Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 40 (1999), No. 2, 399-415.

Orthogonal Gelfand-Zetlin Algebras I

Volodymyr Mazorchuk

Mechanics and Mathematics Department, Kyiv Taras Shevchenko University
64, Volodymyrska st., 252033 Kyiv, Ukraine
e-mail: mazorchu@uni-alg.kiev.ua

Abstract. We define a class of associative algebras which are similar to the enveloping algebra of $gl(n,\mathbb{C})$. We construct a family of simple Verma and generalized Verma modules over such algebras. We also investigate the two simplest classes of such algebras in detail. For the first class we construct all finite-dimensional modules and investigate an $\mathcal{O}$-category and for the second, we classify all simple weight modules.

1. Introduction

Several examples of algebras which are similar to the enveloping algebras of simple finite-dimensional complex Lie algebras appeared during recent years (see for example [1], [10], [13]). So far most examples were obtained by generalization of $sl(2,\mathbb{C})$ algebra as in [1], [13]. Careful study of such algebras made it possible to obtain deep results even for more difficult algebras [2] which are tensor products of some copies of algebras similar to $sl(2,\mathbb{C})$. Recently, some attempts to define a reasonable generalization for higher rank algebras [12] also appeared.

In the present paper we define a new family of associative algebras that arose from the outstanding paper [8]. Using the formulae describing an action of generating elements of $gl(n,\mathbb{C})$ on finite-dimensional modules we define new associative algebras that we call Orthogonal Gelfand-Zetlin algebras (OGZ-algebras). It seems that the representation theory of such algebras possesses several features in common with the representation theory of $gl(n,\mathbb{C})$.

We show that the family of OGZ-algebras contains $U(gl(n,\mathbb{C}))$, polynomial algebras and extended Heisenberg algebra. We define OGZ-algebra as an operator algebra generated by

0138-4821/93 \$ 2.50

special operators in infinite-dimensional space. Regretfully, this definition of OGZ-algebras is not quite handy. It is not too easy to work with it since there is no analogue of PBW-theorem.

Positive feature of such definition is that any OGZ-algebra appears both with a huge family of modules obtained by a specialization of defining formulae. This allows us to investigate weight modules over OGZ-algebras in the two simplest cases: the first one is an analogue to the case of $sl(2,\mathbb{C})$ and the second one is the case of algebras, when defining formulae contain only polynomial coefficients (see formulae (1), (2)).

In the subsequent papers we will discuss some questions concerning submodule structure of Verma and generalized Verma modules over arbitrary OGZ-algebras and describe all finite-dimensional simple modules in terms of highest weight.

Let us briefly describe the structure of the paper. In Section 2 we collect all basic preliminaries. In Section 3 we define OGZ-algebras and describe their basic properties. In Section 4 we prove that $U(gl(n,\mathbb{C}))$ is an OGZ-algebra (Theorem 1). In Section 5 we obtain some identities in $\mathcal{U}$ and describe the canonical Harish-Chandra subalgebra in $\mathcal{U}$ (Corollary 1). In Section 6 we give some examples of OGZ-algebras. In Sections 7–8 we construct two families of simple $\mathcal{U}$-modules which are analogues of Verma and generalized Verma modules over Lie algebras. In Sections 9–11 we investigate first special class of OGZ-algebras which are similar to $sl(2,\mathbb{C})$. We describe all simple finite-dimensional modules (Theorem 4) and prove the BGG-duality in the corresponding category $\mathcal{O}$ (Theorem 5) in this case. In the last Section 12 we classify simple weight modules for another special class of OGZ-algebras (Theorem 6).

2. Preliminaries

Let $\mathbb{C}$ denote the field of complex numbers; $\mathbb{Z}$, the ring of integers; $\mathbb{N}$, the set of all positive integers and $\mathbb{Z}_+$, the set of all non-negative integers. Throughout this paper we fix a field $\mathbb{F}$ of zero characteristic. *Associative algebra* will mean *associative algebra with identity.*

Fix $n \in \mathbb{N}$ and $r = (r_1, r_2, \dots, r_n) \in \mathbb{N}^n$ and set $k = |r| = \sum_{i=1}^{n} r_i$. Consider a vector space $\mathcal{L} = \mathcal{L}(\mathbb{F}, r)$ of dimension k. We will call the elements of $\mathcal{L}$ *tableaux* and will consider them as double indexed families

$$[\,l\,] = \{l_{ij} \mid i = 1, \dots, n; j = 1, \dots, r_i\}.$$

For $[\,l\,] \in \mathcal{L}$ and $i \in \{1, 2, \dots, n\}$ we will denote by $[\,l\,]_i = \{l_{ij} \mid j = 1, \dots, r_i\}$ the i-th row of $[\,l\,]$. The element r will be called the *signature* of $[\,l\,]$. By the *rank* of $[\,l\,]$ we will mean $\mathrm{rank}([\,l\,]) = n - 1$. We will denote by $\delta^{ij} = [\,\delta^{ij}\,]$, $1 \le i \le n$, $1 \le j \le r_i$, the Kronecker tableau, i.e. $\delta^{ij}_{ij} = 1$ and $\delta^{ij}_{pq} = 0$ for $p \ne i$ or $q \ne j$. Denote by $\mathcal{L}_0$ the subset of $\mathcal{L}$ that consists of all $[\,l\,]$ satisfying the following conditions:

1. $l_{nj} = 0$, $j = 1, \dots, r_n$;
2. $l_{ij} \in \mathbb{Z}$, $1 \le i \le n-1$, $1 \le j \le r_i$.

3. Orthogonal Gelfand-Zetlin algebras

Fix some $n \in \mathbb{N}$ and $r = (r_1, \dots, r_n) \in \mathbb{N}^n$. Set $k = |r|$. Consider a field Λ of rational functions in k variables λ_{ij}, $1 \leq i \leq n$, $1 \leq j \leq r_i$. Let $[\mathfrak{l}] \in \mathcal{L}(\Lambda, r)$ be the tableau defined by $\mathfrak{l}_{ij} = \lambda_{ij}$, $1 \leq i \leq n$, $1 \leq j \leq r_i$.

Consider a vector space M over Λ with the base $v_{[\mathfrak{i}]}$, $[\mathfrak{i}] \in [\mathfrak{l}] + \mathcal{L}_0$ (here $[\mathfrak{i}]$ means formal index and thus M is infinite-dimensional over Λ). For $[\mathfrak{i}] \in [\mathfrak{l}] + \mathcal{L}_0$, $1 \leq i \leq n-1$ and $1 \leq j \leq r_i$ denote

$$a_{ij}^{\pm}([\mathfrak{i}]) = \mp \frac{\prod_{m} (\mathfrak{i}_{i\pm 1\, m} - \mathfrak{i}_{ij})}{\prod_{m \neq j} (\mathfrak{i}_{im} - \mathfrak{i}_{ij})}.$$

Define Λ-linear operators $X_i^{\pm} : M \to M$, $1 \leq i \leq n-1$ by

$$X_i^{\pm} v_{[\mathfrak{i}]} = \sum_{j=1}^{r_i} a_{ij}^{\pm}([\mathfrak{i}]) v_{[\mathfrak{i}]+[\delta^{ij}]} \tag{1}$$

where $[\mathfrak{i}] \in [\mathfrak{l}] + \mathcal{L}_0$.

