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# Flag-transitive Extensions of Dual Affine Spaces

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**Abstract.** We construct a simply connected flag-transitive circular extension of the dual affine space $AG^*(3,4)$, with $2^{13}{:}3^{\cdot}M_{22}.2$ as full automorphism group and we show that this geometry as well as its unique flag-transitive proper quotient are the unique flag-transitive circular extensions of finite thick dual affine spaces whose full automorphism group does not induce a 1-dimensional affine group on their point- or plane-residues. Moreover, we observe that our theorem has some nice repercussions for a problem in projective spaces.

## 1. Introduction

### 1.1. Definitions and notation

Let $c$ and $AG^*$ denote respectively the class of circular spaces (namely, linear spaces with all lines of size 2) and the class of dual affine spaces (namely, line-point truncations of affine geometries). A $c{\cdot}AG^*$-*geometry* is a geometry $\Gamma$ over the type set $\{0,1,2\}$ belonging to the following diagram:

$$(c \cdot AG^*) \qquad \overset{0}{\bullet}\overset{c}{\text{———}}\overset{1}{\bullet}\overset{AG^*}{\text{———}}\overset{2}{\bullet}$$

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The elements of $\Gamma$ of type 0, 1, 2 are also called *points*, *lines* and *planes* respectively. If there are integers $n$ and $q$, with $n > 1$ and $q > 1$, such that the dual of every point residue of $\Gamma$ is the point-line truncation of an $n$-dimensional affine space of order $q$, then we say that $\Gamma$ has *order* $q$ and *residual dimension* $n$.

Let $G \leq Aut(\Gamma)$. Given an element $x$ of $\Gamma$, the elementwise stabilizer in $G_x$ of the residue $\Gamma_x$ is denoted by $K_x$ and we set $\overline{G}_x = G_x/K_x$. If $G$ is flag-transitive on $\Gamma$, then the stabilizers of any two elements of the same type are isomorphic. In that case, the isomorphism types of $G_x$, $K_x$ and $\overline{G}_x$, with $x$ of type $i$, will be denoted by $G_i$, $K_i$ and $\overline{G}_i$.

We recall that a permutation group $X$ of degree $t$ is said to be 1-*dimensional affine* if $t$ is a prime power and $X \leq A\Gamma L_1(t)$.

### 1.2. A survey of known results

$c \cdot AG^*$-geometries of order $q$ and residual dimension $n$ have been studied in the two particular cases where $q = 2$ or $n = 2$.

When $q = 2$, they are $c \cdot c^*$-geometries with $2^n$ points in every plane. A complete classification of these $c \cdot c^*$-geometries seems difficult to obtain, even assuming flag-transitivity. The reader is referred to Baumeister and Pasechnik [1] and Baumeister and Pasini [2] for some partial results on this problem.

$c \cdot AG^*$-geometries of residual dimension 2 are also called extended dual affine planes. They have been classified by Baumeister, Del Fra, Meixner and Pasini [3] in the finite and flag-transitive case. The next table summarizes that classification.

| | order | $G = Aut(\Gamma)$ | $G_0 = \overline{G}_0$ | $\overline{G}_2$ | $K_2$ |
|---|---|---|---|---|---|
| (1) | 2 | $L_3(2)$ | $2^2.Sym(3)$ | $2^2.Sym(3)$ | 1 |
| (2) | 2 | $(2^3 \times 2^2).Sym(3)$ | $2^2.Sym(3)$ | $2^2.Sym(3)$ | 1 |
| (2′) | 2 | $2^4.Sym(3)$ | $2^2.Sym(3)$ | $2^2.Sym(3)$ | 1 |
| (3) | 4 | $2^6.3.Sym(6)$ | $A\Gamma L_2(4)$ | $Sym(6)$ | 3 |
| (3′) | 4 | $2^5.Sym(5)$ | $2^4.5.4 < A\Gamma L_1(16)$ | $Sym(5)$ | 1 |
| (3″) | 4 | $2^4.3.Sym(5)$ | $2^4.15.4 = A\Gamma L_1(16)$ | $Sym(5)$ | 3 |
| (4) | 4 | $L_3(4).2$ | $A\Gamma L_2(4)$ | $Sym(6)$ | 1 |
| (5) | 4 | $2^4.Sym(6)$ | $2^4.Sym(5)$ | $Sym(6)$ | 1 |
| (6) | 16 | $2^9.L_2(17)$ | $2^8.17.8 < A\Gamma L_1(2^8)$ | $L_2(17)$ | 1 |
| (6′) | 16 | $2^8.L_2(17)$ | $2^8.17.8 < A\Gamma L_1(2^8)$ | $L_2(17)$ | 1 |

**Note.** (2′), (3′) and (6′) are 2-fold quotients of (2), (3) and (6), respectively; (3″) is a 4-fold quotient of (3).

### 1.3. The main result of this paper

In view of the results quoted in Subsection 1.2, we focus on flag-transitive $c \cdot AG^*$-geometries of order $q > 2$ and residual dimension $n > 2$ (whence $q$ is a prime power). We prove the following:

**Theorem 1.** *Let $\Gamma$ be a flag-transitive $c \cdot AG^*$-geometry of order $q > 2$ and residual dimension $n > 2$ and let $G = Aut(\Gamma)$. Then one of the following occurs:*

(i) *$n = 3$, $q = 4$ and either*
  (i.a) *$G = 2^{13}{:}3^{\cdot}Aut(M_{22})$ and $\Gamma \simeq \Gamma(\mathcal{S})$, where $\Gamma(\mathcal{S})$ is described in Section 2, or*
  (i.b) *$G = 2^{12}{:}3^{\cdot}Aut(M_{22})$ and $\Gamma \simeq \Gamma(\mathcal{S})/2$, where $\Gamma(\mathcal{S})/2$ is a uniquely determined quotient of $\Gamma(\mathcal{S})$ (see Section 2);*

(ii) *$q$ is odd, $q^{n-1} + q^{n-2} + \cdots + q + 2$ is a prime power and both $G_0 = \overline{G}_0$ and $\overline{G}_2$ are 1-dimensional affine groups.*

Theorem 1 will be proved in Section 3.

