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Galleries in Poincaré Half-spaces and Diophantine Approximation

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Abstract. In this paper, we present a multidimensional generalization of a geometrical method of approximation devised by G. Humbert. Introducing Clifford algebras and hyperbolic reflection groups with unbounded fundamental domains of finite volume, we obtain a complete set of results which are traditionally connected with the modular group and the usual continued fraction expansions in $\mathbb{R}$.

Keywords: Hyperbolic reflection groups, Diophantine approximation

1. Introduction

For z in $\mathbb{C} \setminus \mathbb{Q}(i)$, C. Hermite (1854) described a way to get an infinite set of (p, q) in the ring of the Gaussian integers $\mathbb{Z}[i]$ such that

$$\left| z - \frac{p}{q} \right| \leq \frac{1}{\sqrt{2}|q|^2}, \tag{1}$$

using method of reduction of binary forms. Following Humbert's works (1916), L. R. Ford (1917) gave a geometric interpretation by considering the action of the Picard group $PSL_2(\mathbb{Z}[i])$ on the Poincaré upper half-space of $\mathbb{R}^3$. Developing his own methods, A. L. Schmidt (1969) got a similar result in the quaternionic framework of the Hurwitz integers. More recently, L. Ya. Vulakh (1995) obtained this kind of inequality by considering the action of geometrically finite groups (acting on the Poincaré half-space) as a multidimensional generalization of the Humbert-Ford approach. Independently, the author chose in his thesis to restrict himself to hyperbolic reflection groups and got some results that we present in part here.

2. Hyperbolic and Euclidean reflection groups

On $\mathbb{R}^p$, with $p \geq 3$, we consider the Lorentz quadratic form B of signature $(p-1,1)$ and we choose a basis (ϵ_i) such that for all $x = \sum_i x_i \epsilon_i$ we have

$$B(x,x) = \sum_{i=1}^{p-1} x_i^2 - x_p^2.$$

Let $\{e_i\}$ be a finite set of unitary vectors such that $B(e_i,e_j) = -\cos(\pi/m_{i,j})$ for some integer $m_{i,j} < \infty$, or $B(e_i,e_j) \leq -1$. Let $A_i = \{x \in \mathbb{R}^n | B(x,e_i) \geq 0\}$. The set S of the orthogonal reflections s_i in the walls ∂A_i generates a discrete subgroup W_S of the Lorentz orthochronous group $O^+(p-1,1)$ formed by the orthogonal transformations which preserve the positive cone

$$C = \{x \in \mathbb{R}^n | B(x,x) < 0, 0 < x_n\}.$$

The relations $s_i^2 = 1$ and $(s_i s_j)^{m_{i,j}} = 1$ are defining relations for W_S which is a Coxeter group. For w in W_S, the length $l_S(w)$ is the smallest integer q such that w is the product of q elements of S.

Replacing eventually e_i by $-e_i$, we can suppose that $\cap_i A_i$ has a non-empty intersection C_S with the closure $\overline{C}$ of C. For the action of W_S on C, C_S is a fundamental domain, i.e. the union of the chambers $w(C_S)$ for w in W_S covers C, and if x and $w(x)$ are in C_S then $w(x) = x$. Furthermore, the isotropy group of any x is $W_{S(x)}$ where $S(x) = \{s \in S | s(x) = x\}$. We know [3] that for $s_i \in S$, we have $w(C_S) \subset A_i$ or $w(C_S) \subset s_i(A_i)$ and

$$w(C_S) \subset A_i \Leftrightarrow l_S(s_i w) = l_S(w) + 1. \tag{2}$$

We denote by B^{p-1} the Euclidean unit $(p-1)$-ball with centre ϵ_p which is the intersection of the affine hyperplane $P = \{x \in \mathbb{R}^p | x_p = 1\}$ with $\overline{C}$. We shall work with the Poincaré half-space $H^{p-1} := \{x \in P | x_{p-1} > 0\}$. The stereographic projection $S^{p-2} := \partial B^{p-1} \to \partial H^{p-1} \cup \{\infty\}$ from the north pole $v_+ = \epsilon_p + \epsilon_{p-1}$ admits a natural extension $\Phi : B^{p-1} \to \overline{H}^{p-1} \cup \{\infty\}$ by mapping a segment $[x_1,x_2]$, with x_1 and x_2 in S^{p-2}, on a half-line or a half-circle perpendicular to $\partial H^{p-1} \approx \mathbb{R}^{p-2}$. This establishes a one-to-one correspondence between geodesics of the affine Klein model for the Euclidean norm $\| \ \|$ and geodesics of the Poincaré model for the hyperbolic distance ρ.

