings. We give them here in a somewhat more complete and general version as usual (cf. e.g. [13] and [6]) and refer to [11] for the proofs.

Theorem 2.1. *Let S be a semiring, T a semiring with an absorbing zero o_T, and $\varphi : S \to T$ a surjective homomorphism. Then $K = \varphi^{-1}(o_T)$ is a k-ideal of S (also called the kernel of φ) and $\psi([s]_K) = \varphi(s)$ for all $s \in S$ defines a semi-isomorphism $\psi : S/K \xrightarrow{\sim} T$ which satisfies $\psi \circ \kappa_K^{\#} = \varphi$, where $\kappa_K^{\#}$ denotes the natural homomorphism of S onto $S/K = S/\kappa_K$.*

Theorem 2.2. *For a semiring S with an absorbing zero o let A be a subsemiring which contains o and B an ideal of S. Then $\varphi([a]_{A\cap\overline{B}}) = [a]_B$ for all $a \in A \subseteq A + B$ defines a semi-isomorphism*

$$\varphi : A/(A \cap \overline{B}) \xrightarrow{\sim} (A + B)/B. \tag{2.2}$$

Theorem 2.3. *Let A, B be ideals of a semiring S with the additional condition $A \subseteq B$. Then $\tilde{\varphi}([s]_B) = [[s]_A]_{\overline{B}/A}$ for all $s \in S$ defines an isomorphism*

$$\tilde{\varphi} : S/B \to (S/A)/(\overline{B}/A). \tag{2.3}$$

Comparing these theorems with the corresponding ones in ring theory, we note that the semi-isomorphisms in Theorem 2.1 and Theorem 2.2 cannot be replaced by isomorphisms, and that one really needs the k-closure $\overline{B}$ of B to formulate the last two theorems correctly.

3. Radical classes

As already announced we assume henceforth that each semiring S has an absorbing zero $o = o_S$, and we denote the class of all these semirings by $\mathbb{H}$. According to [13], we only consider semiring homomorphsims $\varphi : S \to T$ which satisfy the supplementary assumption $\varphi(o_S) = o_T$. A non-empty subclass $\mathbb{U}$ of $\mathbb{H}$ is called *universal* if $\mathbb{U}$ is *hereditary* (i.e. $S \in \mathbb{U}$ implies $\mathcal{I}(S) \subseteq \mathbb{U}$) and *homomorphically closed* (i.e. $S \in \mathbb{U}$ implies $\varphi(S) \in \mathbb{U}$ for each homomorphism φ of S). Additionally, we define $\mathbb{T} = \{S \in \mathbb{H} \mid |S| = 1\}$ as the class of all *trivial semirings*, and we call a class $\mathbb{M} \subseteq \mathbb{H}$ *admissible* if $\mathbb{T} \subseteq \mathbb{M}$ holds and if $\mathbb{M}$ is isomorphically closed (cf. [19]).

The papers [12]–[15] by D. M. Olson and his co-authors are based on the following concept of a radical class for semirings:

Definition 3.1. *A non-empty subclass $\mathbb{R}$ of a fixed universal class $\mathbb{U}$ is called a radical class of $\mathbb{U}$ if $\mathbb{R}$ satisfies the following two axioms:*

(R1) *$\mathbb{R}$ is homomorphically closed.*

(R2) *For each $S \in \mathbb{U}\backslash\mathbb{R}$ there is a k-ideal $K \in \mathcal{K}(S)\setminus\{S\}$ such that $\mathcal{I}(S/K) \cap \mathbb{R} = \{(o)\}$.*

We show at first that this concept can be equivalently defined by the following two axioms (Ra) and (Rb), which correspond directly to axioms mostly used to define radical classes in ring-theoretical considerations (cf. e.g. [20]). For this we denote by $S \longrightarrow\!\!\!\!\triangleright\, T$ a surjective semiring homomorphism such that $|T| \neq 1$ holds and by $A \lhd S$ that A is a semiring ideal of S satisfying $|A| \neq 1$. These notations allow to compare statements of different Kurosh-Amitsur radical theories, where for instance $A \lhd S$ may refer to ring ideals or to certain invariant subgroups of a group or a proper semifield (cf. the corresponding remarks at the end of Section 1). If we use more concrete formulations it is convenient to call each ideal $A \in \mathcal{I}(S) \cap \mathbb{R}$ an $\mathbb{R}$-*ideal of* S.

Theorem 3.2. *A subclass $\mathbb{R}$ of $\mathbb{U}$ is a radical class of $\mathbb{U}$ iff $\mathbb{R}$ satisfies the following two axioms which refer to*

$$\forall A(S \longrightarrow\!\!\!\!\triangleright\, A)\exists B(B \lhd A \wedge B \in \mathbb{R}). \tag{3.1}$$

(Ra) *For all* $S \in \mathbb{U}$, (3.1) *implies* $S \in \mathbb{R}$.

(Rb) *Each* $S \in \mathbb{R}$ *satisfies* (3.1).

Proof. At first we state the implications (R1) $\Longrightarrow$ (Rb) and (R2) $\Longrightarrow$ (Ra). The former is clear since $S \in \mathbb{R}$ and $S \longrightarrow\!\!\!\triangleright\, A$ even yields $A \in \mathbb{R}$ by (R1). For the latter we use the contraposition

$$S \notin \mathbb{R} \Longrightarrow \exists A(S \longrightarrow\!\!\!\triangleright\, A)\forall B(B \triangleleft A \Longrightarrow B \notin \mathbb{R}) \tag{3.2}$$

of (Ra), where (R2) ensures the existence of a particular homomorphic image of S, namely $A = S/K$, which satisfies (3.2). So it remains to show

$$\text{(Ra)} \wedge \text{(Rb)} \Longrightarrow \text{(Ra)} \wedge \text{(R1)} \Longrightarrow \text{(R2)} \wedge \text{(R1)}$$

for each $\mathbb{R} \subseteq \mathbb{U}$. For the first implication, assume $S_1 \in \mathbb{R}$ and let $\varphi : S_1 \to S_2$ be a surjective homomorphism. If $|S_2| = 1$ is true, then $S_2 \in \mathbb{R}$ follows from (Ra). So we have to show $S_2 \in \mathbb{R}$ for $S_1 \longrightarrow\!\!\!\triangleright\, S_2$. Now $S_2 \longrightarrow\!\!\!\triangleright\, A$ implies $S_1 \longrightarrow\!\!\!\triangleright\, A$. The latter yields by (Rb) the existence of an ideal $B \triangleleft A$ satisfying $B \in \mathbb{R}$. Hence (3.1) holds for S_2 which yields $S_2 \in \mathbb{R}$ by (Ra).

To show (Ra) $\wedge$ (R1) $\Longrightarrow$ (R2) we assume $S \in \mathbb{U} \backslash \mathbb{R}$. Then, by the contraposition of (Ra), there is a homomorphism $\varphi : S \longrightarrow\!\!\!\triangleright\, A$ which satisfies (3.2). We use Theorem 2.1 and prove that (R2) holds for the k-ideal $K = \ker\varphi = \varphi^{-1}(o_A)$ of S. Now $K \neq S$ is clear, and the semi-isomorphism $\psi : S/K \to A$ maps each ideal $C \triangleleft S/K$ onto an ideal $B = \psi(C) \triangleleft A$. Thus $B \notin \mathbb{R}$ according to (3.2) implies $C \notin \mathbb{R}$, since $C \in \mathbb{R}$ would imply $B \in \mathbb{R}$ by (R1). □

Note that each radical class $\mathbb{R}$ is admissible, where e.g. (Ra) implies $\mathbb{T} \subseteq \mathbb{R}$. The next lemma is a convenient tool for the following considerations, where we always deal with a universal class $\mathbb{U}$ of $\mathbb{H}$.

Lemma 3.3. *Let* $\mathbb{R}$ *be a radical class and* S *a semiring of* $\mathbb{U}$. *Then* $A \in \mathcal{I}(S) \cap \mathbb{R}$ *implies* $(A + B)/B \in \mathbb{R}$ *for all* $B \in \mathcal{I}(S)$.

