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The Maximum Number of Real Roots of a Multihomogeneous System of Polynomial Equations

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Abstract. A multihomogeneous system of polynomial equations, with as many equations as degrees of freedom, has instances for which there are as many regular real roots, in the relevant product of projective spaces, as are allowed, for the corresponding dehomogenized system, by Bernshtein's [1] theorem. One may, in addition, require that all roots lie in a prescribed open subset of the solution space. The maximal number of roots is characterized as a multiple (by the inverse of a product of factorials) of the permanent of a certain matrix, and recursive formulas are given for this number.

This note characterizes the maximal number of regular real roots of a system of polynomial equations that is *multihomogeneous*, in the sense that the variables of the system are partitioned into groups, with each equation being homogeneous separately in the variables of each group. (In this note a root is a vector of *nonzero* numbers that satisfy the equations.) The main result is that, for systems in this class, there are open sets of coefficient vectors on which the number of real regular roots is the generic (in the space of complex coefficient vectors) number of complex roots, as given by Bernshtein's [1] theorem. To see that this will not be the case for general sparse systems one may consider that the univariate

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example $a_0 + a_3x^3$ has (for $a_0, a_3 \neq 0$) one real root and two other complex roots [1]. It is easy to find systems outside the multihomogeneous class that can have as many regular real roots as Bernshtein's theorem allows, e.g., $ax + by^2 = 0 = cx + dy$, but we have little else to say about the class of systems with this property. Our method is constructive, and it does allow the result to be sharpened in one respect: for any nonempty open set U in the solution space we can find an open set of coefficient vectors on which U contains the maximal number of roots.

We also express the maximal number of roots in terms of the permanent of a certain matrix, and provide a recursive formula that serves as the basis of efficient algorithms for computing the number.

Examples of multihomogeneous equations include the (singly) homogeneous equations considered in Bezout's theorem, multilinear functions, and the equations characterizing totally mixed Nash equilibrium of a normal form game. The systems occurring in the definition of totally mixed Nash equilibrium are studied by McKelvey and McLennan [8]; the results and proofs here are straightforward generalizations of that work. The mean (with respect to a natural distribution on coefficient vectors) number of totally mixed Nash equilibria is studied in [9], and that analysis is extended to general multihomogeneous systems in [10].

The general form of a sparse system of n polynomial equations in n variables is

$$f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x})) = 0, \tag{1}$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ and, for each $i = 1, \ldots, n$, there is a nonempty finite $\mathcal{A}_i \subset \mathbf{N}^n$ such that $f_i(\mathbf{x}) = \sum_{a \in \mathcal{A}_i} f_{ia}\mathbf{x}^a$ for some coefficients f_{ia}. (Here $\mathbf{x}^a = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$.) We will be most interested in real coefficients f_{ia}, but for several points in the argument we consider the space of vectors of complex coefficients. Let $\mathcal{H}^{\mathbb{R}}$ ($\mathcal{H}^{\mathbb{C}}$) be the set of f with real (complex) coefficients in which no f_i vanishes identically. Let $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$ and $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. Consider $(f, \mathbf{x}) \in \mathcal{H}^{\mathbb{R}} \times (\mathbb{R}^*)^n$ ($\mathcal{H}^{\mathbb{C}} \times (\mathbb{C}^*)^n$). Since $\mathbf{x}$ has only nonzero components, no monomial $\mathbf{x}^a$ vanishes. Since the coefficients of each f_i can be varied freely near f_i without leaving $\mathcal{H}^{\mathbb{R}}$ ($\mathcal{H}^{\mathbb{C}}$), it follows that $(f, \mathbf{x})$ is a regular point of the map $(f, \mathbf{x}) \mapsto f(\mathbf{x})$. The regular value theorem implies that

$$V^{\mathbb{R}} := \{\, (f, \mathbf{x}) \in \mathcal{H}^{\mathbb{R}} \times (\mathbb{R}^*)^n : f(\mathbf{x}) = 0 \,\} \text{ and } V^{\mathbb{C}} := \{\, (f, \mathbf{x}) \in \mathcal{H}^{\mathbb{C}} \times (\mathbb{C}^*)^n : f(\mathbf{x}) = 0 \,\}$$

are smooth manifolds of the same dimensions as $\mathcal{H}^{\mathbb{R}}$ and $\mathcal{H}^{\mathbb{C}}$ respectively.