Let γ_{ij}, $1 \leq i \leq n$, $1 \leq j \leq r_i$ denote the j-th elementary symmetrical polynomial in $\lambda_{i1}, \dots, \lambda_{ir_i}$. Define Λ-linear operators $d_{ij} : M \to M$, $1 \leq i \leq n$, $1 \leq j \leq r_i$ by

$$d_{ij} v_{[\mathfrak{i}]} = \gamma_{ij}(\mathfrak{i}_{i1}, \dots, \mathfrak{i}_{ir_i}) v_{[\mathfrak{i}]}. \tag{2}$$

Consider an associative operator algebra $\mathcal{U}$ (over $\mathbb{F}$), generated by elements $X_i^{\pm}$, $i = 1, \dots, n-1$ and d_{ij}, $1 \leq i \leq n$, $1 \leq j \leq r_i$. We will call it the *Orthogonal Gelfand-Zetlin algebra (OGZ-algebra) of signature* r. Equivalently, $\mathcal{U}$ is the minimal $\mathbb{F}$-subalgebra in the algebra of all Λ-linear transformations of M, containing $X_i^{\pm}$ and d_{ij} for all $1 \leq i \leq n$, $1 \leq j \leq r_i$. The reason of this name will be explained in Section 4.

The subalgebra $\Gamma \subset \mathcal{U}$ generated by elements d_{ij}, $1 \leq i \leq n$, $1 \leq j \leq r_i$ will be called *basic Gelfand-Zetlin subalgebra (basic GZ-subalgebra)* of $\mathcal{U}$. Clearly, Γ is a commutative polynomial algebra in k variables.

We will denote by Γ_i, $i = 1, \dots, n$ the subalgebra of Γ generated by d_{ij}, $j = 1, \dots, r_i$. Clearly, Γ_i is a polynomial algebra in r_i variables.

Consider a polynomial algebra Ω in k variables χ_{ij}, $1 \leq i \leq n$, $1 \leq j \leq r_i$. Identify Ω with the algebra of polynomial functions on $\mathcal{L}(\Lambda, r)$ putting $\chi_{ij}([\mathfrak{i}]) = \mathfrak{i}_{ij}$ for all possible i and j. Define a homomorphism $\varphi : \Gamma \to \Omega$ by $d_{ij} \mapsto \gamma_{ij}(\chi_{i1}, \dots, \chi_{ij})$ for all possible i and j.

The symmetrical group S_{r_i} acts on Ω in natural way permuting χ_{ij} with fixed i. Thus, the direct product $S = \prod_{i=1}^{n} S_{r_i}$ acts on Ω. Clearly, φ is a monomorphism and its image coincides with the algebra of S-invariants in Ω. We identify Γ with $\varphi(\Gamma)$.

We denote by $\mathcal{U}^{\pm}$ the subalgebras of $\mathcal{U}$ generated by $X_i^{\pm}$, $1 \leq i \leq n-1$.

4. $U(gl(n,\mathbb{C}))$ is an OGZ-algebra

In this section we will prove that the class of OGZ-algebras is big enough to contain all enveloping algebras $U(gl(n,\mathbb{C}))$. To prove this we need some auxiliary notations and lemmas.

Let $\mathcal{U}$ be the OGZ-algebra of signature $r=(1,2,\dots,n)$, $\mathfrak{G}=gl(n,\mathbb{C})$ be reductive finite-dimensional complex Lie algebra of $n\times n$ matrices, $U(\mathfrak{G})$ be the universal enveloping algebra of $\mathfrak{G}$ and $\Gamma(\mathfrak{G})$ be the Gelfand-Zetlin subalgebra of $\mathfrak{G}$, [7]. We will also denote by e_{ij}, $i,j\in\{1,\dots,n\}$ the standard matrix units.

Lemma 1. *The map $e_{ii+1}\mapsto X_i^+$, $e_{i+1\,i}\mapsto X_i^-$, $i=1,\dots,n-1$; $e_{11}\mapsto d_{11}+1$ defines a structure of $\mathfrak{G}$-module ($=U(\mathfrak{G})$-module) on M.*

Proof. Follows from [7, Theorem 30]. □

Denote by Λ' the $\mathbb{C}$-subalgebra of Λ, generated by λ_{ij}, $1\le i\le n$, $1\le j\le r_i$ and $(\lambda_{ij}-\lambda_{im}-t)^{-1}$, $1\le i\le n-1$, $1\le j<m\le r_i$, $t\in\mathbb{Z}$. For $[\,l\,]\in\mathcal{L}(\mathbb{C},r)$ with $l_{im}-l_{ij}\notin\mathbb{Z}$ for $1\le i\le n-1$, $1\le j<m\le r_i$ consider a homomorphism $\Phi:\Lambda'\to\mathbb{C}$ defined as follows: $\Phi(\lambda_{ij})=l_{ij}$, $1\le i\le n$, $1\le j\le r_i$. Clearly, Φ induces a homomorphism from a free Λ'-module M' with the base $v_{[\mathfrak{i}]}$, $[\,\mathfrak{i}\,]\in[\,\mathfrak{l}\,]+\mathcal{L}_0$ in $\mathcal{L}(\mathbb{C},r)$, moreover, $\Phi([\,\mathfrak{l}\,])=[\,l\,]$. Since $\Phi|_{[\mathfrak{l}]+\mathcal{L}_0}$ is a monomorphism it follows that it induces a homomorphism from the algebra of all Λ'-linear operators on M' into the algebra of $\mathbb{C}$-linear operators on the $\mathbb{C}$-space $M_{[l]}$ with the base $v_{[t]}$, $[\,t\,]\in[\,l\,]+\mathcal{L}_0$. We will call $M_{[l]}$ the *specialization* of M with respect to $[\,l\,]$. Clearly, Φ defines on $M_{[l]}$ a structure of $\mathcal{U}$-module.

Lemma 2. *Φ defines a structure of $\mathfrak{G}$-module on $M_{[l]}$.*

Proof. Follows immediately from Lemma 1. □

The following statement is the key observation of this section. One can consider it as an analogue of the well-known Harish-Chandra theorem [6, Theorem 2.5.7].

Lemma 3. *For any $0\ne u\in U(\mathfrak{G})$ there exists $[\,l\,]\in\mathcal{L}(\mathbb{C},r)$, $l_{ij}-l_{im}\notin\mathbb{Z}$, $1\le i\le n-1$, $1\le j<m\le r_i$ such that u acts non-trivially on $M_{[l]}$.*

Proof. Consider $0\ne u\in U(\mathfrak{G})$. Let $\mathfrak{G}=\mathfrak{N}_-\oplus\mathfrak{H}\oplus\mathfrak{N}_+$ be the standard triangular decomposition of $\mathfrak{G}$ and $\Delta=\Delta_-\cup\Delta_+$ be the corresponding decomposition of the root system. Denote by X_α, $\alpha\in\Delta$ the elements from the fixed Weyl-Chevalle basis.

We can assume that there exists $\mu\in\mathfrak{H}^*$ such that $[h,u]=\mu(h)u$ for all $h\in\mathfrak{H}$. Clearly, there exists a sequence $X_{\alpha_1},\dots,X_{\alpha_t}$ such that $\mu+\alpha_1+\dots+\alpha_t=\sum_{\beta\in\Delta_+}n_\beta\beta$, where $n_\beta\in\mathbb{Z}_+$.

By Harish-Chandra theorem [6, Theorem 2.5.7] an element $u'=uX_{\alpha_1}\dots X_{\alpha_t}$ acts non-trivially on some finite-dimensional simple $\mathfrak{G}$-module F. By [8] each simple finite-dimensional module possesses a base indexed by the special set of tableaux of signature r. Moreover, the action of generating elements in this base is defined by formulae (1), (2). One can choose $\alpha_1,\dots,\alpha_t$ such that there exists $v\in F$, $uv\ne 0$ and the action of u on v is defined only by GZ-formulae [8].