The following arithmetical question arises from case (ii) of Theorem 1.

**Problem.** What are the solutions $(n, q, q')$ of the following equation, where $q$ and $q'$ are prime powers, $q$ is odd and $n$ is an integer $\geq 3$ ?

$$q^{n-1} + \ldots + q + 2 = q' \tag{E1}$$

We do not know if the equation (E1) admits a solution for $q'$ odd. If $q'$ is even, then the equation (E1) is a particular case of the following equation, which seems to be unsolved and for which only the solution $(n, q, m, q_0) = (3, 5, 5, 2)$ is known (see Makowski and Schinzel [23] for more information).

$$q^{n-1} + \ldots + q = q_0^{m-1} + \ldots + q_0 \tag{E2}$$

where $q$ and $q_0$ are distinct prime powers and $n$ and $m$ are positive integers. Therefore, for $q'$ even, the equation (E1) has only one known solution, namely $(n, q, q') = (3, 5, 32)$.

As we will show in Section 4, for every solution $(n, q, q')$ of the equation (E1), with $q'$ even, there is at least one flag-transitive $c \cdot AG^*$-geometry of order $q$ and residual dimension $n$, obtained as a gluing of a circular space of $q'$ points with the point-line system of $AG(n, q)$. For $(n, q) = (3, 5)$, we obtain in this way exactly two non-isomorphic flag-transitive $c \cdot AG^*$-geometries of order 5 and residual dimension 3 since, in that case, it can be shown (see Section 4) that there are exactly two non-isomorphic flag-transitive gluings.

The material of the present paper is part of an attempt to classify flag-transitive $(c \cdot L^*)$-geometries with finite point residues. We refer to Huybrechts-Pasini [16, 17] for some other works in that direction and to [18] for a survey of the situation (see also Huybrechts [14]).

### 1.4. A consequence of Theorem 1

In Section 5, we exploit Theorem 1 to study some families of subspaces in projective spaces. More precisely, we prove the following:

**Theorem 2.** *Let $\mathcal{S}$ be a family of $d$-dimensional subspaces of $PG(m, q)$ $(1 \leq d < m)$ such that*

(1) *every point of $PG(m, q)$ belongs to either 0 or 2 members of $\mathcal{S}$;*

(2) *any two members of $\mathcal{S}$ have precisely one point in common;*

(3) *the set of the points of* $PG(m,q)$ *belonging to some member of* $\mathcal{S}$ *spans* $PG(m,q)$.

*Furthermore, let* $G_{\mathcal{S}}$ *be the stabilizer of* $\mathcal{S}$ *in* $G = P\Gamma L_{m+1}(q)$ *and assume that* $G_{\mathcal{S}}$ *acts* 2*-transitively on* $\mathcal{S}$*. Then one of the following holds:*

(i) $q = 2$;

(ii) $q = 4$, $m = 2$, $d = 1$, $G_{\mathcal{S}} = Sym(6)$ *and* $\mathcal{S}$ *is the dual of a regular hyperoval of* $PG(2,4)$;

(iii) $q = 4$, $m = 5$, $d = 2$ *and* $G_{\mathcal{S}} = Aut(M_{22})$;

(iv) $q$ *is odd,* $q^d + q^{d-1} + \cdots + q + 2$ *is a prime power and* $G_{\mathcal{S}}$ *is* 1*-dimensional affine.*

A family $\mathcal{S}$ of $d$-dimensional subspaces of $PG(m,q)$ satisfying the above properties (1)–(3) is called a *$d$-dimensional dual hyperoval.* If furthermore $G_{\mathcal{S}}$ acts 2-transitively on $\mathcal{S}$, then we say that $\mathcal{S}$ is 2*-transitive.*

According to this terminology, Theorem 2 reads as follows: 2-transitive $d$-dimensional dual hyperovals only exist in cases (i)–(iv).

J. Thas and B. Cooperstein (Thas [28]) made a construction providing a lot of $d$-dimensional dual hyperovals of $PG(2d,2)$, for any positive integer $d$. Some of them are 2-transitive. The family $\mathcal{S}$ of 22 planes of $PG(5,4)$ we consider in Section 2.1 in order to define the geometry $\Gamma(\mathcal{S})$ of Theorem 1 as an example for (iii).

It seems unlikely that case (iv) of Theorem 2 may occur. No example is known for that case and, more generaly, no $d$-dimensional dual hyperoval is known in $PG(m,q)$ with $q$ odd.

We refer to Huybrechts-Pasini [19] for further comments on Theorem 2 and to Huybrechts [15] for some geometric results on $d$-dimensional dual hyperovals.

### 1.5. More definitions and notation

We end this introduction recalling some basic definitions and stating some notation, to be used in the sequel of the paper.

**The properties (LL) and (T).** The following properties hold in some $c \cdot AG^*$-geometries:

(LL) no two distinct lines are incident to the same pair of points;

(T) every triple of pairwise collinear points is contained in some plane.

For instance, the geometries (1), (2), (3) and (4) of the list of Subsection 1.2 satisfy (LL), whereas (T) holds in all geometries of that list but (1), (4) and (6).

Notice that, in the context of $c \cdot AG^*$-geometries, (LL) is equivalent to the Intersection Property [26, Lemma 7.25].

**Flatness.** Following [26], we say that a $c \cdot AG^*$-geometry $\Gamma$ is *flat* if all points of $\Gamma$ are incident to all planes of $\Gamma$. (For instance, the geometries (2′), (3″), (5) and (6′) of the list of Subsecion 1.2 are flat.) Clearly, the property (LL) is false in flat geometries, but (T) trivially holds in them.