Proof. Clearly, $A/A \cap \overline{B}$ is a homomorphic image of A and, according to Theorem 2.2, there is a semi-isomorphism $\varphi : A/A \cap \overline{B} \xrightarrow{\sim} (A + B)/B$. Applying twice (R1) we get $(A + B)/B \in \mathbb{R}$. □

Theorem 3.4. *Each radical class* $\mathbb{R}$ *of* $\mathbb{U}$ *is closed under extensions within* $\mathbb{U}$, *i.e., for all* $S \in \mathbb{U}$ *and* $A \in \mathcal{I}(S)$ *the following implication holds true:*

(Re) $A \in \mathbb{R} \wedge S/A \in \mathbb{R} \Longrightarrow S \in \mathbb{R}$.

Proof. We proceed by contradiction and assume $S \notin \mathbb{R}$. Then there is a k-ideal $K \in \mathcal{K}(S) \backslash \{S\}$ satisfying $\mathcal{I}(S/K) \cap \mathbb{R} = \{(o)\}$ because of (R2). Now $A \in \mathbb{R}$ yields $(A + K)/K \in \mathcal{I}(S/K) \cap \mathbb{R} = \{(o)\}$ by Lemma 3.3. Since $K \in \mathcal{K}(S)$ holds, we get $A + K \subseteq K$ and so $A \subseteq K$. This yields $S/K \cong (S/A)/(K/A)$ by Theorem 2.3. Therefore $(o) \neq S/K$ is a homomorphic image of S/A, where $S/A \in \mathbb{R}$ holds by assumption. Hence $(o) \neq S/K \in \mathbb{R}$ follows from (R1), which contradicts $\mathcal{I}(S/K) \cap \mathbb{R} = \{(o)\}$. □

Lemma 3.5. *Let* $\mathbb{R}$ *be a radical class and* S *a semiring of* $\mathbb{U}$. *Then* $A, B \in \mathcal{I}(S) \cap \mathbb{R}$ *implies* $A + B \in \mathcal{I}(S) \cap \mathbb{R}$.

Proof. We obtain $(A+B)/B \in \mathbb{R}$ from Lemma 3.3. This and $B \in \mathbb{R}$ yield $A+B \in \mathbb{R}$ by (Re). □

For later use we only assume some of the properties of a radical class for the class $\mathbb{R}$ considered in the next lemma:

Lemma 3.6. *Let $\mathbb{R}$ be a subclass of $\mathbb{U}$ which contains $\mathbb{T}$ and satisfies* (Re). *Then $\overline{A} \in \mathbb{R}$ holds for all ideals $A \in \mathbb{R}$ of a semiring $S \in \mathbb{U}$.*

Proof. Since $\mathbb{U}$ is hereditary, $S \in \mathbb{U}$ yields $\overline{A} \in \mathbb{U}$. Moreover, $A \in \mathcal{I}(\overline{A}) \cap \mathbb{R}$ and $\overline{A}/A = (o) \in \mathbb{R}$ hold. Now we can apply (Re) and get $\overline{A} \in \mathbb{R}$. □

The following property, denoted here by (Rϱ), is essential for each radical class $\mathbb{R}$ of $\mathbb{U}$. It implies that all $\mathbb{R}$-ideals of a semiring $S \in \mathbb{U}$ are contained in a greatest one, called the $\mathbb{R}$-*radical of* S. We write $\varrho_{\mathbb{R}}(S)$ for this radical and define in this way an operator $\varrho_{\mathbb{R}}$, the topic of the next section.

Theorem 3.7. *Let $\mathbb{R}$ be a radical class and S be a semiring of $\mathbb{U}$. Then the union $\varrho_{\mathbb{R}}(S)$ of all $\mathbb{R}$-ideals of S is again an $\mathbb{R}$-ideal of S, i.e.*

(Rϱ) $\quad \varrho_{\mathbb{R}}(S) = \bigcup \{A \in \mathcal{I}(S) \cap \mathbb{R}\} \in \mathcal{I}(S) \cap \mathbb{R}$.

This yields that $\varrho_{\mathbb{R}}(S)$ is also the supremum $\bigvee\{A \in \mathcal{I}(S) \cap \mathbb{R}\}$ in the complete lattice $(\mathcal{I}(S), \subseteq)$ of all $\mathbb{R}$-ideals of S and thus the greatest $\mathbb{R}$-ideal of S.

Proof. First we show that the union $M = \varrho_{\mathbb{R}}(S)$ is an ideal of S. For all $a, b \in M$ it follows $a \in A \in \mathcal{I}(S) \cap \mathbb{R}$ and $b \in B \in \mathcal{I}(S) \cap \mathbb{R}$. This yields $sa, as \in A \subseteq M$ for all $s \in S$ and $a + b \in A + B \subseteq M$, the latter since $A + B \in \mathcal{I}(S) \cap \mathbb{R}$ holds by Lemma 3.5.

Suppose now $M \notin \mathbb{R}$. Then there exists a k-ideal K of M which satisfies $K \neq M$ and $\mathcal{I}(M/K) \cap \mathbb{R} = \{(o)\}$ by (R2). Furthermore $A \in \mathcal{I}(M)$ holds for all $A \in \mathcal{I}(S) \cap \mathbb{R}$. We apply Lemma 3.3 to the semiring $M \in \mathbb{U}$ and get $(A + K)/K \in \mathcal{I}(M/K) \cap \mathbb{R} = \{(o)\}$. Since $K \in \mathcal{K}(M)$ holds, we obtain $A + K \subseteq K$ and thus $A \subseteq K$. This yields $M = \bigcup \{A \in \mathcal{I}(S) \cap \mathbb{R}\} \subseteq K$, which contradicts $K \subset M$. □

Since each radical class $\mathbb{R}$ satisfies the assumptions of Lemma 3.6, we obtain immediately:

Corollary 3.8. *Let $\varrho_{\mathbb{R}}$ be defined as in Theorem 3.7. Then $\varrho_{\mathbb{R}}(S) = \overline{\varrho_{\mathbb{R}}(S)}$ is a k-ideal of S and $\varrho_{\mathbb{R}}(S)$ can also be obtained by*

$$\varrho_{\mathbb{R}}(S) = \bigcup \{A \in \mathcal{K}(S) \cap \mathbb{R}\}.$$

4. Radical operators

Let again $\mathbb{U}$ be a fixed universal class of semirings. As a second approach to a Kurosh-Amitsur radical theory we define now radical operators in $\mathbb{U}$ in an independent way and show that each radical class of $\mathbb{U}$ determines a radical operator in $\mathbb{U}$ and conversely. This yields a bijective correspondence between these concepts, and we use radical operators to obtain further statements on radical classes.

Definition 4.1. *A mapping* $\varrho : \mathbb{U} \to \mathbb{U}$ *is called a radical operator in* $\mathbb{U}$ *if each semiring* $S \in \mathbb{U}$ *has an image* $\varrho(S) \in \mathcal{K}(S) \subseteq \mathbb{U}$ *such that the following conditions are satisfied for all* $S, T \in \mathbb{U}$*:*

(ϱa) $\varphi(\varrho(S)) \subseteq \varrho(\varphi(S))$ *for each homomorphism* $\varphi : S \to T$.

(ϱb) $\varrho(S/\varrho(S)) = (o)$.

(ϱc) $\varrho(T) = T \lhd S \implies T \subseteq \varrho(S)$.

(ϱd) $\varrho(\varrho(S)) = \varrho(S)$.

By results of Section 3, the following theorem applies in particular to each radical class $\mathbb{R}$ of $\mathbb{U}$.

Theorem 4.2. *Let* $\mathbb{R}$ *be a non-empty subclass of* $\mathbb{U}$ *which satisfies* (R1), (Rϱ) *and* (Re) *and define*

$$\varrho_{\mathbb{R}}(S) = \bigcup \{A \in \mathcal{I}(S) \cap \mathbb{R}\} \text{ for each } S \in \mathbb{U} \tag{4.1}$$

as in Theorem 3.7. Then $\varrho_{\mathbb{R}}$ *is a radical operator in* $\mathbb{U}$ *according to Definition 4.1. Moreover,* $\varrho_{\mathbb{R}}$ *characterizes the class* $\mathbb{R}$ *by* $\mathbb{R} = \{S \in \mathbb{U} \mid \varrho_{\mathbb{R}}(S) = S\}$.