Thus, there are non-trivial rational functions in coefficients of the tableau corresponding to v defining this action. It follows that any of these functions should have a non-trivial value in some specialization $M_{[l]}$, since the set $[\,l\,] \in \mathcal{L}(\mathbb{C}, r)$, $l_{ij} - l_{im} \notin \mathbb{Z}$, $1 \leq i \leq n-1$, $1 \leq j < m \leq r_i$ cannot be zero set of a non-trivial rational function. This completes the proof. □

Theorem 1. *$\mathcal{U} \simeq U(\mathfrak{G})$.*

Proof. By Lemma 1 we have $\mathcal{U} \simeq U(\mathfrak{G})/I$, where I is an annihilator of M. By Lemma 3 we obtain $I = 0$ and the statement follows. □

Remark 1. According to Lemma 3 it is clear why we call our algebras OGZ-algebras. The reason is in GZ-formulae [8], which define an action of generating elements of $U(gl(n, \mathbb{C}))$ on simple finite-dimensional module in some orthogonal base.

5. Some identities in $\mathcal{U}$

It follows from the definition of $\mathcal{U}$ that the generating elements satisfy several canonical identities. We describe some of them in this section.

Lemma 4. *For any $1 \leq i \leq n-1$, $[X_i^+, X_i^-] \in \Gamma$.*

Proof. Fix some $i \in \{1, \dots, n-1\}$. Consider an element $[X_i^+, X_i^-]v_{[\mathfrak{i}]}$ for $[\,\mathfrak{i}\,] \in [\,\mathfrak{l}\,] + \mathcal{L}_0$. By the definition

$$
\begin{aligned}
[X_i^+, X_i^-]v_{[\mathfrak{i}]} = (X_i^+ X_i^- - X_i^- X_i^+)v_{[\mathfrak{i}]} = \\
= \sum_{m=1}^{r_i}\sum_{j=1}^{r_i} a_{im}^+([\,\mathfrak{i}\,] - [\,\delta^{ij}\,])a_{ij}^-([\,\mathfrak{i}\,])v_{[\mathfrak{i}]-[\delta^{ij}]+[\delta^{im}]} - \\
- \sum_{j=1}^{r_i}\sum_{m=1}^{r_i} a_{ij}^-([\,\mathfrak{i}\,] + [\,\delta^{im}\,])a_{im}^+([\,\mathfrak{i}\,])v_{[\mathfrak{i}]-[\delta^{ij}]+[\delta^{im}]} = \\
= \sum_{j=1}^{r_i}(a_{ij}^+([\,\mathfrak{i}\,] - [\,\delta^{ij}\,])a_{ij}^-([\,\mathfrak{i}\,]) - a_{ij}^-([\,\mathfrak{i}\,] + [\,\delta^{ij}\,])a_{ij}^+([\,\mathfrak{i}\,]))v_{[\mathfrak{i}]}.
\end{aligned}
$$

Set

$$
\begin{aligned}
f(\mathfrak{i}_{i-1\,1}, \dots, \mathfrak{i}_{i-1\,r_{i-1}}, \mathfrak{i}_{i1}, \dots, \mathfrak{i}_{ir_i}, \mathfrak{i}_{i+1\,1}, \dots, \mathfrak{i}_{i+1\,r_{i+1}}) = \\
= \frac{f_1(\mathfrak{i}_{i-1\,1}, \dots, \mathfrak{i}_{i-1\,r_{i-1}}, \mathfrak{i}_{i1}, \dots, \mathfrak{i}_{ir_i}, \mathfrak{i}_{i+1\,1}, \dots, \mathfrak{i}_{i+1\,r_{i+1}})}{f_2(\mathfrak{i}_{i1}, \dots, \mathfrak{i}_{ir_i})} = \\
= \sum_{j=1}^{r_i}(a_{ij}^+([\,\mathfrak{i}\,] - [\,\delta^{ij}\,])a_{ij}^-([\,\mathfrak{i}\,]) - a_{ij}^-([\,\mathfrak{i}\,] + [\,\delta^{ij}\,])a_{ij}^+([\,\mathfrak{i}\,])).
\end{aligned}
$$

Clearly, $f_2 = \prod_{1 \leq j < k \leq r_i} (\mathfrak{i}_{ik} - \mathfrak{i}_{ij})$. By the direct calculations we obtain that

$$
f_1(\mathfrak{i}_{i-1\,1}, \dots, \mathfrak{i}_{i-1\,r_{i-1}}, \mathfrak{i}_{i1}, \dots, \mathfrak{i}_{ij-1}, \mathfrak{i}_{im}, \mathfrak{i}_{ij+1}, \dots, \mathfrak{i}_{im-1}, \mathfrak{i}_{im}, \dots, \mathfrak{i}_{ir_i}, \mathfrak{i}_{i+1\,1}, \dots, \mathfrak{i}_{i+1\,r_{i+1}}) = 0
$$

and thus f is a polynomial function in $\mathfrak{i}_{i-1\,1}, \ldots, \mathfrak{i}_{i+1\,r_{i+1}}$. One can easily see that f does not change after elementary transpositions of variables $\mathfrak{i}_{i-1\,j} \leftrightarrow \mathfrak{i}_{i-1\,m}$, $\mathfrak{i}_{ij} \leftrightarrow \mathfrak{i}_{im}$, $\mathfrak{i}_{i+1\,j} \leftrightarrow \mathfrak{i}_{i+1\,m}$.

The statement of the lemma now follows from the main theorem on symmetrical polynomials. □

Lemma 5. *For* $1 \leq i \leq n-1$,

1) $[d_{i1}, X_i^{\pm}] = \pm X_i^{\pm}$.
2) $[d_{ij}, X_i^{\pm}] = f_{ij}^{\pm}(\gamma_{i1}, \ldots, \gamma_{ij-1}) X_i^{\pm}$ *for some polynomial* $f_{ij}^{\pm}$.

Proof. For $1 \leq i \leq n-1$ and $[\,\mathfrak{i}\,] \in [\,\mathfrak{l}\,] + \mathcal{L}_0$ we have

$$[d_{i1}, X_i^{\pm}] v_{[\mathfrak{i}]} = \sum_{j=1}^{r_i} \left((\mathfrak{i}_{i1} + \cdots + \mathfrak{i}_{ir_i} \pm 1) a_{ij}^{\pm}([\,\mathfrak{i}\,]) - (\mathfrak{i}_{i1} + \cdots + \mathfrak{i}_{ir_i}) a_{ij}^{\pm}([\,\mathfrak{i}\,]) \right) v_{[\mathfrak{i}]+[\delta^{ij}]} =$$

$$= \pm \sum_{j=1}^{r_i} a_{ij}^{\pm}([\,\mathfrak{i}\,]) v_{[\mathfrak{i}]+[\delta^{ij}]} = \pm X_i^{\pm} v_{[\mathfrak{i}]}$$

and the first statement follows.

The second follows analogously from the fact that for any non-trivial polynomial $f \in \mathbb{C}[x]$ and for any $a \in \mathbb{C}$ holds $\deg(f(x-a) - f(x)) < \deg f(x)$. □

The following identities are obtained at the same way as in [7, Lemma 25]. Any $x \in M$ can be written in the form $x = \sum_{[\mathfrak{i}] \in [\mathfrak{l}] + \mathcal{L}_0} x_{[\mathfrak{i}]} v_{[\mathfrak{i}]}$, where $x_{[\mathfrak{i}]} \in \Lambda$. For $u \in \mathcal{U}$ and $[\,\mathfrak{i}\,] \in [\,\mathfrak{l}\,] + \mathcal{L}_0$ denote by $\mathcal{L}(u, [\,\mathfrak{i}\,])$ the set of all tableaux $[\,\tau\,] \in [\,\mathfrak{l}\,] + \mathcal{L}_0$ such that $(uv_{[\mathfrak{i}]})_{[\tau]} \neq 0$.

Lemma 6. *The set* $\mathcal{L}(u, [\,\mathfrak{i}\,]) - [\,\mathfrak{i}\,]$ *does not depend on* $[\,\mathfrak{i}\,]$.