For all f in $O^+(n-1,1)$ the following diagram commutes

$$\begin{array}{ccc} B^{n-1} & \xrightarrow{\Phi} & \overline{H}^{n-1} \cup \{\infty\} \\ {\scriptstyle pf}\downarrow & & \downarrow{\scriptstyle \tilde{f} := \Phi p f \Phi^{-1}} \\ B^{n-1} & \xrightarrow{\Phi} & \overline{H}^{n-1} \cup \{\infty\} \end{array}$$

where $p(x) = x/x_n$. The map $f \mapsto \tilde{f}$ is an isomorphism from $O^+(p-1,1)$ onto the Möbius group $\text{Möb}(H^{p-1})$ acting on the Poincaré half-space.

For s_i in S, the corresponding Möbius transformation $\tilde{s}_i$ is a reflection in hyperplane or an inversion in a half-sphere orthogonal to ∂H^{p-1}. The chamber $\widetilde{C_S} := \Phi(C_S)$ becomes a fundamental domain for the action of W_S on the Poincaré half-space.

If the set Fix(W_S) of points fixed by W_S in $\overline{H}^{p-1} \cup \{\infty\}$ is empty then we say that W_S is an hyperbolic reflection group.

If Fix$(W_S) \neq \emptyset$ then W_S is conjugate to a discrete subgroup of Euclidean affine motions and we say that W_S is a Euclidean reflection group. Two cases may occur: Fix(W_S) lies in $\partial H^{p-1} \cup \{\infty\}$ (W_S is infinite) or intersects H^{p-1} (W_S is finite).

In [10], Vinberg uses the projective Klein model $C/\mathbb{R}_+^*$ and the Lobachevskii space model

$$\Lambda^p := \{x \in \mathbb{R}^p | B(x,x) = -1, x_p > 0\}.$$

Here we work with three other models for W_S: the linear model $(\mathbb{R}^p, B)$, the affine Klein model $(B^{p-1}, \| \ \|)$ and the Poincaré model (H^{p-1}, ρ).

3. The Cliffordian framework

We consider the Clifford algebra $(Cl_{p-2}, -\| \ \|^2)$ with dimension 2^{p-2} generated by $\epsilon_1, \cdots, \epsilon_{p-2}$ such that

$$\begin{cases} \epsilon_i^2 = -1 \\ \epsilon_i\epsilon_j + \epsilon_j\epsilon_i = 0 \text{ for } i \neq j \end{cases}$$

By mapping ϵ_0 on the unit 1 of Cl_{p-2}, ∂H^{p-1} may be viewed as the subspace $\mathbb{R}.1 \oplus V \subset Cl_{p-2}$ where $V = \oplus_{i=1}^{p-1} \mathbb{R}\epsilon_i$. The corresponding elements of ∂H^{p-1} in Cl_{p-2} will be termed vectors. We have $Cl_0 = \mathbb{R}$, $Cl_1 = \mathbb{C}$ and $Cl_2 = \mathbb{H}$.

We give the definitions of three involutions of Cl_{p-2}:

- The main automorphism $x \mapsto x'$ defined by $\epsilon_i' = -\epsilon_i$ for $2 \leq i \leq p-1$
- The main anti-automorphism $x \mapsto x^*$ defined by $\epsilon_i^* = \epsilon_i$
- The anti-automorphism $x \mapsto \overline{x} := x'^*$ called conjugation.