Let $\pi^{\mathbb{R}} : V^{\mathbb{R}} \to \mathcal{H}^{\mathbb{R}}$ and $\pi^{\mathbb{C}} : V^{\mathbb{C}} \to \mathcal{H}^{\mathbb{C}}$ be the restrictions of the natural projections. Evidently $(f, \mathbf{x})$ is a regular point of $\pi^{\mathbb{R}}$ ($\pi^{\mathbb{C}}$) if and only if there are no vectors in $T_{(f,\mathbf{x})}V^{\mathbb{R}}$ ($T_{(f,\mathbf{x})}V^{\mathbb{C}}$) that are parallel to $(\mathbb{R}^*)^n$ ($(\mathbb{C}^*)^n$), which is precisely the condition that $\mathbf{x}$ is a regular point of f. Let $\mathcal{U}^{\mathbb{R}}$ and $\mathcal{U}^{\mathbb{C}}$ be the sets of f in $\mathcal{H}^{\mathbb{R}}$ and $\mathcal{H}^{\mathbb{C}}$ having only regular roots. Applications of Sard's theorem to $\pi^{\mathbb{R}}$ and $\pi^{\mathbb{C}}$ show that $\mathcal{U}^{\mathbb{R}}$ and $\mathcal{U}^{\mathbb{C}}$ are dense.

[1] Observing that the support of this polynomial, namely the set of monomials $\{1, x^3\}$ is not "convex" in the sense that it does not also contain x or x^2, one might wonder whether convexity of the supports was sufficient for there to exist coefficient vectors with the number of regular real roots allowed by Bernshtein's theorem. The system $ax + by^2 = 0 = cx^2 + dy$ shows that this is not the case.

The convex hull of $\mathcal{A}_i$ is denoted by Q_i, and is called the *Newton polytope* of f_i. Generalizing a result of Kushnirenko [6], Bernshtein [1] shows that, for the general system (1), every f in a dense subset of $\mathcal{U}^{\mathbb{C}}$ has $\mathcal{MV}(Q_1, \ldots, Q_n)$ roots in $(\mathbb{C}^*)^n$, where $\mathcal{MV}(Q_1, \ldots, Q_n)$ is the mixed volume of the n-tuple of Newton polytopes. (The concept of mixed volume is due to Minkowski; Betke [2] is a recent treatment.)

Let $\rho(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ be the maximal number of real regular roots of $f \in \mathcal{U}^{\mathbb{R}}$. Other work studying $\rho(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ includes the theory of fewnomials of Khovanski [5], in which arguments are based on the number $|\mathcal{A}_i|$ of monomials in each equation, and Sturmfels [12] and Itenberg and Roy [4], who use the methods of the theory of sparse systems of polynomials to derive bounds on $\rho(\mathcal{A}_1, \ldots, \mathcal{A}_n)$. (This line of research is very clearly described in [13].) Supports $\mathcal{A}_1, \ldots, \mathcal{A}_n$ for which $\rho(\mathcal{A}_1, \ldots, \mathcal{A}_n) = \mathcal{MV}(Q_1, \ldots, Q_n)$, as in our main result, are interesting from the point of view of the conjectures presented by Itenberg and Roy (for which a counterexample has been developed by Li and Wang [7]) since the assertions of these conjectures can hold only in quite particular ways.