Proof. By definition of $\mathcal{U}$ all $(uv_{[\mathfrak{i}]})_{[\tau]}$ are non-trivial elements in Λ. Then the statement of the lemma follows from the fact that a shift of variables maps a non-trivial rational function to a non-trivial rational function. □

We will denote the set $\mathcal{L}(u, [\,\mathfrak{i}\,]) - [\,\mathfrak{i}\,]$ by $\mathcal{L}_u$. Clearly, $\mathcal{L}_u$ is finite and $\mathcal{L}_u \subset \mathcal{L}_0$. Thus, we have

$$uv_{[\mathfrak{i}]} = \sum_{[\tau] \in \mathcal{L}_u} \theta(u, [\,\mathfrak{i}\,], [\,\tau\,]) v_{[\mathfrak{i}]+[\tau]},$$

where $\theta(u, [\,\mathfrak{i}\,], [\,\tau\,]) = (uv_{[\mathfrak{i}]})_{[\mathfrak{i}]+[\tau]} \neq 0$.

Any $[\,\tau\,] \in \mathcal{L}(\Lambda, r)$ defines an automorphism $\chi \mapsto \chi^{[\tau]}$ of Ω, where $\chi_{ij}^{\tau} = \chi_{ij} + \tau_{ij}$ for all possible i and j. For any $z \in \Gamma$ and $u \in \mathcal{U}$ set

$$F_{u,z}(T, \chi) = \prod_{[\tau] \in \mathcal{L}_u} (T - z^{\tau}).$$

Clearly, $F_{u,z} \in \Gamma(T)$.

Lemma 7. *Let* $z \in \Gamma$ *and* $F_{u,z} = \sum_{i\in I} T^i g_i$, $g_i \in \Gamma$. *Then* $\sum_{i\in I} z^i u g_i = 0$.

Proof. For $[\mathfrak{i}] \in [\mathfrak{l}] + \mathcal{L}_0$ we have

$$\sum_{i\in I} z^i u g_i v_{[\mathfrak{i}]} = \sum_{i\in I} z^i u g_i([\mathfrak{i}]) v_{[\mathfrak{i}]} = \sum_{i\in I} z^i g_i([\mathfrak{i}]) \sum_{[\tau]\in\mathcal{L}_u} \theta(u,[\mathfrak{i}],[\tau]) v_{[\mathfrak{i}]+[\tau]} =$$
$$\sum_{[\tau]\in\mathcal{L}_u} \theta(u,[\mathfrak{i}],[\tau]) \sum_{i\in I} z([\mathfrak{i}]+[\tau])^i g_i([\mathfrak{i}]) v_{[\mathfrak{i}]+[\tau]} =$$
$$\sum_{[\tau]\in\mathcal{L}_u} \theta(u,[\mathfrak{i}],[\tau]) F_{u,z}(z^{[\tau]}([\mathfrak{i}]),[\mathfrak{i}]) v_{[\mathfrak{i}]+[\tau]} = 0.$$

Thus $\sum_{i\in I} z^i u g_i = 0$. □

Corollary 1. Γ *is a Harish-Chandra subalgebra in* $\mathcal{U}$ *in the sense of* [7] *(i.e.* $\Gamma u \Gamma$ *is finitely generated both as right and left* Γ*-module for any* $u \in \mathcal{U}$*).*

Proof. Let $s = \deg F_{u,z}$. By Lemma 7 we have $z^s \in \sum_{i=1}^{s-1} z^i u \Gamma$ and thus $\Gamma u \Gamma$ is a finitely generated right Γ module since Γ is finitely generated.

The map $X_i^{\pm} \mapsto X_i^{\mp}$ can be continued to the canonical involution on $\mathcal{U}$ which maps Γ to Γ. Thus $\Gamma u \Gamma$ is also finitely generated as left Γ-module. □

6. Examples

In this section we give some familiar examples of OGZ-algebras.

6.1. $U(gl(n,\mathbb{C}))$

As shown in Section 4 an algebra $U(gl(n,\mathbb{C}))$ is OGZ-algebra of signature $r = (1,2,\dots,n)$.

6.2. Polynomial algebra

A polynomial algebra is an OGZ-algebra of rank 0. By the main theorem on symmetrical polynomials an OGZ-algebra $\mathcal{U}$ of signature $r = (r_1)$ is a polynomial algebra in r_1 variables. Clearly, any simple irreducible module over $\mathcal{U}$ is defined (up to the action of S_{r_1}) by a vector $(l_j)_{j=1}^{r_1} \in \mathbb{C}^n$ which defines the eigenvalues of the generating elements d_{1j}, $1 \le j \le r_1$.

6.3. Extended Heisenberg algebra

One can show that in the case $r = (1,1)$ an algebra $\mathcal{U}$ is a trivial central extension of an algebra $\mathcal{U}'$ generated by X, Y, H with relations $[X,Y] = -1$, $[H,X] = X$, $[H,Y] = Y$ which is an extended Heisenberg algebra. Clearly, this algebra has no simple finite-dimensional modules. This fact is generalized in Section 12 to OGZ-algebras of signature $(1,1,\dots,1)$.

7. Verma modules in general position

OGZ-algebras appear together with a huge family of modules. In this section we construct some analogue of Verma modules over an OGZ-algebra. These modules could be obtained directly from the definition of an algebra.

At the same way as in Section 4 to each $[l] \in \mathcal{L}(\mathbb{C}, r)$, $l_{ij} - l_{im} \notin \mathbb{Z}$, $1 \le i \le n-1$, $1 \le j < m \le r_1$ (such $[l]$ will be called *quasi-generic*) corresponds some $\mathcal{U}$-module $M_{[l]}$ with a $\mathbb{C}$-base $v_{[t]}$, $[t] \in [l] + \mathcal{L}_0$ and the action of generated elements given by (1), (2). In the mentioned formulae it is necessary to replace $\mathfrak{l}_{ij}$ with l_{ij} for all i and j.

Lemma 8. *Suppose that* $[l] \in \mathcal{L}(\mathbb{C}, r)$ *is quasi-generic. Let* $K \subset [t] + \mathcal{L}_0$ *be such that for any* $[t] \in K$ *and any* $1 \le i \le n-1$,

$$X_i^{\pm} v_{[t]} = \sum_{[s] \in K} a_{[s]} v_{[s]}.$$

Then the space N *spanned by* $v_{[s]}$, $[s] \in K$ *is an* $\mathcal{U}$*-submodule in* $M_{[l]}$.

Proof. Follows from the definition of $\mathcal{U}$. □

A non-zero element v from a $\mathcal{U}$-module V will be called *primitive* provided $X_i^+ v = 0$ for all $1 \le i \le n-1$. In this section we construct a family of simple $\mathcal{U}$-modules generated by primitive elements, as submodules in $M_{[l]}$ for special $[l]$. From now on we assume that the signature vector r satisfies the condition $r_j \ge r_k$ for all possible $j > k$.

Consider a map $\Phi : \mathbb{C}^{r_n} \to \mathcal{L}(\mathbb{C}, r)$ defined as follows: $\Phi((a_1, \dots, a_{r_n})) = [t]$, where $t_{ij} = a_j$, $1 \le i \le n$, $1 \le j \le r_i$. This map is well-defined since $r_j \ge r_k$ for $j > k$.

Let $[l] = \Phi((a_1, \dots, a_{r_n}))$, $a_j - a_m \notin \mathbb{Z}$ for $1 \le j < m \le r_n$. Consider the corresponding specialization $M_{[l]}$. Let $K = K([l])$ be a subset in $[l] + \mathcal{L}_0$ consisting of all elements $[t]$ which satisfy the following conditions:

1. $l_{ij} - t_{ij} \in \mathbb{Z}_+$, $1 \le i \le n-1$, $1 \le j \le r_i$;
2. $t_{ij} \le t_{i+1\,j}$, $1 \le i \le n-1$, $1 \le j \le r_i$.

Lemma 9. *Suppose that* $[t] \in K$ *and* $a_{ij}^+([t]) \neq 0$ *for some* i *and* j. *Then* $[t] + [\delta^{ij}] \in K$.