**Collinearity, distance and diameter.** Given two distinct points $a$ and $b$ of a $c \cdot AG^*$-geometry $\Gamma$, we write $a \sim b$ to mean that $a$ and $b$ are collinear. Accordingly, the set of points collinear with (or equal to) a given point $a$ will be denoted by $a^{\sim}$.

We recall that the symbol $\perp$ is often used in the literature instead of $\sim$ to denote the collinearity relation. We prefer the notation $\sim$ in this paper, keeping $\perp$ to denote the orthogonality relation in a projective space with respect to a given projective sesquilinear form (Subsection 2.2).

The *distance* between two points of $\Gamma$ is their distance in the collinearity graph of $\Gamma$. The *distance* of a point $a$ from a plane $u$ is the minimal distance between $a$ and points of $u$. The *diameter* of $\Gamma$ is the diameter of its collinearity graph.

## 2. A geometry for $2^{13}{:}3^{\cdot}Aut(M_{22})$

### 2.1. A remarkable family of 22 planes of $PG(5,4)$

Let $\Delta_0$ be the rank 3 polar space for $U_6(2)$. It is known ([21]; see also [11, page 39]) that $\Delta_0$ contains a family $\mathcal{S}$ of 22 maximal singular subspaces such that any two of them meet in precisely one point, every point of $S_0 = \bigcup_{X \in \mathcal{S}} X$ is contained in precisely two members of $\mathcal{S}$, and the stabilizer of $\mathcal{S}$ in $\Sigma U_6(2^2)$ is isomorphic to $3^{\cdot}Aut(M_{22})$ (the unique non-trivial central non-split extension of $Aut(M_{22})$), acting 2-transitively as $Aut(M_{22})$ on $\mathcal{S}$. (We warn the reader that a misformulation occurs in [11, page 39], in the paragraph where $\mathcal{S}$ is described: the authors claim that every point of $\Delta_0$ is in exactly two members of $\mathcal{S}$, but they meant 'in either 0 or exactly 2 members'; compare [12].)

It is clear from the above that the pair $(\mathcal{S}, S_0)$, with the incidence relation inherited from $\Delta_0$, is a circular space with $\mathcal{S}$ as the set of points and $S_0$ as the set of lines. Furthermore,

**Lemma 2.1.** *For every point $a$ of $\Delta_0$, at least one line of $\Delta_0$ through $a$ contains at least two points of $S_0$.*

*Proof.* The statement is obvious when $a \in S_0$ (take a line through $a$ contained in a member of $\mathcal{S}$). Assume $a \notin S_0$. Then $|X \cap a^{\perp}| = 5$ for every member $X$ of $\mathcal{S}$. Hence $|S_0 \cap a^{\perp}| = 55$, as every point of $S_0$ belongs to precisely two members of $\mathcal{S}$. On the other hand, 45 is the number of lines of $\Delta_0$ through $a$. Therefore some of those lines meet $S_0$ in at least two points. □

**Corollary 2.2.** *The set $S_0$ spans $\Delta_0$.*

*Proof.* Obvious, by Lemma 2.1. □

**Remark.** According to the above corollary, $\mathcal{S}$ satisfies (3) of Theorem 2. Hence $\mathcal{S}$ is a 2-dimensional dual hyperoval of $PG(5,4)$.

### 2.2. The geometry $\Gamma(\mathcal{S})$

Let $\Delta$ be the polar space for $U_8(2)$ and, given a point $p$ of $\Delta$, let $H = p^{\perp}$ and let $\Delta \setminus H$ be the affine polar space obtained by removing the hyperplane $H$ from $\Delta$. The residue $\Delta_p = H/p$ of $p$ in $\Delta$ is isomorphic to $\Delta_0$ and we may assume that the above mentioned family $\mathcal{S}$ is a set of planes of $\Delta_p$.

Let $\Gamma^*(\mathcal{S})$ be the substructure of $\Delta \setminus H$ defined as follows. There are three types of elements in $\Gamma(\mathcal{S})$, which we call *points*, *lines* and *planes*. The points of $\Gamma^*(\mathcal{S})$ are the ones of $\Delta \setminus H$. The lines of $\Gamma^*(\mathcal{S})$ are the lines of $\Delta \setminus H$ that, viewed as lines of $\Delta$, meet $H$ in a point belonging to a member of $\mathcal{S}$. The planes of $\Gamma^*(\mathcal{S})$ are the 3-spaces of $\Delta \setminus H$ that, viewed as maximal singular subspaces of $\Delta$, meet $H$ in a plane contained in a member of $\mathcal{S}$. The incidence relation is the natural one, inherited from $\Delta$.

**Lemma 2.3.** *The residues of the planes of* $\Gamma^*(\mathcal{S})$ *are isomorphic to the point-line system of* $AG(3,4)$. *The residues of the points of* $\Gamma^*(\mathcal{S})$ *are isomorphic to the dual of the circular space* $(\mathcal{S}, S_0)$.

*Proof.* The first claim is obvious. Let us prove the second one. Given a point $a$ of $\Delta \setminus H$, the set $a^\perp$ meets each member of $\mathcal{S}$ (resp. $S_0$) in a plane (a point). For every $X \in \mathcal{S}$ the 3-space $X$ is the unique member of $\mathcal{S}$ containing the plane $X \cap a^\perp$ and, for every member $Y$ of $S_0$, the point $Y \cap a^\perp$ is contained in just two members of $\mathcal{S}$. The conclusion follows. □

**Lemma 2.4.** *The structure* $\Gamma^*(\mathcal{S})$ *is connected.*

*Proof.* In view of Lemma 2.3, we only need to prove that the point-line system of $\Gamma^*(\mathcal{S})$ is connected.