The set of products of non-zero vectors of $\mathbb{R}^{p-2}$ form the Clifford group denoted by Cg_{p-3}. We know [1] that a map $\widetilde{f} : H^{p-1} \to H^{p-1}$ is an homography (even product of inversions) of Möb(H^{p-1}) if and only if there exist a, b, c and d in $Cg_{p-3} \cup \{0\}$ such that

$$\begin{array}{rl} \text{i)} & ab^*, cd^* \in \mathbb{R}^{p-2} \\ \text{ii)} & ad^* - bc^* = 1 \\ \text{iii)} & \widetilde{f}(x) = (ax+b)(cx+d)^{-1} \end{array}$$

For the anti-homographies, ii) and iii) are replaced by

$$\begin{array}{rl} \text{ii)'} & ad^* - bc^* = -1 \\ \text{iii)'} & \widetilde{f}(x) = (ax'+b)(cx'+d)^{-1} \end{array}$$

It is convenient to write $\widetilde{f}$ using Cliffordian matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ or } \begin{pmatrix} a & b \\ c & d \end{pmatrix}'$$

according that it is an homography or an anti-homography. We have to note that (a,b,c,d) is unique up to a multiplication by ± 1.

For x in Cl_{p-2}, we denote by $\Re(x)$ the coordinate on the subspace $\mathbb{R}.1$ (the component of degree 0). We denote by $|\ |$ the norm on Cl_{p-2} which corresponds to the Euclidean structure given by

$$(x,y) \mapsto \Re(x\overline{y}).$$

In particular, for x in Cg_{p-3} we have $|x|^2 = x\overline{x}$.

For $y \in \overline{H}^{p-1}$, we consider the "form" F_y defined on $Cl_{p-2} \times Cl_{p-2}$ by

$$\mathsf{F}_y(p,q) = |p - yq|^2.$$

We use the same letter to write the Möbius transformation and the corresponding motion in $O^+(p-1,1)$. For $f \in \text{Möb}(H^{p-1})$, we have (see (9) in [8])

$$\mathsf{F}_y f(1,0) = -\frac{1}{2}(1+|y|^2)B(f(v_+), \Phi^{-1}(y)). \tag{3}$$

A point x in $\partial H^{p-1} \cup \{\infty\}$ is said to be parabolic if its isotropy group $W_{S(x)}$ is conjugate to an infinite affine Euclidean group. From now on, we suppose that $\widetilde{C}_S$ is the convex closure of a finite number of points (vertices), and that at least one, say α, is parabolic. Our aim is to approximate a given point in ∂H^{p-1} by some elements in the orbit $W_S(\alpha)$ of α. For w in W_S, $pq^{-1} = w(\alpha)$ will mean that $(p,q) = w(1,0)$ or $w(\alpha,1)$ for $\alpha \neq \infty$ using the Cliffordian matrix of w.

Theorem 1. *Let α be a parabolic vertex of $\widetilde{C}_S$ and y be in H^{p-1}. The minimum of F_y on $W_S(\alpha)$ is attained at the parabolic point of the chambers which contain y.*

Proof. First, we suppose that $y \in \widetilde{C}_S$. From (3), one can show that

$$\mathsf{F}_y w s_i(\alpha) - \mathsf{F}_y w(\alpha) = -2\frac{B(\Phi^{-1}(\alpha), e_i)B(w(e_i), \Phi^{-1}(y))}{B(v_+, \Phi^{-1}(y))} \tag{4}$$

for any $s_i \in S$ and $w \in W_S$. We have $B(v_+, \Phi^{-1}(y)) < 0$ because $\Phi^{-1}(y) \in \overline{C}$. We know that $B(\Phi^{-1}(\alpha), e_i) \geq 0$ because $\Phi^{-1}(\alpha) \in C_S$. From (2) and the fact that $\Phi^{-1}(y) \in C_S$, we obtain

$$l_S(ws_i) > l_S(w) \Rightarrow \mathsf{F}_y w s_i(\alpha) \geq \mathsf{F}_y w(\alpha).$$

Suppose that $y \in w(\widetilde{C}_S)$. We use

$$\mathsf{F}_y w f(\alpha) = \mathsf{F}_{w^{-1}(y)} f(\alpha) \times \mathsf{F}_y w(1,0)$$

which holds for all f in W_S. We have $w^{-1}(y) \in \widetilde{C}_S$ hence

$$\inf\{\mathsf{F}_y(p,q)\} = \mathsf{F}_{w^{-1}(y)}(\alpha) \times \mathsf{F}_y w(1,0) = \mathsf{F}_y w(\alpha). \qquad \square$$