Proof. Since $a_{ij}^+([t]) \neq 0$ it follows that $t_{ij} \neq t_{i+1\,m}$ for all possible m and thus $t_{ij} < t_{i+1\,j}$. This implies that $[t] + [\delta^{ij}] \in K$. □

Theorem 2. $M([l]) = \langle v_{[t]} \,|\, [t] \in K \rangle$ *is a simple* $\mathcal{U}$*-submodule of* $M_{[l]}$ *generated by the primitive element* $v_{[l]}$.

Proof. $M([l])$ is a $\mathcal{U}$-submodule by Lemma 9 and Lemma 8. It follows from direct calculation using formula (1), that $M([l])$ is generated by $v_{[l]}$. Clearly, $v_{[l]}$ is primitive.

Since $\mathcal{U}^+ v$ is finite-dimensional for any $v \in M([l])$, it follows that each submodule of $M([l])$ contains a primitive element. From formula (1) and definition of K we deduce that the only primitive elements in $M([l])$ are $const \cdot v_{[t]}$. Thus, $M([t])$ is simple. □

It is rather natural to call $M([l])$ a Verma module over $\mathcal{U}$ since when $\mathcal{U} \simeq U(gl(n, \mathbb{C}))$ it follows that $M([l])$ is a Verma module over $gl(n, \mathbb{C})$. Note, that we do not construct the complete family of Verma modules but only a class of simple ones arising directly from the definition of $\mathcal{U}$.

8. Generalized Verma modules in general position

In this section we construct a family of simple modules over $\mathcal{U}$ which are similar to generalized Verma modules induced from a parabolic subalgebra of a Lie algebra (see [5], [11]). Our arguments are analogues to those in Section 7.

Fix $m < n$ and let $\mathcal{U}'$ be a subalgebra in $\mathcal{U}$ generated by $X_i^{\pm}$, $1 \leq i \leq m-1$ and d_{ij}, $1 \leq i \leq m$, $1 \leq j \leq r_i$. We will construct a family of $\mathcal{U}$-modules which can be viewed as modules induced from $\mathcal{U}'$-modules. Analogously to the previous section, we have to assume that $r_j \geq r_k$ for $j > k \geq m$.

Set $r' = (r_1, \dots, r_{m-1})$. Consider a map $\Phi : \mathbb{C}^{r_n} \oplus \mathcal{L}(\mathbb{C}, r') \to \mathcal{L}(\mathbb{C}, r)$ defined as follows: $\Phi((a_1, \dots, a_{r_n}), [\,t\,]) = [\,l\,]$, where $[\,t\,] \in \mathcal{L}(\mathbb{C}, r')$ and $l_{ij} = t_{ij}$, $1 \leq i \leq m-1$, $1 \leq j \leq r_i$; $l_{ij} = a_j$, $m \leq i \leq n$, $1 \leq j \leq r_i$.

Let $[\,l\,] = \Phi(a, [\,t\,])$ for some $a \in \mathbb{C}^{r_i}$, $[\,t\,] \in \mathcal{L}(\mathbb{C}, r')$. For the rest of the section we suppose that $[\,l\,]$ satisfies the following conditions:

1. $l_{ij} - l_{im} \notin \mathbb{Z}$, $1 \leq i \leq n-1$, $1 \leq j < m \leq r_i$;
2. $l_{ij} - l_{i+1\,m} \notin \mathbb{Z}$, $1 \leq i \leq m-1$, $1 \leq j \leq r_i$, $1 \leq m \leq r_{i+1}$.

Consider a subset $K = K([\,l\,]) \subset [\,l\,] + \mathcal{L}_0$ consisting of all elements $[\,t\,]$ such that

1. $l_{ij} - t_{ij} \in \mathbb{Z}$, $1 \leq i \leq m-1$, $1 \leq j \leq r_i$;
2. $l_{ij} - t_{ij} \in \mathbb{Z}_+$, $m \leq i \leq n-1$, $1 \leq j \leq r_i$;
3. $t_{ij} \leq t_{i+1\,j}$, $m \leq i \leq n-1$, $1 \leq j \leq r_i$.

Lemma 10. *Suppose that $[\,t\,] \in K$ and $a_{ij}^{+}([\,t\,]) \neq 0$ for some i and j. Then $[\,t\,] + [\,\delta^{ij}\,] \in K$.*

Proof. For $1 \leq i \leq m-1$ the statement is obvious. For $i > m-1$ the proof is analogous to that of Lemma 9. □

Theorem 3. *The $\mathbb{C}$-subspace $M([\,l\,]) \subset M_{[l]}$ with the base $v_{[t]}$, $[\,t\,] \in K$ is a simple $\mathcal{U}$-submodule in $M_{[l]}$. Moreover, $M([\,l\,])$ is generated by $v_{[l]}$.*

Proof. Analogous to that of Theorem 2. □

It is natural to call the modules constructed in this section generalized Verma modules. In the case $\mathcal{U} \simeq U(gl(n, \mathbb{C}))$ the module $M([\,l\,])$ is a generalized Verma module induced from a generic Gelfand-Zetlin module [11]. Note, that the family of modules constructed in this section is not a complete family of generalized Verma modules over $\mathcal{U}$.

9. Orthogonal Gelfand-Zetlin algebras of signature $(1, m)$, basic properties

For the next three sections we assume that $n = 2$ and $r = (1, m)$. In this case it is possible to construct a theory of $\mathcal{U}$-modules almost in the same way as for the Lie algebra $sl(2, \mathbb{C})$. The most interesting feature is that for $m > 2$ there appear some non-trivial effects just as in the case of other algebras considered in [10], [13].

It is shown that an OGZ-algebras of signature $(1, m)$ is a non-trivial extension of algebras similar to enveloping algebras [10], [13]. But it happened that most problems on finite-dimensional modules over OGZ-algebras could be solved in a very simple way using our terminology of tableaux and omitting difficult calculations (as in [13]). Note, that Smith algebras [13] as good as OGZ-algebras of signature $(1, m)$ are the so-called generalized Weyl algebras (in the sense of [1]). For Smith algebras the corresponding ground ring is $\mathbb{C}[H]$ and for OGZ-algebras the corresponding ground ring is $\mathbb{C}[d_{11}, d_{21}, \dots, d_{2m}]$. Complete classification of simple modules over generalized Weyl algebras is known [1]. Nevertheless, we obtain the classification of simple finite-dimensional modules over OGZ-algebras in easier terms than in [1], [13].

Let A_i, $i = 1, \dots, m+3$ be the generating elements $X_1^{\pm}$, d_{11}, d_{2j}, $j = 1, \dots, m$ ordered in arbitrary way. For the non-negative integers $i_1, \dots, i_{m+3}$ set $F(i_1, \dots, i_{m+3}) = A_1^{i_1} \cdot \dots \cdot A_{m+3}^{i_{m+3}}$. To simplify the notations we write $X = X_1^+$ and $Y = X_1^-$.

Proposition 1. *The elements $F(i_1, \dots, i_{m+3})$ span $\mathcal{U}$ as a vector space.*

Proof. Follows from Lemma 4 and Lemma 5 by standard arguments. □

Lemma 11. *The center Z of $\mathcal{U}$ coincides with Γ_2.*

Proof. Since $[X, Y] \in \Gamma$ by Lemma 4, it follows that $Z \subset \Gamma$. Clearly, $\Gamma_2 \subset Z$, any polynomial in d_{11} does not commute with X and the lemma follows. □

Lemma 12. *$\mathcal{U}$ is almost commutative, i.e. there is a standard filtration on $\mathcal{U}$ such that the associated graded algebra is commutative.*

Proof. One can prove this using the following $\mathbb{Z}$-grading on $\mathcal{U}$: $X^{i_1} Y^{i_2} d_{11}^{i_3} \dots d_{2m}^{i_{m+3}} \in \mathcal{U}_{i_1 - i_2}$. □

10. Orthogonal Gelfand-Zetlin algebras of signature $(1, m)$, finite-dimensional modules

In this section we establish the complete theory of Verma modules and describe all finite-dimensional $\mathcal{U}$-modules in the case $r = (1, m)$. We obtain an answer in terms of some group action as in the classical case of Verma modules over complex simple finite-dimensional Lie algebras.