Some of these authors also consider the maximal number of roots in the various orthants of $\mathbb{R}^n$. Let $\rho_+(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ be the maximal number of regular roots in the (strictly) positive orthant of $\mathbb{R}^n$. We will say that a set $\mathcal{A} \subset \mathbf{N}^n$ is *comprehensive* if $a \in \mathcal{A}$ whenever $a, b \in \mathbf{N}^n$ with $a \leq b \in \mathcal{A}$. The next result shows that $\rho_+(\mathcal{A}_1, \ldots, \mathcal{A}_n) = \rho(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ when $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are comprehensive, as is the case for the class of systems studied here.

Lemma 1. *If $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are comprehensive, then, for any $C \in \mathbb{R}$, it is possible to find $f \in \mathcal{U}^{\mathbb{R}}$ for which there are $\rho(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ regular roots in $(C, \infty)^n$.*

Proof. Choose $f' \in \mathcal{U}^{\mathbb{R}}$ with $\rho(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ regular roots in $\mathbb{R}^*$. Let D be a number such that all roots are in $(D, \infty)^n$, let $\underline{\mathbf{x}} = (D - C, \ldots, D - C) \in \mathbb{R}^n$, and let $f(\mathbf{x}) = f'(\mathbf{x} + \underline{\mathbf{x}})$. Since each $\mathcal{A}_i$ is comprehensive, each f_i has no monomials with nonzero coefficients that are not in $\mathcal{A}_i$. The roots of f are the roots of f' translated by $\underline{\mathbf{x}}$, so all of them are in $(C, \infty)^n$. In addition $f \in \mathcal{U}^{\mathbb{R}}$, since if it had irregular roots, so would f', contrary to the requirement that $f' \in \mathcal{U}^{\mathbb{R}}$. □

This note focuses on the specialization of the general framework in which the variables in $\mathbf{x}$ are divided into k groups, so that $\mathbf{x} = (\mathbf{y}^1, \ldots, \mathbf{y}^k)$ where $\mathbf{y}^j = (y^j_1, \ldots, y^j_{n_j})$, and there are nonnegative integers δ_{ij} ($i = 1, \ldots, n$, $j = 1, \ldots, k$) such that

$$\mathcal{A}_i = \mathcal{A}_{i1} \times \ldots \times \mathcal{A}_{ik}, \quad \text{where} \quad \mathcal{A}_{ij} = \{\, a \in \mathbf{N}^{n_j} : a_1 + \ldots + a_{n_j} \leq \delta_{ij} \,\}. \tag{2}$$

To see the significance of this we construct the associated "multihomogenized" system of equations. A new collection of variables is given by $\hat{\mathbf{x}} = (\hat{\mathbf{y}}^1, \ldots, \hat{\mathbf{y}}^k)$ where $\hat{\mathbf{y}}^j = (y^j_0, y^j_1, \ldots, y^j_{n_j})$. Let

$$\hat{\mathcal{A}}_i = \hat{\mathcal{A}}_{i1} \times \ldots \times \hat{\mathcal{A}}_{ik}, \quad \text{where} \quad \hat{\mathcal{A}}_{ij} = \{\, \hat{a} \in \mathbf{N}^{n_j+1} : \hat{a}_0 + \hat{a}_1 + \ldots + \hat{a}_{n_j} = \delta_{ij} \,\}.$$

There are bijections $\beta_{ij} : \hat{\mathcal{A}}_{ij} \to \mathcal{A}_{ij}$ and $\beta_i : \hat{\mathcal{A}}_i \to \mathcal{A}_i$ given by $\beta_{ij}(\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_{n_j}) = (\hat{a}_1, \ldots, \hat{a}_{n_j})$ and $\beta_i(\hat{a}^1, \ldots, \hat{a}^k) = (\beta_{i1}(\hat{a}^1), \ldots, \beta_{ik}(\hat{a}^k))$. For each instance of (1) there is an associated multihomogeneous system of equations

$$\hat{f}(\hat{\mathbf{x}}) = (\hat{f}_1(\hat{\mathbf{x}}), \ldots, \hat{f}_n(\hat{\mathbf{x}})) = 0 \in \mathbb{R}^n \quad \text{where} \quad \hat{f}_i(\hat{\mathbf{x}}) = \sum_{\hat{a} \in \hat{\mathcal{A}}_i} f_{i\beta_i(\hat{a})} \hat{\mathbf{x}}^{\hat{a}}. \tag{3}$$