We say that $pq^{-1} \in W_S(\alpha)$ is a convergent for $z \in \partial H^{p-1}$ if it satisfies the best approximation condition

$$\begin{cases} |Pz - Q| \leq |pz - q| \\ |Q| \leq |q| \end{cases} \Rightarrow \begin{cases} |Pz - Q| = |pz - q| \\ |Q| = |q| \end{cases}$$

for $PQ^{-1} \in W_S(\alpha)$. From Theorem 1, we get

Corollary 1. *Let z be in ∂H^{p-1}. A parabolic vertex of a chamber which intersects (z,∞) is a convergent for z.*

Proof. Indeed, for $y \in (z,\infty)$ in the chamber, $|Pz-Q| \le |pz-q|$ and $|Q| \le |q|$ implies $\mathsf{F}_y(P,Q) \le \mathsf{F}_y(p,q)$, hence $\mathsf{F}_y(P,Q) = \mathsf{F}_y(p,q)$. □

We denote by $B_z(r)$ the Euclidean ball in $\overline{H}^{p-1}$ tangent to ∂H^{p-1} at z, with radius $r>0$. We also write $B_\infty(r)$ for $\{x \in P | x_{p-1} \ge 1/2r\} \cup \{\infty\}$.

Corollary 2. *For each parabolic vertex $\alpha \in \widetilde{C}_S$, let $r_\alpha > 0$ be such that $\widetilde{C}_S$ is in $\cup_\alpha B_\alpha(r_\alpha)$. Let y be in H^{p-1}. The minimum of F_y over $\cup_\alpha W_S(\alpha)$ is smaller than $2y_{p-1}/\gamma_1$ where $\gamma_1 = 1/\sup\{r_\alpha\}$.*

Proof. There are $w \in W_S$ and a parabolic vertex $\alpha \in \widetilde{C}_S$ such that y is in the ball $w(B_\alpha(r_\alpha)) = B_{w(\alpha)}(r_\alpha/|q|^2)$ where $pq^{-1} = w(\alpha)$. Let c be the centre of this ball and $z = y - y_{p-1}\epsilon_{p-1} \in \partial H^{p-1}$. We have

$$|y-c|^2 = |z-w(\alpha)|^2 + (y_{p-1} - r_\alpha/|q|^2)^2 \le (r_\alpha/|q|^2)^2.$$

This implies that

$$|y-w(\alpha)|^2 = |z-w(\alpha)|^2 + y_{p-1}^2 \le 2y_{p-1}r_\alpha/|q|^2$$

and

$$\mathsf{F}_y w(\alpha) = |q|^2|y-w(\alpha)|^2 \le 2y_{p-1}r_\alpha.$$ □

From the inequality

$$2y_{p-1}|q||p-zq| \le |p-zq|^2 + y_{p-1}^2|q|^2$$

we get that

Corollary 3. *Let z be in ∂H^{p-1}. A chamber which intersects the line $[z,\infty)$ has at least one parabolic vertex pq^{-1} satisfying*

$$|pq^{-1} - z| \le \frac{1}{\gamma_1|q|^2}. \tag{5}$$

We have convergents satisfying (5). We now study a reciprocal result.

Proposition 1. *For each parabolic vertex $\alpha \in \widetilde{C}_S$, let $r_\alpha > 0$ be such that $B_\alpha(r_\alpha)$ is in $\cup_{S(\alpha)} w(\widetilde{C}_S)$. Let z be in ∂H^{p-1}. If pq^{-1} satisfies*

$$|pq^{-1} - z| \le \frac{1}{\gamma_2|q|^2}, \tag{6}$$

where $\gamma_2 = 1/\inf\{r_\alpha\}$, then pq^{-1} is a convergent.

Proof. Suppose that $pq^{-1} = w(\alpha)$ satisfies (6). The line $[z,\infty)$ intersects the ball $w(B_\alpha(r_\alpha))$. According to the definition of the $B_\alpha(r_\alpha)$, there is $f \in W_{S(\alpha)}$ such that $[z,\infty)$ intersects the chamber $wf(\widetilde{C}_S)$ with parabolic vertex pq^{-1}. □

4. Reduced gallery and covering of a geodesic

We adopt the Tits vocabulary for buildings. The faces of the chambers $w(C_S)$ are $w(F_i) = w(C_S \cap \partial A_i)$ and two chambers are adjacent if they have a common face. A gallery is a sequence (finite, or infinite indexed by $\mathbb{N}$ or $\mathbb{Z}$) of adjacent chambers. The length of a gallery is the number of its chambers. The distance $d(C, C')$ between two chambers C and C' is the smallest length of a gallery with extremities C and C' minus 1. It is clear that $d(w_1(C_S), w_2(C_S)) = l_S(w_1^{-1}w_2)$ for all w_1 and w_2. We shall say that a gallery (C_i) is reduced if $d(C_i, C_j) = |i - j|$ for all i and j.