Proof. The assumption of (Rϱ) for $\mathbb{R}$ means that $\varrho_{\mathbb{R}}(S)$ is, for each $S \in \mathbb{U}$, the greatest $\mathbb{R}$-ideal of S. This yields $S \in \mathbb{R} \iff \varrho_{\mathbb{R}}(S) = S$ and thus the last statement. Further, since $\mathbb{R} \neq \emptyset$ and (R1) imply $\mathbb{T} \subseteq \mathbb{R}$, we obtain from Lemma 3.6 that $\varrho_{\mathbb{R}}(S)$ is even a k-ideal of S. So it remains to show (ϱa)–(ϱd) for $\varrho_{\mathbb{R}}$.

Proof of (ϱa): Since $\varrho_{\mathbb{R}}(S) \in \mathcal{I}(S) \cap \mathbb{R}$ implies $\varphi(\varrho_{\mathbb{R}}(S)) \in \mathcal{I}(\varphi(S)) \cap \mathbb{R}$ by (R1), $\varphi(\varrho_{\mathbb{R}}(S))$ is contained in the greatest $\mathbb{R}$-ideal $\varrho_{\mathbb{R}}(\varphi(S))$ of $\varphi(S)$.

Proof of (ϱb): We show $A = (o)$ for each $A \in \mathcal{I}(S/\varrho_{\mathbb{R}}(S)) \cap \mathbb{R}$. Now each ideal A of $S/\varrho_{\mathbb{R}}(S)$ is the image $A = B/\varrho_{\mathbb{R}}(S)$ of an ideal B of S satisfying $\varrho_{\mathbb{R}}(S) \subseteq B$. Applying (Re) to $B \in \mathbb{U}$, from $A \in \mathbb{R}$ and $\varrho_{\mathbb{R}}(S) \in \mathbb{R}$ we get $B \in \mathbb{R}$. Hence $B \in \mathcal{I}(S) \cap \mathbb{R}$ yields $B \subseteq \varrho_{\mathbb{R}}(S)$ by (Rϱ). So we obtain $B = \varrho_{\mathbb{R}}(S)$ and thus $A = \{o\}$.

Proof of (ϱc): For all $T \in \mathbb{U}$ we get $\varrho_{\mathbb{R}}(T) \in \mathbb{R}$ by (Rϱ). Now $\varrho_{\mathbb{R}}(T) = T \lhd S$ implies $T \in \mathcal{I}(S) \cap \mathbb{R}$ and thus $T \subseteq \varrho_{\mathbb{R}}(S)$, again by (R$\varrho$).

Proof of (ϱd): It follows from (Rϱ) that $\varrho_{\mathbb{R}}(S)$ itself, as an $\mathbb{R}$-ideal of $\varrho_{\mathbb{R}}(S)$, is the greatest $\mathbb{R}$-ideal $\varrho_{\mathbb{R}}(\varrho_{\mathbb{R}}(S))$ of $\varrho_{\mathbb{R}}(S)$. □

Now we show that, conversely, each radical operator defines a radical class.

Theorem 4.3. *Let* $\varrho : \mathbb{U} \to \mathbb{U}$ *be a radical operator. Then*

$$\mathbb{R}_{\varrho} = \{S \in \mathbb{U} \mid \varrho(S) = S\} \tag{4.2}$$

is a radical class of $\mathbb{U}$.

Proof. Clearly, $\mathbb{R}_\varrho \neq \emptyset$ holds since $S \in \mathbb{R}_\varrho$ is true for all $S \in \mathbb{T}$. We show that $\mathbb{R}_\varrho$ satisfies the conditions (R1) and (R2).

Proof of (R1): Let $S \in \mathbb{R}_\varrho$ and φ a homomorphism of S. Then $S = \varrho(S)$ and (ϱa) imply $\varphi(S) = \varphi(\varrho(S)) \subseteq \varrho(\varphi(S)) \subseteq \varphi(S)$. Hence $\varrho(\varphi(S)) = \varphi(S)$ yields $\varphi(S) \in \mathbb{R}_\varrho$.

Proof of (R2): Assume $S \in \mathbb{U}\backslash\mathbb{R}_\varrho$. Then $\varrho(S) \subset S$ holds and we show that $\varrho(S) = \overline{\varrho(S)}$ is a k-ideal of S which satisfies (R2). By way of contradiction we assume that there exists $C \in \mathcal{I}(S/\varrho(S)) \cap \mathbb{R}_\varrho$ with $C \neq (o)$. This yields $C = \varrho(C) \triangleleft (S/\varrho(S))$ and in turn $C \subseteq \varrho(S/\varrho(S)) = (o)$ by (ϱc) and (ϱb), which contradicts our assumption $C \neq (o)$. □

Remark 4.4. In the previous proof we did not use (ϱd). Nevertheless, we need (ϱd) for the second part of the following theorem.

Theorem 4.5. *For each universal class $\mathbb{U}$, the concepts $\varrho_\mathbb{R}$ and $\mathbb{R}_\varrho$ defined in the Theorems 4.2 and 4.3 yield a bijective correspondence between radical classes and radical operators by the following statements:*

1) $\mathbb{R}_{\varrho_\mathbb{R}} = \mathbb{R}$ *holds for each radical class* $\mathbb{R}$.
2) $\varrho_{\mathbb{R}_\varrho} = \varrho$ *holds for each radical operator* ϱ.

Proof. 1) Each radical class $\mathbb{R}$ determines by Theorem 4.2 a radical operator $\varrho_\mathbb{R}$ which satisfies $\mathbb{R} = \{S \in \mathbb{U} \mid \varrho_\mathbb{R}(S) = S\}$. This radical operator $\varrho_\mathbb{R}$ determines by Theorem 4.3 a radical class $\mathbb{R}_{\varrho_\mathbb{R}} = \{S \in \mathbb{U} \mid \varrho_\mathbb{R}(S) = S\}$. Hence $\mathbb{R} = \mathbb{R}_{\varrho_\mathbb{R}}$ is obvious.

2) Each radical operator ϱ determines the radical class $\mathbb{R}_\varrho = \{S \in \mathbb{U} \mid \varrho(S) = S\}$ by (4.2). This and (ϱd), i.e. $\varrho(\varrho(S)) = \varrho(S)$, imply $\varrho(S) \in \mathbb{R}_\varrho$ for each $S \in \mathbb{U}$. The radical class $\mathbb{R}_\varrho$ determines by Theorem 3.7 a radical operator $\varrho_{\mathbb{R}_\varrho}$ such that $\varrho_{\mathbb{R}_\varrho}(S)$ is the greatest $\mathbb{R}_\varrho$-ideal of S. Hence $\varrho(S) \in \mathbb{R}_\varrho$ already implies $\varrho(S) \subseteq \varrho_{\mathbb{R}_\varrho}(S)$ for each $S \in \mathbb{U}$. The converse inclusion is trivial for $\varrho_{\mathbb{R}_\varrho}(S) = (o)$. Otherwise we have $\varrho(\varrho_{\mathbb{R}_\varrho}(S)) = \varrho_{\mathbb{R}_\varrho}(S) \triangleleft S$ and thus $\varrho_{\mathbb{R}_\varrho}(S) \subseteq \varrho(S)$ by (ϱc). □

Now we use radical operators to show statements on radical classes:

Corollary 4.6. *A non-empty subclass $\mathbb{R}$ of $\mathbb{U}$ is a radical class iff $\mathbb{R}$ satisfies* (R1), (Rϱ) *and* (Re).