Remark 1. The case $k = 1$ recovers the general homogeneous system considered by Bezout's theorem. If $\delta_{ij} = 1$ for all i and j, then each f_i is a multilinear functional. The definition of a totally mixed Nash equilibrium of a normal form game gives rise to a multihomogeneous system in which, for each $j = 1, \ldots, k$, there are n_j indices i for which $\delta_i^j = 0$ and $\delta_{ij'} = 1$ for all $j' \neq j$.

The natural spaces in which to look for, and count, solutions of (3), are the products of projective spaces $\mathcal{P}^K = P^{n_1}(K) \times \ldots \times P^{n_k}(K)$, for $K = \mathbb{R}^*, \mathbb{R}, \mathbb{C}^*, \mathbb{C}$. In the obvious way we regard these spaces as nested: $\mathcal{P}^{\mathbb{R}^*} \subset \mathcal{P}^{\mathbb{R}} \subset \mathcal{P}^{\mathbb{C}}$ and $\mathcal{P}^{\mathbb{R}^*} \subset \mathcal{P}^{\mathbb{C}^*} \subset \mathcal{P}^{\mathbb{C}}$. For

$$\hat{\mathbf{x}} = (\hat{\mathbf{y}}^1, \ldots, \hat{\mathbf{y}}^k) \in (\mathbb{C}^{n_1+1} \backslash \{0\}) \times \ldots \times (\mathbb{C}^{n_k+1} \backslash \{0\})$$

let $[\hat{\mathbf{x}}] = ([\hat{\mathbf{y}}^1], \ldots, [\hat{\mathbf{y}}^k])$ denote the corresponding element of $\mathcal{P}^{\mathbb{C}}$. If $\mathbf{x} = (\mathbf{y}^1, \ldots, \mathbf{y}^k)$ is a solution of (1), where $\mathbf{y}^j = (y_1^j, \ldots, y_{n_j}^j)$, then an associated solution $\hat{\mathbf{x}}$ of (3) is derived by setting $\hat{\mathbf{y}}^j = (1, y_1^j, \ldots, y_{n_j}^j)$ and $[\hat{\mathbf{x}}] = ([\hat{\mathbf{y}}^1], \ldots, [\hat{\mathbf{y}}^k])$. Although Theorem 1 below is most naturally understood as pertaining to (3), for several reasons it will be more convenient to argue in terms of system (1).

For systems satisfying (2), a root $[\hat{\mathbf{x}}] \in \mathcal{P}^{\mathbb{C}}$ of $\hat{f}$ is said to be *regular* if $\hat{\mathbf{x}}$ is a regular point of $\hat{f}$. It is easy to show that if $\hat{\mathbf{x}} = (\hat{\mathbf{y}}^1, \ldots, \hat{\mathbf{y}}^k)$ is a regular point of $\hat{f}$, then so is $(\alpha^1 \hat{\mathbf{y}}^1, \ldots, \alpha^k \hat{\mathbf{y}}^k)$ for any $\alpha^1, \ldots, \alpha^k \in \mathbb{C}^*$, so this definition is meaningful. Our main result is that, when (2) holds, there exists a real coefficient vector for which there are as many regular real roots as are allowed by Bernshtein's theorem.

Theorem 1. *When (2) holds, for any open $\hat{U} \subset \mathcal{P}^{\mathbb{R}}$ there is an $f \in \mathcal{U}^{\mathbb{R}}$ for which the corresponding $\hat{f}$ has $\mathcal{MV}(Q_1, \ldots, Q_n)$ regular roots in $\hat{U}$.*

Remark 2. Theorem 1 suggests that multihomogeneous systems are well behaved in the sense of sharing properties of the general homogeneous system. Additional evidence for this point of view is given by Sturmfels and Zelevinsky [11], who give Sylvester type (that is, determinantal) formulas for the resultant of the unmixed ($\mathcal{A}_0 = \mathcal{A}_1 = \ldots = \mathcal{A}_n$) multihomogeneous system in which the number of equations exceeds the number of degrees of freedom by one.