We shall see that distances d and ρ are closely related by introducing galleries along geodesics.

We work with the Klein model. Let x_1 and x_2 be different in S^{p-2}. We denote by $S_{succ}(x_1, x_2)$ the set of reflections $s_i \in S$ satisfying

- $x_1 \in A_i$ and $x_2 \in s_i(A_i)$
- $[x_1, x_2]$ is not in ∂A_i
- $[x_1, x_2] \cap F_i \neq \emptyset$

A necessary and sufficient condition for $S_{succ}(x_1, x_2) \neq \emptyset$ is $[x_1, x_2] \cap C_S \neq \emptyset$. We define $S_{pred}(x_1, x_2) := S_{succ}(x_2, x_1)$.

For different x_1 and x_2 in S^{p-2}, we denote by $\mathrm{Arc}(x_1, x_2)$ the set of $w^{-1}[x_1, x_2]$ which intersect C_S. We introduce the oriented graph $\Gamma(x_1, x_2)$ whose set of vertices is $\mathrm{Arc}(x_1, x_2)$ and edges $(a, s, s(a))$ for $s \in S_{succ}(a)$, or equivalently $(s(a), s, a)$ for $s \in S_{pred}(a)$. The graph is simple and without loops.

Remark 1. The graph $\Gamma(x_2, x_1)$ can be viewed as the reverse of $\Gamma(x_1, x_2)$ by replacing the vertices by their opposites.

Remark 2. One can show that there is a one-to-one correspondence between finite graphs and closed geodesics in H^{n-1}/W_S.

Let $\sigma = (\sigma_0, \cdots, \sigma_{N-1})$ be a finite path in $\Gamma(x_1, x_2)$ where $\sigma_n = (a_n, t_{n+1}, a_{n+1})$. To this path, we associate the gallery

$$G(\sigma) := (C_S, t_1(C_S), t_1t_2(C_S), \ldots, t_1 \cdots t_N(C_S)).$$

From (2), we deduce that the gallery $G(\sigma)$ is reduced. In others words, $l_S(w)$ is the number of hyperplanes between C_S and $w(C_S)$.

From now on, we suppose that ∞ is a parabolic vertex of the fundamental chamber $\widetilde{C}_S$. Let z be in ∂H^{p-1} and σ be an infinite path in $\Gamma(v_+, \Phi^{-1}(z))$ indexed by $\mathbb{N}$ with starting vertex $a_0 = [v_+, \Phi^{-1}(z)]$. Setting $w_0(\sigma) = Id$ and $w_n(\sigma) = t_1 \cdots t_n$ for $n > 0$, we obtain an infinite reduced gallery in the Poincaré half-space which covers the geodesic (z, ∞). We say that $(w_n(\sigma))$ in $W_S^{\mathbb{N}}$ is an H-sequence associated to z (H for Hermite) or that we get $(w_n(\sigma))$ from the Hermite's algorithm.

We need the following result

Lemma 1. *Let α be a parabolic vertex of $\widetilde{C}_S$ and $t \in \mathbb{R}$. The set*

$$\{|q| \mid pq^{-1} \in W_S(\alpha), |q| \leq t\}$$

is finite.

Proof. Let $B_\alpha(r_\alpha)$ be in the interior of $\cup w(\widetilde{C}_S)$ where w runs over $W_{S(\alpha)}$. To each $w \in W_S$, we associate the ball $w(B_\alpha(r_\alpha)) = B_{pq^{-1}}(r_\alpha/|q|^2)$. These balls are disjoint. Using the affine Euclidean subgroup $W_{S(\infty)}$, we can suppose that pq^{-1} is in the bounded fundamental domain $\widetilde{C}_{S(\infty)} \cap \partial H^{p-1}$ for the action of $W_{S(\infty)}$ on ∂H^{p-1}. The set $\{|q| \mid pq^{-1} \in W_S(\alpha), |q| \leq t\}$ is finite because only a finite number of disjoint balls with radius $\geq r_\alpha/t^2$ can intersect the chimney $\widetilde{C}_{S(\infty)}$. □