Proof. By Section 3, each radical class $\mathbb{R}$ satisfies $\mathbb{R} \neq \emptyset$, (R1), (R$\varrho$) and (Re). The converse implication follows as in the first part of the last proof: since Theorem 4.2 is applicable to each class $\mathbb{R}$ with these properties, such a class $\mathbb{R}$ equals the radical class $\mathbb{R}_{\varrho_\mathbb{R}}$. □

Theorem 4.7. *A non-empty subclass $\mathbb{R}$ of $\mathbb{U}$ is a radical class iff $\mathbb{R}$ satisfies* (R1), (Re) *and the following inductive property:*

(Ri) *Each ascending chain $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_\alpha \subseteq \ldots$ of $\mathbb{R}$-ideals A_α of any semiring $S \in \mathbb{U}$ satisfies $B = \bigcup A_\alpha \in \mathbb{R}$.*

Proof. Note that B is always an ideal of S (in fact a k-ideal if each A_α is one). We show at first that each radical class $\mathbb{R}$ satisfies (Ri), where we may assume $A_\alpha \neq (o)$ for all occuring ideals because of $B = (o) \in \mathbb{R}$. Let $\varrho_\mathbb{R}$ be the radical operator determined by $\mathbb{R}$. Then $A_\alpha \in \mathbb{R}$

yields $\varrho_{\mathbb{R}}(A_\alpha) = A_\alpha \triangleleft B$ and thus $A_\alpha \subseteq \varrho_{\mathbb{R}}(B)$ by (ϱc). This implies $B = \bigcup A_\alpha \subseteq \varrho_{\mathbb{R}}(B) \subseteq B$, and $B = \varrho_{\mathbb{R}}(B)$ states $B \in \mathbb{R}$. Conversely, we assume (R1), (Re) and (Ri) for a subclass $\mathbb{R} \neq \emptyset$ of $\mathbb{U}$. Then (Ri) allows to apply Zorn's Lemma to $\mathcal{I}(S) \cap \mathbb{R}$ such that each semiring $S \in \mathbb{U}$ has a maximal $\mathbb{R}$-ideal, say M. We show that M is even the greatest $\mathbb{R}$-ideal of S, which yields (Rϱ) for $\mathbb{R}$ and that $\mathbb{R}$ is a radical class by Corollary 4.6. For this let C be any $\mathbb{R}$-ideal of S. We apply Theorem 2.2 to the ideals M and C of S and obtain $M/(M \cap \overline{C}) \xrightarrow{\sim} (M + C)/C$. Now $M \in \mathbb{R}$ and (R1) imply $M/(M \cap \overline{C}) \in \mathbb{R}$ and in turn $(M + C)/C \in \mathbb{R}$. The latter and $C \in \mathbb{R}$ yield $M + C \in \mathbb{R}$ by (Re). Since M is a maximal $\mathbb{R}$-ideal of S, from $M \subseteq M + C$ it follows $M = M + C$ and thus $C \subseteq M$ as we were to show. □

Remark 4.8. a) In Theorem 4.7 the inductive property (Ri) can be replaced by the corresponding condition, say $\overline{\text{(Ri)}}$, which only deals with ascending chains of k-ideals $A_\alpha \in \mathbb{R}$. To show the non-trivial part of this statement, one proceeds as in the second part of the above proof. Then one obtains a maximal element $M = \overline{M}$ of $\mathcal{K}(S) \cap \mathbb{R}$ and $M + C \in \mathbb{R}$ for each $C \in \mathcal{I}(S) \cap \mathbb{R}$. Since Lemma 3.6 is applicable to $\mathbb{R}$, also $\overline{M + C} \in \mathbb{R}$ holds, and $\overline{M + C} = M$ yields $C \subseteq M$.

b) Let $\mathbb{R} = \mathbb{NIL}(\mathbb{U})$ be the class of all semirings $S \in \mathbb{U}$ whose elements are nilpotent. It is immediate that $\mathbb{R}$ satisfies (R1), (Re) and (Ri). Hence Theorem 4.7 provides a short proof that $\mathbb{R} = \mathbb{NIL}(\mathbb{U})$ is a radical class of $\mathbb{U}$ (cf. Section 4 in [13] and Section V in [15]).

In the same way one can prove that the class $\mathbb{R} = \mathbb{L}(\mathbb{U})$ of all locally nilpotent semirings $S \in \mathbb{U}$ is a radical class, corresponding to the Levitzki radical $\varrho_{\mathbb{L}}(S)$ introduced for semirings in [2]. Indeed, (R1) and (Ri) are again obvious, and (Re) follows from 6.2 in [17]. For completeness we mention in this context two further well-known radical classes. At first, let $\mathbb{R} = \mathbb{B}(\mathbb{U})$ be the class of all $S \in \mathbb{U}$ with the property that each non-zero homomorphic image of S has a non-zero nilpotent ideal. It is the smallest radical class of $\mathbb{U}$ which contains all nilpotent semirings, and the radical $\varrho_{\mathbb{B}}(S)$ is the pendant to the lower Baer radical in ring theory (cf. [13]). Secondly, the Jacobson radical $\varrho_{\mathbb{J}}(S)$ for semirings already introduced by S. Bourne in [3] corresponds to the radical class $\mathbb{R} = \mathbb{J}(\mathbb{U})$ of all right semiregular semirings $S \in \mathbb{U}$. The latter means that for all $s, t \in S$ there are $x, y \in S$ satisfying $s + x + sx + ty = t + y + sy + tx$. Moreover, $\varrho_{\mathbb{B}}(S) \subseteq \varrho_{\mathbb{L}}(S) \subseteq \varrho_{\mathbb{J}}(S)$ holds for each $S \in \mathbb{U}$ (cf. [13] and [17]).

Finally we state that, for each radical class $\mathbb{R}$ of $\mathbb{U}$ the greatest $\mathbb{R}$-ideal $\varrho_{\mathbb{R}}(S)$ of S can also be characterized "from above". This follows by applying the next theorem to the radical operator $\varrho_{\mathbb{R}}$ corresponding to $\mathbb{R}$.

Theorem 4.9. *Let ϱ be a radical operator in $\mathbb{U}$ and $S \in \mathbb{U}$. Then $\varrho(S)$ is the intersection of all k-ideals of S which contain $\varrho(S)$, i.e.*

$$\varrho(S) = \bigcap \{K \in \mathcal{K}(S) \mid \varrho(S/K) = (o)\}. \tag{4.3}$$

Proof. Let K be a k-ideal of S as in (4.3) and $\varphi : S \to S/K$ the natural homomorphism. Then (ϱa) yields $\varphi(\varrho(S)) \subseteq \varrho(S/K) = (o)$ and thus $\varrho(S) \subseteq K$. The converse inclusion follows since $K = \varrho(S) \in \mathcal{K}(S)$ satisfies $\varrho(S/\varrho(S)) = (o)$ by (ϱb). □

Remark 4.10. The intersection $\delta(S) = \bigcap \{I \in \mathcal{I}(S) \mid \varrho(S/I) = (o)\}$ of all ideals of S satisfying $\varrho(S/I) = (o)$ may be properly contained in $\varrho(S)$. This was already stated in [8] in the context of Theorem 21 of [15]. For a simple example, let ϱ be the radical operator corresponding to the radical class $\mathbb{R} = \mathbb{NIL}(\mathbb{H})$ (cf. Remark 4.8 b)) and define a semiring $S \in \mathbb{H}$ by the tables:

$+$	o	a	b
o	o	a	b
a	a	a	b
b	b	b	b

$\cdot$	o	a	b
o	o	o	o
a	o	o	o
b	o	o	b

Then $\varrho(S) = \{o, a\} = A$ is clear. But the ideal $B = \{o, b\}$ has the k-closure $\overline{B} = S$, which yields $\varrho(S/B) = \varrho(S/S) = (o)$ and thus $\delta(S) = S \cap A \cap B = (o) \subset \varrho(S)$.

5. Semisimple classes

As a third starting point for a Kurosh-Amitsur radical theory we define now semisimple classes of a universal class $\mathbb{U}$ of semirings. Then we show, among other results, that each semisimple class $\mathbb{S}$ of $\mathbb{U}$ determines a radical class $\mathbb{R}$ of $\mathbb{U}$ as well as the corresponding radical operator ϱ and vice versa.

Definition 5.1. *A subclass $\mathbb{S}$ of a universal class $\mathbb{U}$ is called a semisimple class of $\mathbb{U}$ iff $\mathbb{S}$ satisfies the following two axioms which refer to*

$$\forall A (A \lhd S) \exists B (A \longrightarrow\!\!\!\!\triangleright\, B \wedge B \in \mathbb{S}). \tag{5.1}$$

(Sa) *For all $S \in \mathbb{U}$,* (5.1) *implies $S \in \mathbb{S}$.*

(Sb) *Each $S \in \mathbb{S}$ satisfies* (5.1).