For a $\mathcal{U}$-module V and $\chi \in \Gamma^*$ we set

$$V_\chi = \{v \in V \mid zv = \chi(z)v \text{ for all } z \in \Gamma\}.$$

If $V_\chi \neq 0$ we will call it a *weight space* of V corresponding to the *weight* χ. A module V will be called a *weight module* if

$$V = \bigoplus_{\chi \in \Gamma^*} V_\chi.$$

Each non-zero element from V_χ will be called a *weight element.* Since we identify Γ with $\varphi(\Gamma)$ (see Section 3) it is possible to parametrize $\chi \in \Gamma^*$ by $[\,t\,] \in \mathcal{L}(\mathbb{C}, r)$ in a natural way. Thus if χ is parametrized by $[\,t\,]$ we will write $V_\chi = V_{[\,t\,]}$ and say that an element $v \in V_{[\,t\,]}$ has tableau $[\,t\,]$. For a weight module V we will denote by $\operatorname{Tsupp} V$ the set of all tableaux which parametrize weights of V. Note, that by definition $\mathrm{Tsupp}\, V$ is invariant under the action of S. We recall that an element $v \in V_\chi$ is called *primitive* if $Xv = 0$.

For $[\,l\,] \in \mathcal{L}(\mathbb{C}, r)$ we will denote $M(([\,l\,]))$ the Verma module [6] generated by a primitive element which has tableau $[\,l\,]$.

Lemma 13. *Let F be a simple finite-dimensional $\mathcal{U}$-module. Then F is a weight module generated by a primitive element.*

Proof. Since $[X, [d_{11}, X]] = [X, X] = 0$ it follows from Klienike-Shirokov theorem that X is nilpotent on F and thus there exists a non-trivial primitive element v in F. Since F is simple it follows that v generates F. Clearly, $Y^i v$, $i \geq 0$ span a $\mathcal{U}$-submodule in F which coincides with F since F is simple. One can show by induction in i that $Y^i v$ is a weight element for each $i \geq 0$ and the lemma follows. □

Corollary 2.

1) *Each simple finite-dimensional $\mathcal{U}$-module F is a quotient of some Verma module.*
2) *The unique irreducible quotient of a Verma module is finite-dimensional if and only if the Verma module is not simple.*

Proof. The first statement follows from Proposition 1 which can be reformulated into the following standard universal property of Verma modules: The module generated by a primitive element is a quotient of a Verma module. The second statement is obvious. □

By Corollary 2 to classify all finite-dimensional modules we have to classify all maximal submodules in non-simple Verma modules.

Recall the construction of Verma modules from Section 7. Let $\Phi : \mathbb{C}^m \to \mathcal{L}(\mathbb{C}, r)$ be a map defined as follows: $\Phi((a_1, \dots, a_m)) = [\,l\,]$ where $l_{ij} = a_j$, $1 \leq i \leq 2$, $1 \leq j \leq r_i$. A group S_m acts on $\mathbb{C}^m$ in a natural way $\sigma(a_1, \dots, a_m) = (a_{\sigma(1)}, \dots, a_{\sigma(m)})$ for $(a_1 \dots, a_m) \in \mathbb{C}^m$ and $\sigma \in S_m$. Clearly, this induces an action of S_m on $P = \Phi(\mathbb{C}^m)$.

For $[\,l\,], [\,t\,] \in P$ we set $[\,t\,] < [\,l\,]$ if there exists $\sigma \in S_m$ such that $[\,l\,]_2 = \sigma([\,t\,]_2)$ and $l_{11} - t_{11} \in \mathbb{N}$.

Let $W \simeq S_{m-1}$ be a subgroup in S_m fixing the first element.

For a given $[\,l\,] \in P$ set $K = K([\,l\,]) = \{[\,l\,] - j[\,\delta^{11}\,] \mid j \geq 0\}$. Then the subspace $M([\,l\,]) \subset M_{[l]}$ spanned by $v_{[t]}$, $[\,t\,] \in K$ is a $\mathcal{U}$-submodule in $M_{[l]}$ and is isomorphic to the Verma module generated by a primitive element which has tableau $[\,l\,]$.

Lemma 14. *For $[\,l\,] \in P$, $M([\,l\,]) \simeq M(([\,l\,]))$.*

Proof. Follows from Proposition 1. □

Lemma 15. *Each $\mathcal{U}^-$-free module generated by a primitive element is of the form $M([\,l\,])$ for some $[\,l\,] \in P$.*

Proof. Follows from the formulae (1), (2). □

Lemma 16. *Let $[\,l\,] \in P$ and $\sigma \in W$ then $M([\,l\,]) \simeq M(\sigma([\,l\,]))$.*

Proof. Follows from the definition of d_{2j}, $1 \leq j \leq m$. □

Lemma 17. *Let $[\,l\,] \in P$. A module $M([\,l\,])$ is simple if and only if $l_{21} - l_{2j} \notin \mathbb{N}$ for all $2 \leq j \leq m$ or, equivalently, $M([\,l\,])$ is simple if and only if $\sigma[\,l\,] \not< [\,l\,]$ for any $\sigma \in S_m$.*

Proof. Follows from the formula (1) and construction of the module $M([\,l\,])$. □

Let P^+ be the subset of P consisting from all $[\,l\,]$ such that there exists $j \in \{2, \dots, m\}$ with $l_{21} - l_{2j} \in \mathbb{N}$.

Theorem 4.

1) *Suppose that $[\,l\,] \in P^+$. Then there exists a unique finite-dimensional module $L([\,l\,])$ which is generated by a primitive vector that has tableau $[\,l\,]$.*
2) *Each finite-dimensional module F is isomorphic to $L([\,l\,])$ for some $[\,l\,] \in P^+$.*
3) *$L([\,l\,]) \simeq L([\,t\,])$, $[\,l\,], [\,t\,] \in P^+$ if and only if there exists $\sigma \in W$ such that $[\,l\,] = \sigma([\,t\,])$.*

Proof. The first statement follows from the uniqueness of the simple quotient in a Verma module. The second follows from Lemma 15. The last one follows from Lemma 16. □

The following theorem which is an analogue of the BGG-criterion for the existence of a submodule in Verma a module follows immediately from the discussion above:

Corollary 3. *The following statements are equivalent:*

1) *$M([\,t\,]) \subset M([\,l\,])$.*
2) *$L([\,t\,])$ is a subquotient in a composition series of $M([\,l\,])$.*
3) *There exists $\sigma \in S_m$ such that $[\,t\,] = \sigma([\,l\,]) < [\,l\,]$.*

Proof. Follows from Lemma 17, Theorem 4 and definition of the S_m-action. □

The last theorem describes the set of all simple finite-dimensional modules over $\mathcal{U}$. We emphasize three different cases:

$m = 1$. In this case there are no finite-dimensional modules over $\mathcal{U}$.

$m = 2$. In this case $\mathcal{U} \simeq U(gl(2, \mathbb{C}))$. By the Weyl theorem each finite-dimensional module is completely reducible.

$m > 2$. In this case an analogue of the Weyl theorem does not hold. For the rest of this section we will investigate this case more carefully and thus assume that $m > 2$.

Denote by (xy), $1 \leq x < y \leq m$ the elementary transposition in S_m.