Let $a_1, a_2$ be collinear points of $\Delta \setminus H$ and let $L$ be the line joining them in $\Delta \setminus H$. The line $L$, viewed as a line of $\Delta$, meets $H$ in a point $b \neq p$. If $b$ belongs to a member of $\mathcal{S}$, then $L$ is a line of $\Gamma^*(\mathcal{S})$ and $a_1$ and $a_2$ are collinear in $\Gamma^*(\mathcal{S})$. Otherwise, it follows from Lemma 2.1 that there is a plane $X$ of $\Delta$ containing the points $p$ and $b$ and at least two lines $L_1, L_2$ of $\mathcal{S}_0$. For $i = 1, 2$, let $b_i$ be the intersection of $L_i$ with the line $M = L^\perp \cap X$ and let $M_i$ be the line of $\Delta$ joining $a_i$ with $b_i$. The lines $M_1, M_2$ belong to $\Gamma^*(\mathcal{S})$. Furthermore, the lines $M$ and $L$ meet in $b$. Hence they span a plane $Y$ of $PG(7,4)$. Both $M_1$ and $M_2$ belong to $Y$. Whence they meet in a point, say $c$. Clearly, $c$ is collinear with both $a_1$ and $a_2$ in $\Gamma^*(\mathcal{S})$.

The connectedness of the point-line system of $\Gamma^*(\mathcal{S})$ now follows from the connectedness of the collinearity graph of $\Delta \setminus H$. □

We denote the dual of $\Gamma^*(\mathcal{S})$ by $\Gamma(\mathcal{S})$. Namely, $\Gamma(\mathcal{S})$ and $\Gamma^*(\mathcal{S})$ are the same structure, but for calling the points and the planes of $\Gamma^*(\mathcal{S})$ *planes* and *points* of $\Gamma(\mathcal{S})$, respectively.

**Proposition 2.5.** *The structure* $\Gamma(\mathcal{S})$ *is a* $c \cdot AG^*$*-geometry of order* 4 *and residual dimension* 3.

*Proof.* Obvious, by Lemmas 2.3 and 2.4. □

### 2.3. The group $G = Aut(\Gamma(\mathcal{S}))$

Let $G$ be the stabilizer of $p$ and $\mathcal{S}$ in $Aut(\Delta)$. Clearly, $G = N{:}G_2$ where $N = 2^{13}$ acts regularly on the set of points of $\Delta \setminus H$ and $G_2 = 3^{\cdot}Aut(M_{22})$ is the stabilizer in $G$ of a point of $\Delta \setminus H$ (namely, a plane of $\Gamma(\mathcal{S})$). Furthermore, $G$ stabilizes $\Gamma(\mathcal{S})$ and acts faithfully and flag-transitively on it, with point-stabilizer $G_0 = A\Sigma L_3(4)$. It will turn out from a theorem of Section 3 (Theorem 3.5) that $G = Aut(\Gamma(\mathcal{S}))$. Thus,

$$Aut(\Gamma(\mathcal{S})) \;=\; G \;=\; 2^{13}{:}3^{\cdot}Aut(M_{22}).$$

Furthermore (compare Section 3, Theorem 3.5(i) and proof of Theorem 3.6),

$$\begin{array}{lll} G_0 &=& \overline{G}_0 \;=\; A\Sigma L_3(4) \\ G_1 &=& ASL_2(4){:}(2\times S_4) \text{ with } K_1 \;=\; ASL_2(4) \;\; (\text{hence } \overline{G}_1 \;=\; 2\times S_4) \\ G_2 &=& 3^{\cdot}Aut(M_{22}) \text{ with } K_2 \;=\; 3 \;\; (\text{hence } \overline{G}_2 \;=\; Aut(M_{22})) \end{array}$$

(notation as in Subsection 1.1). The commutator subgroup $G' = 2^{13}{:}3^{\cdot}M_{22}$ of $G$ also acts flag-transitively in $\Gamma(\mathcal{S})$. According to Theorem 3.5 of Section 3, $G'$ is the unique flag-transitive proper subgroup of $G$.

Writing $G_i'$ for $G' \cap G_i$, we have $K_i \leq G_i'$ for $i = 1, 2$ and

$$G_0' \;=\; ASL_3(4), \quad G_1' \;=\; ASL_2(4){:}(2\times A_4), \quad G_2' \;=\; 3^{\cdot}M_{22}.$$

### 2.4. The quotient $\Gamma(\mathcal{S})/2$

The hyperplane $H$ of $\Delta$ spans a hyperplane $\langle H\rangle$ of $PG(7,4)$. Let $Z_0$ be the intersection of $Aut(\Delta)$ with the group of elations of $PG(7,4)$ with axis $\langle H\rangle$ and center $p$. It is well-known that $Z_0$ has order 2 and it is the center of the stabilizer of $p$ in $Aut(\Delta)$.

**Proposition 2.6.** *We have $Z(G') = Z(G) = Z_0$.*

*Proof.* Clearly, $Z_0 \leq Z(G')$. Assume that $Z_0 < Z(G')$. Then $G'/Z_0$ has a non-trivial center. However, $G'/Z_0$ is a subgroup of $V{:}SU_6(2)$, with $V$ the additive group of the natural $GF(4)$-module $V(6,4)$ of $SU_6(2)$ (see [11]). Furthermore, $G'/Z_0 = V{:}3^{\cdot}M_{22}$, with $3^{\cdot}M_{22}$ acting faithfully on $V$. Hence $Z(G'/Z_0) \leq V$. Consequently, some non-trivial element of $V$ is central in $G'/Z_0$. However, the centralizer in $SU_6(2)$ of a non-trivial element of $V$ does not involve $M_{22}$ (see [11]); contradiction.

Therefore, $Z(G') = Z_0$. As $|Z_0| = 2$, $Z(G')$ is central in $G$, too. Hence $Z(G) = Z(G')$. □

It is well-known [26, 8.4.7] that $Z_0$ defines a quotient of the affine polar space $\Delta \setminus H$. Accordingly, $Z_0$ ($= Z(G)$) defines a quotient of $\Gamma(\mathcal{S})$. We denote that quotient by $\Gamma(\mathcal{S})/2$. Clearly, $\Gamma(\mathcal{S})/2$ is flag-transitive with

$$Aut(\Gamma(\mathcal{S})/2) \;=\; G/Z_0 \;=\; 2^{12}{:}3^{\cdot}Aut(M_{22})$$

(but $Z(G) \leq G'$ and the factor group $G'/Z(G) = 2^{12}{:}3^{\cdot}M_{22}$ is also flag-transitive on $\Gamma(\mathcal{S}/2)$).