Theorem 1 follows from Lemmas 2–4 below.

Lemma 2. *If (2) holds, there is an $f \in \mathcal{U}^{\mathbb{R}}$ for which the corresponding $\hat{f}$ has no roots in $\mathcal{P}^{\mathbb{C}} \setminus \mathcal{P}^{\mathbb{R}^*}$.*

Proof. For each i let

$$f_i(\mathbf{x}) = \prod_{j=1}^{k} \prod_{\ell=1}^{\delta_{ij}} \mu_i^{j\ell}(\mathbf{y}^j)$$

for some choice of affine functionals $\mu_i^{j\ell} : \mathbb{R}^{n_j} \to \mathbb{R}$. It is easy to see that

$$\hat{f}_i(\hat{\mathbf{x}}) = \prod_{j=1}^{k} \prod_{\ell=1}^{\delta_{ij}} \hat{\mu}_i^{j\ell}(\hat{\mathbf{y}}^j),$$

where, for each i, j, ℓ, $\hat{\mu}_i^{j\ell}$ is the linear functional on $\mathbb{R}^{n_j+1}$ that reduces to $\mu_i^{j\ell}$ under the specialization $y_0^j = 1$.

For each j let

$$E^j = \{\, \mu_i^{j\ell} : 1 \le i \le n, 1 \le \ell \le \delta_{ij} \,\}.$$

We may choose the affine functionals $\mu_i^{j\ell}$ in general position in the sense that, for any subset $E \subset E^j$, the intersection of the kernels of the elements of E is an affine subspace of $\mathbb{R}^{n_j}$ of dimension $n_j - \#E$. We may also insist that the intersection of the kernels has an intersection with each coordinate subspace of the expected dimension, so that when $\#E = n_j$, the unique common root has nonvanishing coordinates, and there are no roots when $\#E > n_j$

Let $[\hat{\mathbf{x}}] = ([\hat{\mathbf{y}}^1], \ldots, [\hat{\mathbf{y}}^k])$ be a projective root of (3). For each i there must be some (j_i, ℓ_i) such that $\hat{\mu}_i^{j_i\ell_i}(\hat{\mathbf{y}}^{j_i}) = 0$. The general position assumption implies that for each j there are at most n_j indices i with $j_i = j$, and since the total number of indices i is $n = n_1 + \ldots + n_k$, for each j there must be exactly n_j such i. Therefore each $\hat{\mathbf{y}}^j$ is a solution of a system of n_j linear equations in $n_j + 1$ variables that has exactly one projective root, which is real, namely $[(1, y_1^j, \ldots, y_{n_j}^j)]$ where $\mathbf{y}^j = (y_1^j, \ldots, y_{n_j}^j)$ is the unique root of the associated system of n_j affine functionals $\mu_i^{j\ell}$. This shows that all roots of (3) are in $\mathcal{P}^{\mathbb{R}^*}$.

Suppose $j_i = j$. By varying $\mathbf{y}^j$ in the intersection of the kernels of the other $\mu_{i'}^{j_{i'}\ell_{i'}}$ with $j_{i'} = j$, we may vary f_i without changing the value of any other $f_{i'}$. Thus the image of $Df(\mathbf{x})$ spans the standard basis of $\mathbb{R}^n$, so $\mathbf{x}$ is a regular root. This shows that all roots are regular, so that $f \in \mathcal{U}^{\mathbb{R}}$. $\square$