These axioms imply in particular that each semisimple class $\mathbb{S}$ contains $\mathbb{T}$ and is closed with respect to isomorphisms, i.e. that $\mathbb{S}$ is admissible. Moreover, as an important feature for each Kurosh-Amitsur radical theory, Definition 5.1 corresponds to the characterization of radical classes in Theorem 3.2 by the interchange of $X \longrightarrow\!\!\!\!\triangleright\, Y$ and $Y \lhd X$ for all $X, Y \in \mathbb{U}$. The same holds for the class operators $\mathcal{U}$ and $\mathcal{S}$ introduced as follows:

Definition 5.2. *For each isomorphically-closed subclass $\mathbb{M}$ of $\mathbb{U}$ we define the subclasses $\mathcal{U}\mathbb{M}$ and $\mathcal{S}\mathbb{M}$ of $\mathbb{U}$ by*

$$\mathcal{U}\mathbb{M} = \{S \in \mathbb{U} \mid \forall A (S \longrightarrow\!\!\!\!\triangleright\, A \Longrightarrow A \notin \mathbb{M})\} \quad \text{and} \tag{5.2}$$

$$\mathcal{S}\mathbb{M} = \{S \in \mathbb{U} \mid \forall A (A \lhd S \Longrightarrow A \notin \mathbb{M})\}. \tag{5.3}$$

Obviously, these operators are antitone (i.e. $\mathbb{M}_1 \subseteq \mathbb{M}_2$ implies $\mathcal{U}\mathbb{M}_1 \supseteq \mathcal{U}\mathbb{M}_2$ and $\mathcal{S}\mathbb{M}_1 \supseteq \mathcal{S}\mathbb{M}_2$), map $\mathbb{T}$ onto $\mathbb{U}$ and conversely, and satisfy $\mathbb{M} \cap \mathcal{U}\mathbb{M} = \mathbb{T}$ as well as $\mathbb{M} \cap \mathcal{S}\mathbb{M} = \mathbb{T}$ for each $\mathbb{M} \subseteq \mathbb{U}$. We shall see that they provide a Galois connection between the semisimple classes and the radical classes of $\mathbb{U}$. Before that we state more generally:

Theorem 5.3. *Let $\mathbb{M}$ be a isomorphically closed and regular subclass of $\mathbb{U}$, where the latter means that $\mathbb{M}$ satisfies* (Sb). *Then $\mathcal{U}\mathbb{M}$ is a radical class of $\mathbb{U}$. Moreover, each radical class $\mathbb{R}$ of $\mathbb{U}$ with the property $\mathbb{M} \cap \mathbb{R} = \mathbb{T}$ satisfies $\mathbb{R} \subseteq \mathcal{U}\mathbb{M}$.*

Proof. From $\mathbb{T} \subseteq \mathcal{U}\mathbb{M}$ and since $S_1 \longrightarrow\!\!\triangleright\, S_2 \longrightarrow\!\!\triangleright\, A$ yields $S_1 \longrightarrow\!\!\triangleright\, A$ one obviously obtains (R1) for $\mathcal{U}\mathbb{M}$. Now we use that (R1) and (Ra) characterize a radical class (cf. the proof of Theorem 3.2) and show (Ra) by way of contradiction. So we assume

$$\forall A(S \longrightarrow\!\!\triangleright\, A)\exists B(B \lhd A \wedge B \in \mathcal{U}\mathbb{M})$$

for some $S \in \mathbb{U}$, but $S \notin \mathcal{U}\mathbb{M}$. The latter yields the existence of an A satisfying $S \longrightarrow\!\!\triangleright\, A$ and $A \in \mathbb{M}$. Since $\mathbb{M}$ is regular, for each $B \lhd A$ there is a C which satisfies $B \longrightarrow\!\!\triangleright\, C$ and $C \in \mathbb{M}$. This implies $B \notin \mathcal{U}\mathbb{M}$ for all $B \lhd A$, a contradiction.

For the last statement we disprove $\mathbb{R} \not\subseteq \mathcal{U}\mathbb{M}$ even for each class $\mathbb{R}$ satisfying (R1) and $\mathbb{M} \cap \mathbb{R} = \mathbb{T}$. Indeed, $S \notin \mathcal{U}\mathbb{M}$ and $S \in \mathbb{R}$ imply $S \longrightarrow\!\!\triangleright\, A$ for some $A \in \mathbb{M}$ and $A \in \mathbb{R}$, which is excluded by $\mathbb{M} \cap \mathbb{R} = \mathbb{T}$. □

Theorem 5.4. *Each semisimple class $\mathbb{S}$ of $\mathbb{U}$ determines a radical class $\mathcal{U}\mathbb{S}$ of $\mathbb{U}$ and each radical class $\mathbb{R}$ of $\mathbb{U}$ a semisimple class $\mathcal{S}\mathbb{R}$ of $\mathbb{U}$. This correspondence $\mathbb{S} \to \mathcal{U}\mathbb{S}$ and $\mathbb{R} \to \mathcal{S}\mathbb{R}$ is bijective because of $\mathcal{S}\mathcal{U}\mathbb{S} = \mathbb{S}$ and $\mathcal{U}\mathcal{S}\mathbb{R} = \mathbb{R}$. In particular, $\mathbb{S} \cap \mathbb{R} = \mathbb{T}$ holds for each semisimple class $\mathbb{S}$ and the corresponding radical class $\mathbb{R} = \mathcal{U}\mathbb{S}$.*

Proof. By Theorem 5.3, $\mathcal{U}\mathbb{S}$ is a radical class of $\mathbb{U}$, and $\mathbb{S} \cap \mathcal{U}\mathbb{S} = \mathbb{T}$ was already stated above. Next we show (Sa) and (Sb) for $\mathcal{S}\mathbb{R}$, i.e.

$$S \in \mathcal{S}\mathbb{R} \iff \forall A(A \lhd S)\exists B(A \longrightarrow\!\!\triangleright\, B \wedge B \in \mathcal{S}\mathbb{R})$$

for all $S \in \mathbb{U}$. Now $S \in \mathcal{S}\mathbb{R} \iff \forall A(A \lhd S \wedge A \notin \mathbb{R})$ holds by (5.3), and $A \notin \mathbb{R} \iff \exists B(A \longrightarrow\!\!\triangleright\, B)\forall C(C \lhd B \Longrightarrow C \notin \mathbb{R})$ by (Ra) and (Rb) for $\mathbb{R}$. Using again (5.3), we replace $\forall C(C \lhd B \Longrightarrow C \notin \mathbb{R})$ by $B \in \mathcal{S}\mathbb{R}$, which shows that $\mathcal{S}\mathbb{R}$ is a semisimple class.

Now we show that (Sb) for $\mathbb{S}$ implies $\mathbb{S} \subseteq \mathcal{S}\mathcal{U}\mathbb{S}$ and assume $S \in \mathbb{S}$. Then, for each $A \lhd S$, there is some B satisfying $A \longrightarrow\!\!\triangleright\, B$ and $B \in \mathbb{S}$. Hence $A \notin \mathcal{U}\mathbb{S}$ holds for all ideals $A \lhd S$ by (5.2) such that $S \in \mathcal{S}\mathcal{U}\mathbb{S}$ follows from (5.3). For the converse inclusion, assume $S \in \mathcal{S}\mathcal{U}\mathbb{S}$, i.e. $A \notin \mathcal{U}\mathbb{S}$ for all $A \lhd S$ by (5.2). This implies by (5.3) the existence of some B satisfying $A \longrightarrow\!\!\triangleright\, B$ and $B \in \mathbb{S}$. So we obtain $S \in \mathbb{S}$ by (Sa) and thus $S = \mathcal{S}\mathcal{U}\mathbb{S}$.