Proposition 2. *Let $[\,l\,] \in P$, $T = \{l_{2j} \,|\, 2 \leq j \leq m, l_{21} - l_{2j} \in \mathbb{N}\}$. Set $t = |T|$ and $T = \{p_1, \dots, p_t\}$, $p_1 > p_2 > \dots > p_t$. Assume that $p_j = l_{2\,j+1}$, $1 \leq j \leq t$. Then the length of $M([\,l\,])$ equals $t + 1$. Moreover, a composition series of $M([\,l\,])$ has the form*

$$0 \subset M((1\,t+1)([\,l\,])) \subset M((1t)([\,l\,])) \subset \dots \subset M((12)([\,l\,])) \subset M([\,l\,]).$$

Proof. Follows from the construction of $M([\,l\,])$ and formula (1). □

Lemma 18. *Let $L([\,l\,])$, $L([\,t\,])$, $[\,l\,], [\,t\,] \in P^+$ be two non-isomorphic simple modules. Suppose that*

$$0 \to L([\,t\,]) \to X \to L([\,l\,]) \to 0$$

is a non-split extension. Then there exists $\sigma \in S_m$ such that $[\,l\,]_2 = \sigma([\,t\,]_2)$.

Proof. Follows from Lemma 11 and the Fitting lemma. □

Lemma 19. *Let $L([\,l\,])$, $L([\,t\,])$ be distinct finite-dimensional modules, $l_{11}-t_{11}\in\mathbb{N}$. Then the following statements are equivalent:*

1) $\mathrm{Ext}^1_{\mathcal{U}}(L([\,l\,]),L([\,t\,]))\neq 0$.
2) *$M([\,t\,])$ is the maximal submodule in $M([\,l\,])$.*
3) $\dim \mathrm{Ext}^1_{\mathcal{U}}(L([\,l\,]),L([\,t\,]))=1$.
4) $\dim \mathrm{Ext}^1_{\mathcal{U}}(L([\,t\,]),L([\,l\,]))=1$.

Proof. Let $0\to L([\,t\,])\to X\to L([\,l\,])\to 0$ be a non-split sequence. Then by Lemma 18 there exists an epimorphism $\psi: M([\,l\,])\to X$. Conversely, one can obtain a non-split extension of $L([\,t\,])$ and $L([\,l\,])$ under the lemma conditions as a quotient of $M([\,l\,])$. This proves that the first and second statements are equivalent. The rest follows from formula (1). □

Proposition 3. *Let $L([\,l\,])$, $[\,l\,]\in P^+$ be a simple finite-dimensional $\mathcal{U}$-module. Assume that $l_{21}-l_{22}\in\mathbb{N}$ and $l_{21}-l_{2j}\in\mathbb{N}$, $3\leq j\leq m$ implies $l_{2j}\leq l_{22}$. Then the following holds:*

1) $\dim \mathrm{Ext}^1_{\mathcal{U}}(L([\,l\,]),L([\,l\,]))\leq 1$.
2) $\dim \mathrm{Ext}^1_{\mathcal{U}}(L([\,l\,]),L([\,l\,]))=1$ *if and only if*

$$\prod_{j=3}^{m}(l_{2j}-l_{21})=-\prod_{j=3}^{m}(l_{2j}-l_{22}).$$

Proof. Let I be an ideal in $\mathcal{U}$ generated by $d_{2j}-\gamma_{2j}(l_{21},\dots,l_{2j})$, $1\leq j\leq m$. Consider an algebra $\mathcal{U}_I=\mathcal{U}/I$. Clearly, $L([\,l\,])$ is a $\mathcal{U}_I$-module. By Lemma 11 it is sufficient to prove the statements for $\mathcal{U}_I$-modules.

Consider a polynomial $g(x)=\prod\limits_{j=1}^{m}(l_{2j}-x+1)$. One can see that $[X,Y]=(g(d_{11}+1)-g(d_{11}))$ mod I. Thus $\mathcal{U}_I$ is a quotient of some algebra similar to the enveloping algebra of $sl(2)$ by a central character [13]. Now all statements follow from [10, Theorem 3.1] by direct calculation. □

From the last two statements it follows that the representation theory of our algebras is rather similar to that of algebras which are close to $U(sl(2))$. We conclude that the category of finite-dimensional modules is not semi-simple for $m>3$, moreover, some of finite-dimensional simple modules can have non-trivial self-extensions.

11. Orthogonal Gelfand-Zetlin algebras of signature $(1,m)$, an analogue of the $\mathcal{O}$ category

In this section we construct an analogue of the well-known category $\mathcal{O}$ for OGZ-algebra of signature $(1,m)$ (see [4] for the classical case). We define $\mathcal{O}$ to be a full subcategory of $\mathcal{U}$-modules V satisfying the following conditions:

1. V is d_{11}-diagonalizable;
2. V is finitely generated;
3. $\mathcal{U}^+v$ is finite-dimensional for any $v\in V$.

Set $\mathcal{L}^+$ to be a subset of $\mathcal{L}_0$ consisting of all $[\,t\,] \in \mathcal{L}_0$ such that $t_{11} \in \mathbb{Z}_+$. Basic properties of $\mathcal{O}$ are described in the following lemma:

Lemma 20. *Suppose that* $V \in \mathcal{O}$. *Then*

1) $\operatorname{Tsupp} V \subset \bigcup_{j=1}^{N}([\,l_j\,] - \mathcal{L}^+)$ *for some* $[\,l_j\,]$, $j = 1, \dots, N$.
2) *All weight spaces of* V *(with respect to* d_{11}*) are finite-dimensional.*
3) V *has a finite length.*
4) $\dim \hom(V, N) < \infty$ *for any* $N \in \mathcal{O}$.

Proof. Follows from the previous section. □

Lemma 21.

1) $\mathcal{O}$ *is closed under the operations of taking submodules, quotients and finite direct sums.*
2) $\{L([\,l\,]) \mid [\,l\,] \in P\}$ *is the complete list of simple objects in* $\mathcal{O}$.

Proof. Follows from Lemma 15. □

For a $\mathcal{U}$-module V and $\chi \in \Gamma_2^*$ set

$$V^\chi = \{v \in V \mid (z - \chi(z))^s v = 0 \text{ for all } z \in \Gamma_2, s > 0\}.$$

For any $M \in \mathcal{O}$, $M = \oplus_\chi M^\chi$ and therefore $\mathcal{O} = \oplus_\chi \mathcal{O}_\chi$, where $\mathcal{O}_\chi$ consists of those $M \in \mathcal{O}$ such that $(z - \chi(z))^s M = 0$ for some $s > 0$ and for all $z \in \Gamma$.

Lemma 22. *Each object in* $\mathcal{O}$ *is a quotient of a projective object in* $\mathcal{O}$.

Proof. Clearly it is enough to prove this for each $\mathcal{O}_\chi$. Fix $[\,t\,] \in \mathcal{L}(\mathbb{C}, r)$.

One can see that there exists $s > 0$ such that $X^s V_{[\,t\,]} = 0$ for any $V \in \mathcal{O}_\chi$. Set I to be a left ideal of $\mathcal{U}$ generated by $d_{11} - t_{11}$ and X^s. Then $Q([\,t\,]) = \mathcal{U}/I \in \mathcal{O}$. One can see that the map $\hom_{\mathcal{U}}(Q([\,t\,]), V) \to V_{[\,t\,]}$ is an isomorphism and thus the functor $V \mapsto \hom_{\mathcal{U}}(Q([\,t\,])^\chi, V)$ is exact. We conclude that $Q([\,t\,])$ is projective which completes the proof since V is finitely generated. □

Corollary 4. *There is a 1-1 correspondence between simple objects in* $\mathcal{O}$ *and indecomposable projective objects in* $\mathcal{O}$.

For $[\,l\,] \in P$ we will denote by $Pr([\,l\,])$ the projective cover of $L([\,l\,])$.