Now we are in position to prove

Theorem 2. *Let $z \in \partial H^{p-1}$ and (w_n) be a H-sequence associated to z. Let α be a parabolic vertex of $\widetilde{C}_S$, and $p_n q_n^{-1} := w_n(\alpha)$. We have*
(i) *$p_n q_n^{-1}$ is a convergent for z,*
(ii) *the sequence $n \mapsto |p_n - zq_n|$ decreases,*
(iii) *the sequence $n \mapsto |q_n|$ increases, and is bounded if and only if z is in $W_S(\alpha)$.*

Proof. (i) See Corollary 1.
(ii) Let u_n be the vector of t_n. From (4)

$$\begin{aligned} |p_{n+1} - zq_{n+1}|^2 - |p_n - zq_n|^2 &= \mathsf{F}_z w_{n+1}(\alpha) - \mathsf{F}_z w_n(\alpha) \\ &= -2\frac{B(\Phi^{-1}(\alpha), u_{n+1}) B(w_n(u_{n+1}), \Phi^{-1}(z))}{B(v_+, \Phi^{-1}(z))} \leq 0. \end{aligned}$$

(iii) We suppose that $|q_n|$ is bounded. Lemma 1 implies that there is an integer N such that $|q_n| = |q_N|$ for $n \geq N$. From (i) and (ii), we get that $|p_n - zq_n| = |p_N - zq_N|$ for $n \geq N$ (see Corollary 1). Hence there is only a finite number of $p_n q_n^{-1}$ (see proof of Lemma 1). One of these, say $x := p_N q_N^{-1}$, belongs to an infinite number of chambers in the gallery. This means that there is an infinite number of hyperplanes of reflection in $W_{S(x)}$ which separate ∞ and z. We deduce that $x = z$.

We suppose that $z = pq^{-1}$. We know that $|q_n| \leq |q|$ from (i). □

It is clear that $\lim_{n\to\infty} p_n q_n^{-1} = z$, and the term "convergent" is now fully justified.

5. Approximation constants

For z in ∂H^{p-1}, we define the approximation constant $\gamma(z)$ by

$$\gamma(z) = \limsup(|q||p - zq|)^{-1},$$

the limsup being taken over $W_S(\infty)$, and $\gamma(z) = \infty$ for $z \in W_S(\infty)$. We get a geometric interpretation of $\gamma(z)$ by noting that $(|q||p-zq|)^{-1}$ is the diameter of the half-circle $w^{-1}(\infty, z)$ in H^{p-1} where $pq^{-1} = w(\infty)$.

We say that z_1 and z_2 in ∂H^{p-1} are equivalent if there is $w \in W_S$ such that $z_2 = w(z_1)$.

Proposition 2. *If z_1 and z_2 are equivalent then $\gamma(z_1) = \gamma(z_2)$.*

Proof. We suppose that z_1 and z_2 are not equivalent to ∞. Let $w \in W_S$ be such that $z_2 = w(z_1)$. Let (w_n) be a sequence in $W_S^{\mathbb{N}}$ such that

$$\gamma(z_2) = \lim_n |w_n^{-1}(z_2) - w_n^{-1}(\infty)|.$$

We have the inequality

$$|w_n^{-1}(z_2) - w_n^{-1}(\infty)| \leq |w_n^{-1}w(z_1) - w_n^{-1}w(\infty)| + |w_n^{-1}w(\infty) - w_n^{-1}(\infty)|. \tag{7}$$

We can write

$$|w_n^{-1}w(\infty) - w_n^{-1}(\infty)| = (|q_n|^2|w_n(\infty) - w(\infty)|)^{-1}$$

where $p_n q_n^{-1} = w_n(\infty)$. From the fact that $\lim_n |q_n| = \infty$ and $\lim_n w_n(\infty) = z_2 \neq w(\infty)$, we get $\lim_n |w_n^{-1}w(\infty) - w_n^{-1}(\infty)| = 0$. Hence from (7), we obtain $\gamma(z_1) \geq \gamma(z_2)$. By symmetry, we have $\gamma(z_1) = \gamma(z_2)$. □

From now on, we suppose that ∞ is the only parabolic vertex of the fundamental chamber.