Using the correspondence (Sa) $\longleftrightarrow$ (Ra), (Sb) $\longleftrightarrow$ (Rb) and (5.3) $\longleftrightarrow$ (5.2) mentioned above, one obtains that (Rb) for $\mathbb{R}$ implies $\mathbb{R} \subseteq \mathcal{U}\mathcal{S}\mathbb{R}$ and the converse inclusion by (Ra). □

Since each radical class $\mathbb{R}$ corresponds uniquely to a radical operator ϱ, Theorem 5.4 yields the same for each semisimple class $\mathbb{S}$, where the interrelation between $\mathbb{S}$ and ϱ can be described in a direct way:

Theorem 5.5. *Let $\mathbb{S}$ be a semisimple class of $\mathbb{U}$, $\mathbb{R} = \mathcal{U}\mathbb{S}$ the corresponding radical class and $\varrho = \varrho_{\mathbb{R}}$ the radical operator corresponding to $\mathbb{R}$. Then ϱ and $\mathbb{S}$ determine each other by*

$$\mathbb{S} = \{S \in \mathbb{U} \mid \varrho(S) = (o)\} \quad \text{and} \tag{5.4}$$

$$\varrho(S) = \bigcap \{K \in \mathcal{K}(S) \mid S/K \in \mathbb{S}\} \quad \text{for all } S \in \mathbb{U}. \tag{5.5}$$

Proof. From (5.2) we get $S \in \mathbb{S} = \mathcal{S}\mathbb{R} \iff \forall A(A \lhd S \Longrightarrow A \notin \mathbb{R})$, where the latter holds iff the greatest $\mathbb{R}$-ideal $\varrho(S)$ of S equals (o). Hence $S \in \mathbb{S} \iff \varrho(S) = (o)$ yields (5.4). Moreover, $S/K \in \mathbb{S} \iff \varrho(S/K) = (o)$ for each $K \in \mathcal{K}(S)$ shows that (5.5) is equivalent to (4.3). □

Note in this context that (ϱb) can be rewritten as $S/\varrho(S) \in \mathbb{S}$ and that $\delta(S) = \bigcap\{I \in \mathcal{I}(S) \mid S/I \in \mathbb{S}\} \subseteq \varrho(S)$ need not be true with equality (cf. Remark 4.10).

A main topic of the papers [12], [14] and [15] are special and weakly special subclasses $\mathbb{M}$ of a universal class $\mathbb{U}$ of semirings. One property of these classes is that they are inverse semi-isomorphically closed, i.e. that each $S \in \mathbb{H}$ satisfies $S \in \mathbb{M}$ if there is any semi-isomorphism $\varphi : S \overset{\sim}{\longrightarrow} T$ onto some $T \in M$. For this reason, the concept of a universal class $\mathbb{U} \subseteq \mathbb{H}$ is restricted in the above papers by the assumption that $\mathbb{U}$ itself is inverse semi-isomorphically closed. However, this assumption is rather strong. For instance, if an inverse semi-isomorphically closed universal class $\mathbb{U}$ contains at least one semiring which is zero-sum free ($a + b = o$ implies always $a = b = o$) and zero-divisor free, then $\mathbb{U}$ already coincides with the class $\mathbb{H}$ of all semirings, cf. Theorem 3.1 in [8]. To stay with hereditary and homomorphically closed subclasses $\mathbb{U} \subseteq \mathbb{H}$ as universal classes, in [11] the following generalization of the above property for a class $\mathbb{M} \subseteq \mathbb{U}$ was introduced:

Definition 5.6. *A subclass $\mathbb{M}$ of a universal class $\mathbb{U}$ is called inverse semi-isomorphically closed in $\mathbb{U}$ if, for each $S \in \mathbb{U}$, the existence of a semi-isomorphism $\varphi : S \overset{\sim}{\longrightarrow} T$ onto some $T \in \mathbb{M}$ implies $S \in \mathbb{M}$.*

Clearly, if $\mathbb{M}$ is inverse semi-isomophically closed in $\mathbb{U}$ and $\mathbb{U}$ is inverse semi-isomorphically closed in $\mathbb{H}$, then $\mathbb{M}$ is inverse semi-isomorphically closed in $\mathbb{H}$ as described above. Here we only show the following result on semisimple classes:

Theorem 5.7. *Each semisimple class $\mathbb{S}$ of a universal class $\mathbb{U}$ is inverse semi-isomorphically closed in $\mathbb{U}$.*

Proof. Assume $S \in \mathbb{U}$ and let $\varphi : S \overset{\sim}{\longrightarrow} T$ be a semi-isomorphism onto some $T \in \mathbb{S}$. Then $\varrho(T) = (o_T)$ holds for the radical operator $\varrho = \varrho_{\mathcal{U}\mathbb{S}}$ corresponding to $\mathbb{S}$, and $\varphi(\varrho(S)) \subseteq \varrho(T) = (o_T)$ follows from (ϱa). Since only o_S is mapped onto o_T by a semi-isomorphism, we obtain $\varrho(S) = (o_S)$ and thus $S \in \mathbb{S}$. $\square$

6. A-D-S-property and hereditariness

The duality of radical classes and semisimple classes based on the interchange of $X \longrightarrow\!\!\!\!\triangleright\, Y$ and $Y \triangleleft X$ has, for rings as well as for semirings, the disadvantage that the former relation is always transitive, whereas $B \triangleleft A \triangleleft S$ does in general not imply $B \triangleleft S$. Therefore it was an important result for the radical theory of (associative) rings by T. Anderson, N. Divinsky and A. Sulinski in [1] that at least each radical $\varrho(A)$ of an ideal A of a ring S is an ideal of S. In this context one speaks about the A-D-S-property of a radical class. In the first part of this section we show that this property also holds true for each radical class of semirings considered here, and we deal with some consequences of the A-D-S-property.

Lemma 6.1. *Assume $B \triangleleft A \triangleleft S$ and $s \in S$ for a semiring S. Then $sB + B$ is an ideal of A and $\varphi(b) = [sb]_B$ defines a surjective homomorphism $\varphi : B \to (sB + B)/B$.*

Proof. Clearly, $sB+B$ is an ideal of A and φ is a surjective mapping which satisfies $\varphi(b_1+b_2) = \varphi(b_1) + \varphi(b_2)$ for all $b_i \in B$. But also $\varphi(b_1 \cdot b_2) = [sb_1b_2]_B$ and $\varphi(b_1) \cdot \varphi(b_2) = [sb_1 \cdot sb_2]$ are equal since $sb_1 \cdot b_2$ and $sb_1s \cdot b_2$ are contained in $AB \subseteq B$, which yields $sb_1b_2 \,\kappa_B\, sb_1sb_2$ by (2.1). $\square$

Theorem 6.2. *Let $\mathbb{R}$ be a radical class of a universal class $\mathbb{U}$ of semirings and $\varrho = \varrho_{\mathbb{R}}$ the corresponding radical operator. Then, for each ideal A of a semiring $S \in \mathbb{U}$, the radical $\varrho(A)$ of A is an ideal of S, which in particular yields $\varrho(A) \subseteq \varrho(S) \cap A$.*

Proof. Since the case $\varrho(A) = (o)$ is trivial, we apply Lemma 6.1 to the k-ideal $\varrho(A) = B$ of A and $s \in S$. Then $B \in \mathbb{R}$ implies $\varphi(B) = (sB + B)/B \in \mathbb{R}$ by (R1) and in turn $sB + B \in \mathbb{R}$ by (Re). Since the $\mathbb{R}$-ideal $sB + B$ of A contains the greatest $\mathbb{R}$-ideal $\varrho(A) = B$ of A, we get $sB + B = B$ and thus $sB \subseteq B$ for each $s \in S$. The left-right-dual of Lemma 6.1 yields in the same way $Bs \subseteq B$ for each $s \in S$, such that the subsemigroup $\varrho(A) = B$ of $(S, +)$ is an ideal of S. Now $\varrho(A) \in \mathcal{I}(S)$ and $\varrho(A) \in \mathbb{R}$ imply that $\varrho(A)$ is contained in the greatest $\mathbb{R}$-ideal $\varrho(S)$ of S, our last assertion. □

Theorem 6.3. *Each semisimple class $\mathbb{S}$ of a universal class $\mathbb{U}$ is hereditary.*

Proof. Let A be an ideal of a semiring $T \in \mathbb{S}$. We use (5.4) for the radical operator $\varrho = \varrho_{\mathcal{U}\mathbb{S}}$ corresponding to $\mathbb{S}$, i.e. $S \in \mathbb{S} \iff \varrho(S) = (o)$. Then $\varrho(A) \subseteq \varrho(T)$ by Theorem 6.2 and $\varrho(T) = (o)$ yield $\varrho(A) = (o)$, i.e. $A \in \mathbb{S}$. □

Now we deal with the analogous question for radical classes.