### 2.5. Another way to construct $\Gamma(\mathcal{S})/2$

Writing $\mathcal{P}$ for $PG(7,4)$, let $\mathcal{P}/p$ (resp. $\langle H\rangle/p$) be the star of $p$ in $\mathcal{P}$ (in $\langle H\rangle$). Clearly, $\langle H\rangle/p \cong PG(5,4)$ and, as $H/p$ contains a copy of $\mathcal{S}$, we may assume that $\mathcal{S}$ is a family of planes of the projective space $\langle H\rangle/p$. Let $\mathcal{A}$ be the complement of $\langle H\rangle/p$ in $\mathcal{P}/p$. Clearly, $\mathcal{A} \cong AG(6,4)$. Furthermore, every line of $\mathcal{P}$ through $p$ not contained in $\langle H\rangle$ meets $\Delta$ in two points different from $p$, which form an orbit of $Z_0$. Thus, the orbits of $Z_0$ on the set of points of $\Delta \setminus H$ bijectively correspond to the points of $\mathcal{A}$. It is now clear that the dual $\Gamma^*(\mathcal{S})/2$ of $\Gamma(\mathcal{S})/2$ can be described as follows:

points: the points of $\mathcal{A}$;

lines: the lines of $\mathcal{A}$ with point at infinity belonging to a member of $\mathcal{S}$;
planes: the 3-spaces of $\mathcal{A}$ with a member of $\mathcal{S}$ as the plane at infinity;
incidence relation: inherited from $\mathcal{A}$.

In short, $\Gamma^*(\mathcal{S})/2$ is the affine expansion of the dual circular space $(S_0, \mathcal{S})$ to $\mathcal{A}$ (see Buekenhout, Huybrechts and Pasini [7, Section 4] for the definition of affine expansions; also [26, Section 2.3]).

The stabilizer of $\mathcal{S}$ in $Aut(\mathcal{A})$ $(= A\Gamma L_6(4))$ is indeed $2^{12}{:}3^{\cdot}Aut(M_{22})$. As the latter is just $Aut(\Gamma(\mathcal{S})/2)$, all automorphisms of $\Gamma(\mathcal{S})/2$ are inherited from $Aut(\mathcal{A})$.

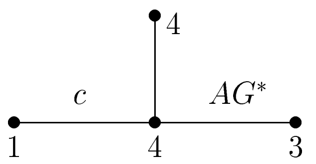
### 2.6. Some properties of $\Gamma(\mathcal{S})$ and $\Gamma(\mathcal{S})/2$

As the points and the lines of $\Gamma^*(\mathcal{S})$ are points and lines of a projective geometry, no two distinct lines of $\Gamma^*(\mathcal{S})$ are incident to the same pair of points. The same property holds in $\Gamma^*(\mathcal{S})/2$. Consequently, $\Gamma^*(\mathcal{S})$ and $\Gamma^*(\mathcal{S})/2$ also satisfy the Intersection Property ([26, Lemma 7.25]). Needless to say, the same holds in $\Gamma(\mathcal{S})$ and $\Gamma(\mathcal{S})/2$. In particular, the property (LL) (Subsection 1.5) holds in $\Gamma(\mathcal{S})$ and $\Gamma(\mathcal{S})/2$.

Both $\Gamma(\mathcal{S})$ and $\Gamma(\mathcal{S})/2$ satisfy (T). (This follows from Proposition 3.11 of Section 3, but it can also be proved straightforwardly.)

The geometry $\Gamma(\mathcal{S})/2$ has 1408 points. The number of points of $\Gamma(\mathcal{S})$ is twice that, namely 2816. As it will be proved in Section 3 (Proposition 3.36), the collinearity graph of $\Gamma(\mathcal{S})/2$ is strongly regular with parameters $k = 336$ and $\lambda = \mu = 80$. The collinearity graph of $\Gamma(\mathcal{S})$ is regular of diameter 4, with valency $k = 336$ and $\mu = 40$ (see Section 3, 3.7).

As both (LL) and (T) hold in $\Gamma(\mathcal{S})$ and in $\Gamma(\mathcal{S})/2$, both $\Gamma(\mathcal{S})$ and $\Gamma(\mathcal{S})/2$ can be erected to geometries of rank 4 with diagram and orders as follows (Huybrechts [15]):




## 3. Proof of Theorem 1

### 3.1. Outline of the proof

We first determine the structure of point- and plane-stabilizers and show in Theorem 3.5 (Subsection 3.2) that we can focus on the case of order $q = 4$ and residual dimension $n = 3$.

In Subsection 3.3 (Theorem 3.6), we prove that there is just one simply connected flag-transitive $c \cdot AG^*$-geometry with $n = 3$ and $q = 4$, say $\widetilde{\Gamma}$. Our next step is to find a geometric characterization of $\widetilde{\Gamma}$ by which we can eventually show that $\widetilde{\Gamma}$ is indeed the geometry $\Gamma(\mathcal{S})$ of Section 2. In view of that, we first investigate properties of a generic flag-transitive $c \cdot AG^*$-geometry $\Gamma$ of order 4 and residual dimension 3. After having proved that $\Gamma$ satisfies (LL) and (T) (Subsection 3.4), we study the common neighbourhood of two points at distance two (Subsection 3.5), eventually proving that $\Gamma$ has diameter at most 4 (Subsection 3.6). Then we assume that $\Gamma$ also admits a flag-transitive proper quotient. As we prove in Subsection

3.7, that additional hypothesis forces $\Gamma = \widetilde{\Gamma}$. At this stage, the conclusion $\widetilde{\Gamma} = \Gamma(\mathcal{S})$ follows quite easily.