Lemma 23. *Each* $Pr([\,l\,])$, $[\,l\,] \in P$ *has a Verma flag (in a sense of* [4]*).*

Proof. Follows from the fact that $Pr([\,l\,])$ is $\mathcal{U}^-$-free. □

For a $\mathcal{U}$-module V we turn V^* into $\mathcal{U}$-module by setting $(X\psi)(v) = \psi(Yv)$, $(Y\psi)(v) = \psi(Xv)$, $(z\psi)(v) = \psi(zv)$, $\psi \in V^*$, $v \in V$, $z \in \Gamma$. Let $\delta V = \{\psi \in V^* \mid \dim(\Gamma\psi) < \infty\}$. Then δ is left exact contravariant functor on $\mathcal{O}$. The following lemma is clear:

Lemma 24. 1) $\delta^2 V = V$, $V \in \mathcal{O}$. 2) $\delta L([\,t\,]) \simeq L[\,t\,]$, $[\,t\,] \in P$.

Proof. The first statement is obvious and the second one follows from the uniqueness of a simple highest weight module with fixed highest weight. □

For a fixed $\chi \in \Gamma_2^*$ let $M([t_j])$, $1 \leq j \leq s$ be the complete list of Verma modules in $\mathcal{O}_\chi$. Set $Pr = \oplus_{j=1}^s Pr([t_j])$. Then $A = \hom_{\mathcal{U}}(Pr, Pr)$ is a finite-dimensional algebra and $\mathcal{O}_\chi$ is equivalent to the category of left A-modules of finite length.

For $[l], [t] \in P$ let $[Pr([l]) : M([t])]$ denote the number of subquotients in Verma flag of $Pr([l])$ isomorphic to $M([t])$. Also set $(M([t]) : L([l]))$ to be the number of subquotients in a composition series of $M([t])$ isomorphic to $L([l])$.

Theorem 5. *A is BGG-algebra in the sense of* [9]. *In particular,*

$$[Pr([l]) : M([t])] = (M([t]) : L([l]))$$

for all $[t], [l] \in P$.

Proof. We only need to prove that any $V \in \mathcal{O}_\chi$ with $\mathrm{Rad}\, V \simeq L([l])$ and any subquotients of the form $L([t])$, $[t] \in [l] - \mathcal{L}^+$ is a quotient of a Verma module. This follows from the universal property of Verma modules. The rest follows from [9]. □

12. Weight modules over OGZ-algebras of signature $(1, 1, \ldots, 1)$

In this section we investigate another family of OGZ-algebras and describe all their simple weight modules. We assume that $r = (1, 1, \ldots, 1)$ and use the notation of the weight module from Section 10.

Set $X_{ii}^{\pm} = X_i^{\pm}$, $1 \leq i \leq n-1$ and define $X_{ij}^{\pm} = [X_{ii}^{\pm}, X_{i+1\, j}^{\pm}]$ for $1 \leq i < j \leq n-1$ by induction.

Lemma 25. *The following identities are valid:*

1) $[X_m^{\pm}, X_{ij}^{\pm}] = 0$, $1 \leq i < j \leq n-1$, $1 \leq m \leq n-1$, $m \neq i-1$, $m \neq j+1$.
2) $[X_m^{\pm}, X_{ij}^{\mp}] = 0$, $1 \leq i < j \leq n-1$, $1 \leq m \leq n-1$, $m \neq i+1$, $m \neq j-1$.
3) $[X_1^{-}, X_{ij}^{+}] = 0$, $1 \leq i < j \leq n-1$;
4) $[X_1^{-}, X_1^{+}] = 1$.

Proof. Follows by direct calculations. □

Consider the set T consisting of all elements $X_{ij}^{\pm}$, $1 \leq i \leq j \leq n-1$, d_{i1}, $1 \leq i \leq n$. Rename the elements from T at the following way: $T = \{t_1, t_2, \ldots, t_p\}$. For $i_1, \ldots, i_p \in \mathbb{Z}_+$ the monomial

$$F(i_1, \ldots, i_p) = t_1^{i_1} \cdot \cdots \cdot t_p^{i_p}$$

will be called standard monomial.

Corollary 5. *The standard monomials span* $\mathcal{U}$ *as a vector space.*

Proof. Follows from Lemma 25 by the standard arguments (see for example [6, Ch. 2]). □

For $[\,l\,] \in \mathcal{L}(\mathbb{C}, r)$ consider a specialization $M_{[\,l\,]}$. Clearly $\dim(M_{[\,l\,]})_{[\,l\,]} = 1$. Let $M([\,l\,])$ be a submodule of $M_{[\,l\,]}$ generated by $(M_{[\,l\,]})_{[\,l\,]}$.

Lemma 26.

1) *$M([\,l\,])$ has the unique maximal submodule and thus the unique simple quotient $L([\,l\,])$.*
2) $\dim L([\,l\,])_{[\,l\,]} = 1$.

Proof. Proof is analogous to that of [7, Theorem 30]. □

Theorem 6.

1) *$L([\,l\,])$, $[\,l\,] \in \mathcal{L}(\mathbb{C}, r)$ is the complete list of simple weight $\mathcal{U}$-modules.*
2) *$L([\,l\,]) \simeq L([\,t\,])$ if and only if $(L([\,l\,]))_{[\,t\,]} \neq 0$.*

Proof. Using Lemma 7 and [7, Proposition 31] one can prove this statement in the same way as [7, Theorem 30]. □

Acknowledgments. This work was done during the visit of the author to SFB 343, Bielefeld University whose financial support and accommodation are gratefully acknowledged. I would like to thank the referee for many helpful suggestions that led to the improvements in the paper.

References

[1] Bavula, V. V.: *Generalized Weyl algebras and their representations.* St. Petersburg Math. J. **4** (1993), 71–92.

[2] Bavula, V.; Bekkert, V.: *Indecomposable representations of generalized Weyl algebras.* Preprint 94-095, SFB-343, Universität Bielefeld, Germany, 1994.

[3] Barut, A. O.; Rachka, R.: *Theory of group representations and applications.* PWN – Polish Scientific Publisher, Warszawa 1977.

[4] Bernstein, J.; Gelfand, I. M.; Gelfand, S. I.: *A category of $\mathfrak{G}$-modules.* Funct. Anal. Appl. **10** (1976), 87–92.

[5] Coleman, A. J.; Futorny, V. M.: *Stratified L-modules.* J. Algebra **163** (1994), 219–234.

[6] Dixmier, J.: *Enveloping algebras.* North-Holland Publishing Company, Amsterdam 1977.

[7] Drozd, Yu. A.; Ovsienko, S. A.; Futorny, V. M.: *Harish-Chandra subalgebras and Gelfand-Zetlin modules.* NATO Adv. Sci. Ser. C Math. Phys. Sci. **424** (1994), 72–89.

[8] Gelfand, I. M.; Zetlin, M. L.: *Finite-dimensional representations of groups of unimodular matrices.* Dokl. Akad. Nauk SSSR **71** (1950), 1014–1016 (in Russian).

[9] Irving, R. S.: *BGG-algebras and BGG-reciprocity principle.* J. Algebra **135** (1990), 362–380.

[10] Jie, X.; van Oystaeyen, F.: *Weight modules and their extensions over a class of algebras similar to the enveloping algebra of $sl(2, \mathbb{C})$.* J. Algebra **175** (1995), 844–864.

[11] Mazorchuk, V.; Ovsienko, S.: *Submodule structure of generalized Verma modules induced from generic Gelfand-Zetlin modules.* Preprint 97-007, SFB-343, Universität Bielefeld, Germany, 1994.

[12] Mazorchuk, V.; Turowska, L.: **-representations of twisted generalized Weyl constructions.* Preprint TRITA-MAT-1998-MA-04 (Jan. 1998), Dept. of Math., Royal Institute of Technology, Stockholm, Sweden.

[13] Smith, S. P.: *A class of algebras similar to the enveloping algebra of* $sl(2)$. Trans. Amer. Math. Soc **322** (1990), 285–314.

Received June 30, 1997