Proposition 3. *Let γ_3 be the smallest diameter of edges of the fundamental chamber. Let z be in ∂H^{p-1} and (w_n) be a H-sequence associated to z. Setting $w_n(\infty) = p_n q_n^{-1}$, we have*

$$\gamma_3 \leq \limsup(|q_n||p_n - zq_n|)^{-1}.$$

Proof. We suppose that there is an integer N such that $|w_n^{-1}(\infty) - w_n^{-1}(z)| < \gamma_3$ for $n \geq N$. There is a one-to-one correspondence between the vertices of $\widetilde{C}_S$ not in $B_\infty(1/\gamma_3)$ and the connected components of $\widetilde{C}_S \setminus (\widetilde{C}_S \cap B_\infty(1/\gamma_3))$. Let $\alpha \in H^{p-1}$ be the vertex of the connected component encountered by $w_N^{-1}(\infty, z)$. We prove by induction on n that the reflections t_n used for the construction of the reduced gallery leave α invariant. But $W_{S(\alpha)}$ is a finite group. This gives the contradiction. □

Now we know that $\gamma_3 \leq \gamma(z)$ for all z. Hence γ_3 is a lower bound of the Hurwitz constant $C(W_S) = \inf\{\gamma(z), z \in \partial H^{p-1}\}$ (see [11] in the general case of the geometrically finite discrete Möbius groups).

By noting that for any orbit $W_S(y)$, $y \in H^{p-1}$, a point of the largest height in the orbit belongs to the fundamental domain $\widetilde{C}_S$, we get immediately (see [12]):

Theorem 3. *Let z be in ∂H^{p-1} and (w_n) be a H-sequence associated to z. We have*

$$\gamma(z) = \limsup_n |w_n^{-1}(\infty) - w_n^{-1}(z)|.$$

6. The main result about equivalence

We recall that, by assumption, ∞ is the only parabolic vertex of the fundamental chamber. We give here a necessary and sufficient condition for equivalence of two points by using the corresponding H-sequences.

Theorem 4. *Let z_1 and z_2 in ∂H^{p-1} be not equivalent to ∞. Let (u_n) and (v_n) be the corresponding H-sequences. Then $z_2 = w(z_1)$ with $w \in W_S$ if and only if the set $\{v_n^{-1} w u_n\}$ is finite.*

Proof. First, we suppose that $z_2 = w(z_1)$ with $w \in W_S$. Let $p_n q_n^{-1} := u_n(\infty)$. We have

$$|u_n^{-1}(\infty) - u_n^{-1} w^{-1}(\infty)| = (|q_n|^2 |u_n(\infty) - w^{-1}(\infty)|)^{-1}.$$

We know that $\lim_n |q_n| = \infty$ and $\lim_n |u_n(\infty) - w^{-1}(\infty)| = |z_1 - w^{-1}(\infty)| \neq 0$ because z_1 is not in $W_S(\infty)$. This proves that $\lim_n |u_n^{-1}(\infty) - u_n^{-1} w^{-1}(\infty)| = 0$. We deduce that for a sufficiently large n, there is a chamber $w_2(\widetilde{C}_S)$, with w_2 in the finite set $W_2 = \{w | w(\widetilde{C}_S) \cap \widetilde{C}_S \cap H^{p-1} \neq \emptyset\}$, such that $w_2(\widetilde{C}_S) \cap u_n^{-1} w^{-1}[\infty, z_2] \neq \emptyset$. There is an integer p such that $w u_n w_2(\widetilde{C}_S) \cap v_p(\widetilde{C}_S) \neq \emptyset$. We deduce the existence of $w_2' \in W_2$ such that $w u_n w_2 = v_p w_2'$. It follows that

$$|n - p| \leq l_S(w) + l_S(w_2) + l_S(w_2')$$

and

$$l_S(v_n^{-1} w u_n) \leq |n - p| + l_S(w_2) + l_S(w_2') \leq l_S(w) + 4|W_2| < \infty.$$

We suppose that $\{v_n^{-1} w u_n\}$ is finite. To simplify, we work with homographies and we set

$$w = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ u_n = \begin{pmatrix} p_n & * \\ q_n & * \end{pmatrix}, \ v_n = \begin{pmatrix} P_n & * \\ Q_n & * \end{pmatrix}, \ v_n^{-1} w u_n = \begin{pmatrix} * & * \\ \gamma_n & * \end{pmatrix}.$$