Theorem 6.4. *Let $\mathbb{R}$ be a radical class of $\mathbb{U}$ and $\varrho = \varrho_{\mathbb{R}}$ the corresponding radical operator. Then $\mathbb{R}$ is hereditary iff $\varrho(A) \supseteq A \cap \varrho(S)$ holds for each ideal A of any semiring $S \in \mathbb{U}$. By Theorem 6.2, this inclusion is equivalent to $\varrho(A) = A \cap \varrho(S)$.*

Proof. Let $\mathbb{R}$ be hereditary and $A \in \mathcal{I}(S)$. Then $A \cap \varrho(S)$ is an ideal of $\varrho(S)$, and $\varrho(S) \in \mathbb{R}$ yields $A \cap \varrho(S) \in \mathbb{R}$. Since $A \cap \varrho(S)$ is also an ideal of A, it follows that $A \cap \varrho(S) \subseteq \varrho(A)$ holds. Conversely, assume the latter for each ideal A of $S \in \mathbb{U}$. Then $S \in \mathbb{R}$, i.e. $\varrho(S) = S$, implies $A \subseteq \varrho(A) \subseteq A$ and thus $A \in \mathbb{R}$. □

The four radical classes of an arbitrary universal class $\mathbb{U} \subseteq \mathbb{H}$ mentioned in Section 4 are hereditary. This is obvious for $\mathbb{NIL}(\mathbb{U})$ and $\mathbb{L}(\mathbb{U})$, and was shown for $\mathbb{B}(\mathbb{U})$ in [13]. For the Jacobson radical class $\mathbb{J}(\mathbb{U})$, the proof of the main theorem in [10] (which states that the semiradical of a semiring $S \in \mathbb{H}$ introduced in [4] coincides with the Jacobson radical of S) shows that $\mathbb{J}(\mathbb{U})$ is hereditary. We give an example of a radical class $\mathbb{R}$ of $\mathbb{U} = \mathbb{H}$ which is not hereditary.

Example 6.5. We use the notation $S^2 = \{\sum_{i=1}^{n} s_i t_i \mid s_i, t_i \in S\}$ for each semiring S. The class $\mathbb{M} = \{S \in \mathbb{H} \mid S^2 = (o)\}$ is clearly hereditary and thus regular. Hence by Theorem 5.3, $\mathbb{R} = \mathcal{U}\mathbb{M} = \{S \in \mathbb{H} \mid \forall A(S \longrightarrow\!\!\!\!\triangleright\, A \Longrightarrow A \notin \mathbb{M})\}$ is a radical class of $\mathbb{H}$. (In passing we note that $\mathbb{R}$ consists of all $S \in \mathbb{H}$ which satisfy $S = \overline{S^2}$ for the k-closure $\overline{S^2}$ of the ideal S^2 of S, but we do not need that.) Then the semiring $S \in \mathbb{H}$ defined by the tables

$+$	o	a	e
o	o	a	e
a	a	a	e
e	e	e	e

$\cdot$	o	a	e
o	o	o	o
a	o	o	a
e	o	a	e

is contained in $\mathbb{R}$ since S has the identity e. However, the ideal $\{o, a\}$ of S is contained in $\mathbb{M}$, which yields $\{o, a\} \notin \mathbb{R}$.

7. Further interrelation between radical and semisimple classes

We start with common characterizations of a radical class $\mathbb{R}$ and its corresponding semisimple class $\mathbb{S}$.

Theorem 7.1. *Let $\mathbb{R}$ and $\mathbb{S}$ be subclasses of a universal class $\mathbb{U}$ of semirings. Then $\mathbb{R}$ is a radical class and $\mathbb{S} = \mathcal{S}\mathbb{R}$ the corresponding semisimple class of $\mathbb{U}$ iff the following statements hold for all $S \in \mathbb{U}$:*

1) $\mathbb{R} \cap \mathbb{S} = \mathbb{T}$,
2) $S \in \mathbb{R} \Longrightarrow \forall A(S \longrightarrow\!\!\!\!\triangleright\, A \Longrightarrow A \notin \mathbb{S})$,
3) $S \in \mathbb{S} \Longrightarrow \forall A(A \triangleleft S \Longrightarrow A \notin \mathbb{R})$,
4) $\exists K(K \in \mathcal{K}(S) \wedge K \in \mathbb{R} \wedge S/K \in \mathbb{S})$.

Proof. First we note that 2) is equivalent to $\mathbb{R} \subseteq \mathcal{U}\mathbb{S}$ and 3) is equivalent to $\mathbb{S} \subseteq \mathcal{S}\mathbb{R}$. Hence each radical class $\mathbb{R}$ of $\mathbb{U}$ and its corresponding semisimple class $\mathbb{S}$ satisfy $1) - 3)$, and 4) holds since, for each $S \in \mathbb{U}$, the radical $K = \varrho_{\mathbb{R}}(S)$ of S has the properties claimed there.

Conversely, assume these conditions for arbitrary subclasses $\mathbb{R}$ and $\mathbb{S}$ of $\mathbb{U}$. Then 1) (in fact already $\mathbb{R} \cap \mathbb{S} \supseteq \mathbb{T}$) yields $\mathbb{R} \neq \emptyset$ and $S \neq \emptyset$. Next we show $\mathcal{U}\mathbb{S} \subseteq \mathbb{R}$ by applying 4) to each $S \in \mathcal{U}\mathbb{S} = \{S \in \mathbb{U} \mid S \longrightarrow\!\!\!\!\triangleright\, A \Longrightarrow A \notin \mathbb{S}\}$. Then $|S/K| \neq 1$ is excluded by $S \longrightarrow\!\!\!\!\triangleright\, S/K \in \mathbb{S}$, and $|S/K| = 1$ for the k-ideal K of S yields $S = K \in \mathbb{R}$. So we have $\mathbb{R} = \mathcal{U}\mathbb{S}$. Hence $A \notin \mathbb{R}$ is equivalent to $\exists B(A \longrightarrow\!\!\!\!\triangleright\, B \wedge B \in \mathbb{S})$. This and 3) yield that $\mathbb{S}$ is regular, such that $\mathbb{R} = \mathcal{U}\mathbb{S}$ is a radical class of $\mathbb{U}$ by Theorem 5.3. It remains to show $\mathcal{S}\mathbb{R} \subseteq \mathbb{S}$. For this we apply 4) to each $S \in \mathcal{S}\mathbb{R} = \{S \in \mathbb{U} \mid A \triangleleft S \Longrightarrow A \notin \mathbb{R}\}$. Then $K \in \mathcal{K}(S) \cap \mathbb{R}$ is only possible for $K = (o)$, and we obtain $S = S/K \in \mathbb{S}$. □

Corollary 7.2. *In the above theorem, the conditions* 2) *and* 3) *can be replaced by*

2′) $\mathbb{R}$ *is homomorphically closed,*
3′) $\mathbb{S}$ *is hereditary.*

Proof. A radical class $\mathbb{R}$ satisfies 2′) and a semisimple class $\mathbb{S}$ is hereditary by Theorem 6.3. For the converse, $\mathbb{R} \cap \mathbb{S} = \mathbb{T}$ yields $A \in \mathbb{R} \Longrightarrow A \notin \mathbb{S}$ and $A \in \mathbb{S} \Longrightarrow A \notin \mathbb{R}$ for all $A \neq (o)$ and thus the implications 2′) $\Longrightarrow$ 2) and 3′) $\Longrightarrow$ 3). □

We note that conditions like 1), 2′), 3′) and 4) for a pair of classes $(\mathbb{R}, \mathbb{S})$ are typical in the context of torsion theories for modules and their generalizations (cf. [20]). A further question is of course to ask for classes $\mathbb{R}$ and $\mathbb{S}$ as above which also satisfy $\mathbb{R} \cup \mathbb{S} = \mathbb{U}$:

Definition 7.3. *A radical class $\mathbb{R}$ of a universal class $\mathbb{U}$ is called complementary if $\mathbb{R} \cup \mathbb{S} = \mathbb{U}$ holds for $\mathbb{R}$ and its corresponding semisimple class $\mathbb{S} = \mathcal{S}\mathbb{R}$. In particular, $\mathbb{R}$ is called a non-trivial complementary radical class of $\mathbb{U}$ if $\mathbb{T} \subset \mathbb{R} \subset \mathbb{U}$ is satisfied and thus $\mathbb{T} \subset \mathbb{S} \subset \mathbb{U}$.*

We mention at first that the Kurosh-Amitsur radical theories for distinct structures differ considerably with respect to the existence of (clearly non-trivial) complementary radical classes. There are cases where even the greatest universal class, the frame of such a theory, has non-trivial complementary radical classes, as for instance for the Kurosh-Amitsur radical theory for modules or for proper semifields and groups (cf. [18] and [19]). For other Kurosh-Amitsur radical theories non-trivial complementary radical classes exists only for some more or less restricted subclasses of the greatest universal class considered in such a theory. This is the case for rings and for semirings according to the following propositions, which analogously hold for rings:

Proposition 7.4. *Let $\mathbb{U} \subset \mathbb{H}$ be a universal class of semirings which is closed with respect to direct products. Then $\mathbb{U}$ has no non-trivial complementary radical classes.*

Proof. By way of contradiction, assume that there is such a class $\mathbb{R}$ of $\mathbb{U}$. Then $\mathbb{T} \subset \mathbb{R} \subset \mathbb{U}$ and $\mathbb{T} \subset \mathbb{S} \subset \mathbb{U}$ for $\mathbb{S} = \mathcal{S}\mathbb{U}$ yield the existence of semirings R and S in $\mathbb{U}$ satisfying $(o) \neq R \in \mathbb{R}$ and $(o) \neq S \in \mathbb{S}$. By our assumption on $\mathbb{U}$, the direct product $T = R \times S$ of these semirings is contained in $\mathbb{U}$. However, $S \notin \mathbb{R}$ is a non-zero homomorphic image of T, and T has also a non-zero ideal which is isomorphic to $R \notin \mathbb{S}$. The former shows $T \notin \mathbb{R}$ by $2'$) and the latter $T \notin \mathbb{S}$ by $3'$). This disproves our assumption $\mathbb{R} \cup \mathbb{S} = \mathbb{U}$. □

Proposition 7.5. *Let $\mathbb{U} \subseteq \mathbb{H}$ be a universal class of semirings and define*

$$\mathbb{U}' = \{S \in \mathbb{U} \mid S \textit{ does not contain ideals different from } (o) \textit{ and } S\ \} .$$

Then $\mathbb{U}'$ is a universal subclass of $\mathbb{H}$ with the property that each homomorphically closed subclass $\mathbb{R} \neq \emptyset$ of $\mathbb{U}'$ is a hereditary radical class of $\mathbb{U}'$. Moreover, if $\mathbb{R}$ can be chosen such that $\mathbb{T} \subset \mathbb{R} \subset \mathbb{U}'$ holds, $\mathbb{R}$ is a non-trivial complementary radical class of $\mathbb{U}'$.

Proof. Clearly, $S \in \mathbb{U}'$ implies that each ideal of S is also contained in $\mathbb{U}'$, and the same is easily checked for each homomorphic image (regardless of the fact that a semiring S without non-trivial ideals may have numerous homomorphic images $\varphi(S) \neq (o)$ which are not isomorphic to S). Hence $\mathbb{U}'$ is a universal subclass of $\mathbb{H}$. From Theorem 4.7 we obtain that each subclass $\mathbb{R} \neq \emptyset$ of $\mathbb{U}'$ which satisfies (R1) is already a (clearly hereditary) radical class of $\mathbb{U}'$. Indeed, the other conditions (Ri) and (Re) of that theorem are trivially true for each subclass $\mathbb{R}$ of $\mathbb{U}'$. So it remains to show $\mathbb{R} \cup \mathbb{S} = \mathbb{U}'$ for each radical class $\mathbb{R}$ of $\mathbb{U}'$ and the semisimple class $\mathbb{S} = \mathcal{S}\mathbb{R}$. This follows from

$$\mathcal{S}\mathbb{R} = \{S \in \mathbb{U}' \mid \forall A (A \triangleleft S \Longrightarrow A \notin \mathbb{R})\} = \{S \in \mathbb{U}' \mid S \notin \mathbb{R}\} \cup \mathbb{T}. \qquad \square$$

Before turning to some examples, we recall that statements analogous to these propositions are obviously also true (and probably known) for the radical theory of rings. However, Proposition 7.5 produces more illustrative examples for non-trivial complementary radical classes of semirings as corresponding considerations restricted to simple rings. One reason is that, as already mentioned in the last proof, a semiring S without ideals except (o) and S may have various homomorphic images $\varphi(S)$ satisfying $(o) \neq \varphi(S) \ncong S$. This is already the case for *proper semifields with an absorbing zero*, i.e. semirings $(S, +, \cdot)$ for which $(S \setminus \{o\}, \cdot)$ is a group, where *proper* excludes that $(S, +, \cdot)$ is even a not necessarily commutative field. (There is a one-to-one correspondence between those semifields $(S, +, \cdot)$ and proper semifields $(T, +, \cdot)$ as defined at the end of Section 1, cf. Satz I.5.8 in [7], and for non-trivial homomorphic images of proper semifields we refer to [9].)

Example 7.6. To avoid assumptions that the universal class $\mathbb{U}$ in Proposition 7.5 contains certain semirings, we choose $\mathbb{U} = \mathbb{H}$. So we only deal with subclasses of the universal class $\mathbb{H}'$ of all semirings $S \in \mathbb{H}$ without non-trivial ideals.

a) The subclass $\mathbb{R} = \{S \in \mathbb{H}' \mid (S, +)$ is an idempotent semigroup$\}$ of $\mathbb{H}'$ is homomorphically closed and hence by Proposition 7.5 a radical class of $\mathbb{H}'$. Since there are even semifields S with an absorbing zero which are additively idempotent as well as those which are not (cf. [16]), we obtain $\mathbb{T} \subset \mathbb{R} \subset \mathbb{H}'$. So $\mathbb{R}$ is a non-trivial complementary radical class of $\mathbb{H}'$.

b) The subclass $\mathbb{R} = \{S \in \mathbb{H} \mid S \setminus \{o\}$ is a group$\} \cup \mathbb{T}$ of $\mathbb{H}'$ is homomorphically closed and thus a radical class of $\mathbb{H}'$, in fact again a non-trivial complementary one. Here $\mathbb{T} \subset \mathbb{R}$ is clear. For $\mathbb{R} \subset \mathbb{H}'$ we can use that each matrix semiring $M_{n,n}(S)$ over such a semifield S, clearly for $n > 1$, is contained in $\mathbb{H}' \setminus \mathbb{R}$ (cf. [7], Aufgabe I.8.7, for the case that S and thus $M_{n,n}(S)$ is not a ring). An example more typical for the semiring-theoretical situation is the semiring S defined by the tables

$+$	o	a	b
o	o	a	b
a	a	a	b
b	b	b	b

$\cdot$	o	a	b
o	o	o	o
a	o	a	a
b	o	b	b

Clearly, this semiring has only the trivial ideals and is not a semifield. Note also that $\varphi(o) = o_{\mathbb{B}}$ and $\varphi(a) = \varphi(b) = e_{\mathbb{B}}$ define a homomorphism of S onto the smallest proper semifield with an absorbing zero $\mathbb{B} = \{o_{\mathbb{B}}, e_{\mathbb{B}}\}$, mostly called the Boolean semiring.

c) Other examples of non-trivial complementary radical classes of $\mathbb{H}'$ are the class $\{S \in \mathbb{H} \mid$ S is a proper semifield with an absorbing zero$\} \cup \mathbb{T}$ (which may be further restricted by assumptions like $(S, +)$ being idempotent), but also the class consisting of $\mathbb{T}$ and all not necessarily commutative fields, say $\mathbb{F}$.

d) As a remark to the Kurosh-Amitsur radical theory of rings, the class $\mathbb{F}$ just introduced is also a non-trivial complementary radical class of the universal class $\mathbb{V}$ of all simple rings, all in the ring-theoretical meaning. As a somewhat curious fact one easily checks that $\mathbb{F}$ is also a semisimple class of $\mathbb{V}$ and that the corresponding semisimple class $\mathcal{S}\mathbb{F} = \{R \in \mathbb{V} \mid \forall A (A \triangleleft R \Longrightarrow A \notin \mathbb{F})\}$ of $\mathbb{V}$ is likewise also a radical class of $\mathbb{V}$. However, the analogous semiring-theoretical case, i.e. the non-trivial complementary radical class $\mathbb{R}$ of $\mathbb{H}'$ considered at *b*), does not share this curiosity. Indeed, $\mathbb{R}$ is not a semisimple class of $\mathbb{H}'$: the semiring $S = \{a, b, c\}$ defined at *b*) satisfies (5.1), but not $S \in \mathbb{R}$, which disproves (Sa) for $\mathbb{R}$.

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