We finish the proof (Subsection 3.8) by showing that the equality $\widetilde{\Gamma} = \Gamma(\mathcal{S})$ implies that $\Gamma(\mathcal{S})/2$ is the unique flag-transitive quotient of $\Gamma(\mathcal{S})$.

### 3.2. Feasible structures of point- and plane-stabilizers

Henceforth $\Gamma$ is a flag-transitive $c \cdot AG^*$-geometry of order $q > 2$ and residual dimension $n > 2$ and $G$ is a flag-transitive subgroup of $Aut(\Gamma)$. The prime which $q$ is a power of and its exponent in $q$ will be denoted by $p$ and $h$, respectively; namely, $q = p^h$.

In the sequel $\{a, u\}$ is a given point-plane flag. It will be linguistically convenient to distinguish between the residue of $a$ and its dual. We denote the latter by $\Gamma_a^*$, calling the planes of $\Gamma$ incident to $a$ the *points* of $\Gamma_a^*$.

We have $K_a = 1$ for every point $a$ (this is true in any $c.L^*$-geometry, as proved in Huybrechts and Pasini [16, Lemma 2.8]). Hence $\overline{G}_a = G_a$, namely $G_a$ acts faithfully on the residue $\Gamma_a$ of $a$. Thus, $G_a$ is a subgroup of $A\Gamma L_n(q) = Aut(AG(n,q))$. As $G_a$ is flag-transitive on the affine space $\Gamma_a^*$, it contains all translations of $\Gamma_a^*$ (see Buekenhout, Delandtsheer, Doyen [5]). Note also that, for every plane $u$, $K_u$ is the kernel of the action of $G_{a,u} = G_a \cap G_u$ on the set of lines incident to $a$ and $u$, for every point $a$ incident to $u$.

According to the notation stated in Subsection 1.1, we denote the stabilizer in $G$ of $u$ (resp. $a$) by $G_2$ (resp. $G_0$), $K_2$ stands for the kernel of the action of $G_2$ on the residue $\Gamma_u$ of $u$ and $\overline{G}_2 = G_2/K_2$.

**Lemma 3.1.** *Let $G_0$ be not 1-dimensional affine. Then $G_0$ is 2-transitive on the set of points of the affine space $\Gamma_a^*$. Furthermore, the following are the possibilities for $G_{0,2} = G_0 \cap G_2$:*

(i) *$SL_m(q_1) \trianglelefteq G_{0,2}$ for some $m$ with $1 < m \leq n$ and some power $q_1 = p^k$ of $p$ with $mk = nh$;*

(ii) *$Sp_{2m}(q_1) \trianglelefteq G_{0,2}$ for some $m$ with $2 < 2m \leq n$ and some power $q_1 = p^k$ of $p$ with $2mk = nh$;*

(iii) *$p = 2$ and $G_2(q_1)' \trianglelefteq G_{0,2}$ for some power $q_1 = 2^k$ of 2 with $6k = nh$;*

*Proof.* As $G_0$ is not 1-dimensional affine, one of the following holds (Buekenhout et al. [6, 3.1]):

(a) $G_0$ is 2-transitive on the set of points of the affine space $\Gamma_a^*$;

(b) $n = 4$, $q = 3$ and $G_{0,2}^{(\infty)} = 2.Alt(5)$.

In case (b), $u$ has 41 points and $G_{0,2}^{(\infty)}/K_2 = Alt(5)$. On the other hand, $G_2$ acts 2-transitively on the set of points incident to $u$. Thus, $\overline{G}_2$ is a 2-transitive group of degree 41 with point stabilizer $(\overline{G}_2)_0$ such that $(\overline{G}_2)_0^{(\infty)} \cong Alt(5)$. On the one hand, such a permutation group, if any, is of affine type. Indeed, $Alt(41)$ and $Sym(41)$ are the unique almost simple 2-transitive permutation groups of degree 41 (Cameron [8]), but they do not fit with the condition $(\overline{G}_2)_0^{(\infty)} \cong Alt(5)$. On the other hand, the condition $(\overline{G}_2)_0^{(\infty)} \cong Alt(5)$ is not satisfied by any affine 2-transitive permutation group of degree 41, since any such group is solvable.

Thus, we are left with case (a): $G_0$ is one of the 2-transitive permutation groups of affine type listed in Liebeck [22, Appendix 1]. In view of the assumption that $n, q > 2$ and recalling that $G_0 \leq A\Gamma L_n(q)$, the possibilities that might occur for $G_{0,2}$ are listed below:

(i) $SL_m(q_1) \trianglelefteq G_{0,2}$ for some $m$ with $1 < m \leq n$ and some power $q_1 = p^k$ of $p$ with $mk = nh$;
(ii) $Sp_{2m}(q_1) \trianglelefteq G_{0,2}$ for some $m$ with $2 < 2m \leq n$ and some power $q_1 = p^k$ of $p$ with $2mk = nh$;
(iii) $p = 2$ and $G_2(q_1)' \trianglelefteq G_{0,2}$ for some power $q_1 = 2^k$ of 2 with $6k = nh$;
(iv) $q = p = 3$, $n = 4$ and $G_{0,2}$ is an extension of a central product of $D_8$ and $Q_8$ by a subgroup of $Sym(5)$;
(v) $p = 3$, $nh = 6$ and $G_{0,2} = SL_2(13)$.

In case (iv) the group $\overline{G}_2$ is 2-transitive of degree 41 with point-stabilizer of order divisible by 16. By looking at the list of 2-transitive groups, one can readily see that no such group exists.