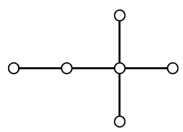
A calculation shows that

$$|\gamma_n| = |Q_n||q_n||w(p_n q_n^{-1}) - P_n Q_n^{-1}||c p_n q_n^{-1} + d|.$$

We have $\lim_n |Q_n| = \lim_n |q_n| = \infty$ and $\lim_n p_n q_n^{-1} = z_1 \neq -c^{-1} d = w^{-1}(\infty)$ because z_1 is not in $W_S(\infty)$. By hypothesis, the set $\{|\gamma_n|\}$ is finite. It follows that $\lim_n |w(p_n q_n^{-1}) - P_n Q_n^{-1}| = 0$. This proves that $z_2 = w(z_1)$. $\square$

7. An example

One can find examples in [7] (modular group, Hecke groups, Picard group...). Here we exhibit the most interesting. Following an idea of Prof. Y. Hellegouarch, we consider the hyperbolic reflection group W_S with Coxeter scheme



This group is a discrete subgroup of $\text{Möb}(H^5)$ acting on the Poincaré half-space of $\mathbb{R}^5$. The affine Weyl group $W_{S(\infty)}$ is of type $\widetilde{D}_4$. The Clifford algebra Cl_4 is generated by i, j, k and

l such that $i^2 = j^2 = k^2 = l^2 = -1$ and $ji = -ij$ ... One can identify the reflections of S with

$$s_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}' : x \mapsto 1/x', \qquad s_2 = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}' : x \mapsto -x' + 1,$$
$$s_3 = \begin{pmatrix} -\xi' & 0 \\ 0 & \xi \end{pmatrix}' : x \mapsto -\xi' x' \xi', \quad s_4 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}' : x \mapsto -ix'i,$$
$$s_5 = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}' : x \mapsto -jx'j, \qquad s_6 = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}' : x \mapsto -kx'k,$$

where $\xi = \frac{1}{2}(1 + i + j + k)$.

The entries of the Cliffordian matrices lie in $\mathbb{Z}[i, j, \xi]$. We denote by Cl_3 the sub-algebra generated by i, j and k, and by $\mathbb{H}$ the sub-algebra generated by i and j. Considering the morphism $\Psi : Cl_3 \to \mathbb{H}$ defined by $\Psi(i) = i$, $\Psi(j) = j$ and $\Psi(k) = ij$, one can show that it is possible to work with the order $\mathfrak{O} = \mathbb{Z}[i, i, \frac{1}{2}(1 + i + j + ij)]$ of the Hurwitz integers. If z is in Cg_3 then $|\Psi(z)| = |z|$. Moreover Ψ establishes a one-to-one correspondence between the lattice $\mathbb{Z}[i, i, \xi] \cap \mathbb{R}^4$ and $\mathfrak{O}$, and between the set of parabolic points and $\mathbb{Q}(i, j) \cup \{\infty\}$. So the results of the previous sections can be stated as follows ($\gamma_1 = \sqrt{2}$ and $\gamma_2 = 2$):

Theorem 5. *Let z be in $\mathbb{H}$.*

(i) *For all $\alpha > 0$, we can find p and q in $\mathfrak{O}$ such that $|p - zq|^2 + \alpha^2 |q|^2 \leq \sqrt{2}\alpha$.*

(ii) *If p and q in $\mathfrak{O}$ satisfy $|z - pq^{-1}| \leq 1/2|q|^2$ then pq^{-1} is a convergent for z.*

A lower bound for the Hurwitz constant $C(W_S)$ is $\gamma_3 = \sqrt{5/2}$. A. Schmidt showed in [9] that this bound is reached. We can find in [11] another proof which uses extremal geodesics. The author has developed in [7] a complete study of pseudo-periodic H-sequences which gives a third proof by showing that

$$\gamma\left(\frac{1}{2} + \frac{1-\sqrt{5}}{4}i + \frac{1+\sqrt{5}}{4}j\right) = \sqrt{\frac{5}{2}}.$$

The main idea is that, for a loxodromic fixed point, we can work with the corresponding closed geodesic.

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