Lemma 3. *If, for some $f \in \mathcal{U}^{\mathbb{R}}$, the corresponding $\hat{f}$ has no roots in $\mathcal{P}^{\mathbb{C}} \setminus \mathcal{P}^{\mathbb{R}^*}$, then*

$$\rho(\mathcal{A}_1, \ldots, \mathcal{A}_n) = \mathcal{MV}(Q_1, \ldots, Q_n).$$

Proof. Consider a root $[\hat{\mathbf{x}}]$ of $\hat{f}$. If $\hat{\mathbf{x}} = (\hat{\mathbf{y}}^1, \ldots, \hat{\mathbf{y}}^k)$ where $\hat{\mathbf{y}}^j = (y_0^j, y_1^j, \ldots, y_{n_j}^j)$, we set $\mathbf{x} = (\mathbf{y}^1, \ldots, \mathbf{y}^k)$ where $\mathbf{y}^j = (y_1^j/y_0^j, \ldots, y_{n_j}^j/y_0^j)$. Applying the (complex) implicit function theorem at $\mathbf{x}$, then multihomogenizing, we see that there exist neighborhoods $U_{[\hat{\mathbf{x}}]} \subset \mathcal{H}^{\mathbb{C}}$ of f and $W_{[\hat{\mathbf{x}}]} \subset \mathcal{P}^{\mathbb{C}}$ of $[\hat{\mathbf{x}}]$ for which

$$\{\, (f', [\hat{\mathbf{x}}']) \in U_{[\hat{\mathbf{x}}]} \times W_{[\hat{\mathbf{x}}]} : [\hat{\mathbf{x}}'] \text{ is a root of } \hat{f}' \,\}$$

is the graph of a C^∞ function $r_{[\hat{\mathbf{x}}]} : U_{[\hat{\mathbf{x}}]} \to W_{[\hat{\mathbf{x}}]}$. At this point we have not used the hypothesis that $\hat{f}$ has no roots outside of $\mathcal{P}^{\mathbb{R}^*}$, and by applying the argument to this point to some $\hat{f}$ with the maximal number of real regular roots, we may conclude that

$$\rho(\mathcal{A}_1, \ldots, \mathcal{A}_n) \le \mathcal{MV}(Q_1, \ldots, Q_n).$$

If every neighborhood of f in $\mathcal{H}^{\mathbb{C}}$ contained an f' for which the associated $\hat{f}'$ had a root in $\mathcal{P}^{\mathbb{C}} \setminus \bigcup_{[\hat{\mathbf{x}}]} W_{[\hat{\mathbf{x}}]}$, then, since this set is compact, it would contain a root of $\hat{f}$, which is impossible. Consequently it must be possible to choose a neighborhood U of f small enough that, for every $f' \in U$, the only roots of the associated $\hat{f}'$ in $\mathcal{P}^{\mathbb{C}}$ are the various $r_{[\hat{\mathbf{x}}]}(f')$. Since U is open in $\mathcal{H}^{\mathbb{C}}$, generic $f' \in U$ must have $\mathcal{MV}(Q_1, \ldots, Q_n)$ roots, so we see that this number is also a lower bound on $\rho(\mathcal{A}_1, \ldots, \mathcal{A}_n)$. □

Lemma 4. *Suppose that (2) holds, and $f \in \mathcal{U}^{\mathbb{R}}$. If the corresponding $\hat{f}$ has no roots in $\mathcal{P}^{\mathbb{C}} \setminus \mathcal{P}^{\mathbb{R}^*}$, then for any open $\hat{U} \subset \mathcal{P}^{\mathbb{R}}$ there is an $f' \in \mathcal{U}^{\mathbb{R}}$ for which the corresponding $\hat{f}'$ has as many roots as f, with all of them lying in $\hat{U}$.*