Let (v) occur. Then either $n = 3$ and $q = 9$ or $n = 6$ and $q = 3$. The group $\overline{G}_2$ is 2-transitive of degree 92 in the first case and 365 in the latter, with point-stabilizer isomorphic to $L_2(13)$ in any case. However, no such permutation group exists. Case (v) is also impossible. □

The group $\overline{G}_2$, being flag-transitive in $\Gamma_u$, is 2-transitive. Its degree (namely, the number of points of $\Gamma_u$) is

$$d = q^{n-1} + q^{n-2} + \cdots + q + 2 \tag{1}$$

and its point-stabilizer is $G_{0,2}/K_2$. Note that

$$d \geq 14 \tag{2}$$

(by (1), and since $n, q \geq 3$ by assumption).

**Lemma 3.2.** *Let $G_0$ be not 1-dimensional affine. Then $\overline{G}_2$ is not of affine type.*

*Proof.* Assume the contrary. Then

$$d = p_1^s \tag{3}$$

for some prime $p_1$ and some integer $s$. Comparing (3) with (1) we obtain

$$p_1^s = 2 + p^h + p^{2h} + \cdots + p^{(n-1)h}. \tag{4}$$

This, together with the assumptions $n, q > 2$, forces $p \neq 2$ and $p \neq p_1$. It is also clear that the largest power of $p_1$ dividing $q^n - 1$ divides $p^h - 1$ and that $p$ is not a divisor of $p_1^s - 1$.

As $p \neq 2$, case (iii) of Lemma 3.1 is impossible. Thus, we are left with cases (i) and (ii) of Lemma 3.1. In both (i) and (ii) of Lemma 3.1, the center of $G_{0,2}$ is the kernel of the action of $G_{0,2}$ on the set of lines of $\Gamma^*_{\underline{a}}$ on $u$, namely $K_2 = Z(G_{0,2})$. Furthermore, $(G_{0,2}/K_2)^{(\infty)}$ is non-trivial and simple. Hence $\overline{G}_2$ is not 1-dimensional affine. (Indeed, 1-dimensional affine permutation groups have solvable point-stabilizers.) Consequently, by Liebeck [22, Appendix 1] and recalling that (4) holds and that $n, q > 2$, the following three possibilities are obtained for $G_{0,2}/K_2$:

(j) $SL_r(p_1^t) \trianglelefteq G_{0,2}/K_2$ for some integers $r, t$ with $rt = s$ and $r > 1$;
(jj) $Sp_{2r}(p_1^t) \trianglelefteq G_{0,2}/K_2$ for some integers $r, t$ with $2rt = s$ and $r > 1$;
(jjj) $p_1 = 2$ and $G_2(2^t)' \trianglelefteq G_{0,2}/K_2$ for some integer $t$ with $6t = s$.

Furthermore, according to Lemma 3.1 and since $K_2 = Z(G_{0,2})$, the center of $G_{0,2}/K_2$ is trivial. This forces $(r, p_1^t - 1) = 1$ in (j) and $p_1 = 2$ in (jj). We should now match each of (j), (jj) and (jjj) with (i) and (ii) of Lemma 3.1. The isomorphisms between groups of Lie type are well-known (see [11, 2, 3], for instance). Exploiting that information and recalling that $n > 2$ by assumption, one can see that no matching as required above is possible. □

**Lemma 3.3.** *Let $G_0$ be not 1-dimensional affine. Then $n = 3$, $q = 4$, $K_2 = 3$ and either $G_0 = ASL_3(4)$ and $G_2 = 3^{\cdot}M_{22}$ or $G_0 = A\Sigma L_3(4)$ and $G_2 = 3^{\cdot}Aut(M_{22})$.*

*Proof.* By Lemma 3.2, the permutation group $\overline{G}_2$ is not of affine type. As noticed in the proof of Lemma 3.2, in each of the cases (i), (ii) and (iii) we have $K_2 = Z(G_{0,2})$ and $(G_{0,2}/K_2)^{(\infty)}$ is non-trivial and simple. Hence $\overline{G}_2$ is not a group of Lie type acting on the set of points of its natural geometry. Indeed, if $X$ is a group of Lie type and $X_0$ is its natural point-stabilizer, then $X_0^{(\infty)}$ is either non-simple or trivial.

Also, $\overline{G}_2$ is not isomorphic to either of $Sym(d)$ and $Alt(d)$, as $(G_{0,2}/K_2)^{(\infty)} \not\cong Alt(d-1)$. Furthermore, there is no $s$ such that $\overline{G}_2 \cong O_{2s+1}(2)$ in its action on one of the two families of non-singular hyperplanes of $PG(2s, 2)$. Indeed, if such an $s$ existed, then either $d = 2^s(2^s+1)$ or $d = 2^s(2^s - 1)$ but, comparing those equalities with (1) and recalling that $n, q > 2$, a contradiction is easily obtained.

Thus, $\overline{G}_2$ falls into one of the following exceptional almost simple cases:

| $\overline{G}_2$ | $d$ |
|---|---|
| $L_2(11)$ | 11 |
| $Alt(7)$ | 15 |
| $M_{11}$ | 12 |
| $M_{11}$ | 11 |
| $M_{12}$ | 12 |
| $M_{22}$ or $Aut(M_{22})$ | 22 |
| $M_{23}$ | 23 |
| $M_{24}$ | 24 |
| $HS$ | 176 |
| $Co_3$ | 276 |

However, it is easily checked that none of the values listed above for $d$ but 22 survives through the conditions (1) and (2). On the other hand, $22 = 4^2 + 4 + 2$. Thus, $d = 22$, $q = 4$, $n = 3$ and $\overline{G}_2 = M_{22}$ or $Aut(M_{22})$. Accordingly, $G_{0,2}/K_2 = L_3(4)$ or $L_3(4).2$ and $G_0 = ASL_3(4)$ or $A\Sigma L_3(4)$. In any case, $K_2 = 3$. □

**Lemma 3.4.** *Let $G_0$ be 1-dimensional affine. Then $q$ is odd, $d$ is a prime power and the permutation group $\overline{G}_2$ is of 1-dimensional affine type.*

*Proof.* We recall that a group $X$ is said to be *metacyclic* if either it is cyclic or it admits a normal subgroup $Y$ such that both $Y$ and $X/Y$ are cyclic. It is well-known (and easy to prove) that, if a group $X$ is metacyclic, then all subgroups and factor groups of $X$ are metacyclic. We also recall that point-stabilizers in 1-dimensional affine groups are metacyclic.