Proof. By Lemma 1, for arbitrary $C \in \mathbb{R}$ we can find $f'' \in \mathcal{U}^{\mathbb{R}}$ with as many regular real roots as f, and with all roots in $(C, \infty)^n$. If a neighborhood of

$$\big([(0,1,\ldots,1)], \ldots, [(0,1,\ldots,1)]\big) \in \mathcal{P}^{\mathbb{R}}$$

is given, then, by choosing C large, we can force the corresponding $\hat{f}''$ to have all its projective roots in this neighborhood. The action of the product group $G = O(n_1+1) \times \ldots \times O(n_k+1)$ on $\mathbb{R}^{n_1+1} \times \ldots \times \mathbb{R}^{n_k+1}$ induces an action on the space of multilinear systems (3) with real coefficients. Under such a transformation, the roots of the transformed system are related to the roots of the given system by the corresponding action of G on $\mathcal{P}^{\mathbb{R}} = P^{n_1}(\mathbb{R}) \times \ldots \times P^{n_k}(\mathbb{R})$. By choosing an appropriate element of G we can move $\big([(0,1,\ldots,1)], \ldots, [(0,1,\ldots,1)]\big)$ to any other point in $\mathcal{P}^{\mathbb{R}}$, and under the induced action $\hat{f}''$ is transformed to a system $\hat{f}'$ having all its roots in the corresponding neighborhood of the chosen point. With C sufficiently large, and the point in $\mathcal{P}^{\mathbb{R}}$ chosen suitably, this results in $\hat{f}'$ having all its roots in $\hat{U}$. Finally, it is easy to check that the transformations considered in this argument preserve the regularity of all roots, so that $f' \in \mathcal{H}^{\mathbb{R}}$. □

Let $\mathbf{n} := (n_1, \ldots, n_k)$, let δ be the $n \times k$ matrix with entries δ_{ij}, and let $B(\mathbf{n}, \delta)$ be the Bernstein bound for the system given by (2). Let J be the set of n-tuples $(j_1, \ldots, j_n)$ such that, for each $1 \le j \le k$, there are n_j indices i with $j_i = j$. For the coefficient vector constructed in the proof of Lemma 2, the roots are evidently in one-to-one correspondence with the set of n-tuples $((j_1, \ell_1), \ldots, (j_n, \ell_n))$ such that $(j_1, \ldots, j_n) \in J$ and $1 \le \ell_i \le \delta_{ij_i}$ for all i. Thus

$$B(\mathbf{n}, \delta) = \sum_{(j_1,\ldots,j_n) \in J} \left(\prod_{i=1}^{n} \delta_{ij_i} \right).$$

Recall that the *permanent* (e.g., [3]) of an $n \times n$ matrix M with entries m_{ij} is

$$\operatorname{per}(M) := \sum_{\sigma \in S_n} \left(\prod_{i=1}^{n} m_{i\sigma(i)} \right).$$

(Here S_n is the symmetric group.) Let $M(\mathbf{n}, \delta)$ be the $n \times n$ matrix whose first n_1 columns are the first column of δ, whose next n_2 columns are the second column of δ, and so forth. The next result follows from the formula above because, for any $(j_1, \ldots, j_n) \in J$, there are $n_1! \cdot \ldots \cdot n_k!$ permutations $\sigma \in S_n$ such that, for each i, column $\sigma(i)$ of $M(\mathbf{n}, \delta)$ is column j_i of δ.

Theorem 2. $B(\mathbf{n},\delta) = \frac{\mathrm{per}(M(\mathbf{n},\delta))}{n_1!\cdot\ldots\cdot n_k!}$.

Let δ^{-i} be the $(n-1)\times k$ matrix obtained by eliminating the i^{th} row of δ. Expansion of the permanent by minors yields the following recursive formulas:

Corollary 1. *For all* $i = 1,\ldots,n$,

$$B(\mathbf{n},\delta) = \sum_{j:n_j>0} \delta_{ij}\cdot B((n_1,\ldots,n_j-1,\ldots,n_k),\delta^{-i}).$$

Corollary 2. *For all* $j = 1,\ldots,k$ *such that* $n_j > 0$,

$$B(\mathbf{n},\delta) = \frac{1}{n_j}\sum_{i=1}^{n} \delta_{ij}\cdot B((n_1,\ldots,n_j-1,\ldots,n_k),\delta^{-i}).$$

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