As $G_0$ is 1-dimensional affine, $G_{0,2}$ is metacyclic. Hence $(\overline{G}_2)_0$ $(= G_{0,2}/K_2)$ is metacyclic. Assume that $\overline{G}_2$ is not of affine type. Then, as $(\overline{G}_2)_0$ is metacyclic, by the the list of 2-transitive permutation groups we see that $\overline{G}_2 = L_2(p_1)$ for some prime $p_1$ with $p_1 + 1 = d$. Thus,

$$p_1 = \frac{q^n - 1}{q - 1} \tag{5}$$

(by (1)) and $G_{0,2}/K_2$ has order $(p_1 - 1)p_1/2$ (notice that $p_1 \neq 2$, by (2)). However, as $G_0$ is 1-dimensional affine, the order of $G_{0,2}/K_2$ divides the order $(q^n - 1)nh$ of $\Gamma L_1(p^{nh})$. Therefore,

$$(p_1 - 1)p_1 \ \text{ divides } \ 2(q^n - 1)nh. \tag{6}$$

Comparing (5) and (6) we see that $q^{n-1} + q^{n-2} + \cdots + q$ divides $2(q-1)nh$, which is impossible, since $n > 2$.

Thus, $\overline{G}_2$ is of affine type; namely, it is one of the groups of [22, Appendix 1]. Actually, the only groups listed there with metacyclic point-stabilizer and degree $d$ satisfying (1) with $n, q \geq 3$ are the 1-dimensional affine ones.

As $\overline{G}_2$ is of affine type, $d$ is a prime power. This, together with (1) and the assumption $n, q > 2$, forces $q$ to be odd. □

The following theorem is an assembling of Lemmas 3.3 and 3.4.

**Theorem 3.5.** *One of the following holds:*

(i) $n = 3$, $q = 4$ *and either*
  - (i.a) $G_0 = ASL_3(4)$ *and* $G_2 = 3^{\cdot}M_{22}$ *with* $K_2 = 3$, *or*
  - (i.b) $G_0 = A\Sigma L_3(4)$ *and* $G_2 = 3^{\cdot}Aut(M_{22})$ *with* $K_2 = 3$;

(ii) $q$ *is odd,* $q^{n-1} + q^{n-2} + \cdots + q + 2$ *is a prime power and both* $G_0 = \overline{G}_0$ *and* $\overline{G}_2$ *are* 1*-dimensional affine.*

### 3.3. Uniqueness of the amalgam when $(n, q) = (3, 4)$

From now on we focus on the case of $q = 4$ and $n = 3$, that is on case (i) of Theorem 3.5 (for indeed, when $(n, q) = (3, 4)$, case (ii) of Theorem 3.5 cannot hold since $4^2 + 4 + 2 = 22$ is not a prime power).

**Theorem 3.6.** *There exists precisely one flag-transitive and simply connected* $c \cdot AG^*$*-geometry of order* 4 *and residual dimension* 3.

*Proof.* At least one such geometry exists, namely the universal cover of the geometry $\Gamma(\mathcal{S})$ described in Section 2. Thus, we only need to prove the uniqueness claim.

As observed above, we are in case (i) of Theorem 3.5. As both cases (i.a) and (i.b) of Theorem 3.5 are realized in $\Gamma(\mathcal{S})$ for $G = Aut(\Gamma(\mathcal{S}))$ and $G = Aut(\Gamma(\mathcal{S}))'$ respectively, we only need to show the uniqueness of the amalgam of the parabolics $G_0$, $G_1$, $G_2$ for each of the cases (i.a) and (i.b) of Theorem 3.5. We will consider case (i.a) first. Then we shall shortly discuss (i.b).

Thus, let (i.a) of Theorem 3.5 hold. The intersections $G_{0,2} = G_0 \cap G_2$, $G_{0,1} = G_0 \cap G_1$, $G_{1,2} = G_1 \cap G_2$ and $G_{0,1,2} = G_0 \cap G_1 \cap G_2$ are uniquely determined inside $G_0$ and $G_2$. In particular, $G_{0,1}$ (resp. $G_{0,2}$, $G_{0,1,2}$) is the stabilizer of a line (a point, a point-line flag) of $AG(3,4)$ in $ASL_3(4) = G_0$. Thus,

$$G_{0,1} \cong (V \times W){:}L$$

with $V = 2^2, W = 2^4, L = GL_2(4) = Z \times L'$, where $|Z| = 3$ and $L' = L_2(4)$. The group $L$ normalizes both $V$ and $W$ and we have

$$WL = Z \times WL' = 3 \times ASL_2(4), \quad VL = L' \times VZ = L' \times GL_1(4).$$

Also,

$$G_{0,1,2} = WL \text{ and } G_{0,2} = SL_3(4) \text{ with center } Z.$$

Note that $Z$ is also the center of $G_2$ ($= Z^{\cdot}M_{22}$.) Finally, $G_{1,2}$ is the stabilizer in $3{:}M_{22}$ of a duad of $S(22,6,3)$ (namely, a point of the subset $S_0$ of $PG(5,4)$ considered in Subsection 2.1). Clearly, $G_{1,2} = G_{0,1,2}2$ and $Z$ is also the center of $G_{1,2}$. (Indeed, it is the center of $G_2$). Furthermore,

$$G_{1,2}/Z = 2^4{:}S_5 \text{ with} G_{0,1,2}/Z = 2^4{:}A_5$$

(see [11]). Therefore,

$$G_{1,2} = \langle G_{0,1,2}, i\rangle$$

for a suitable involution $i$ normalizing $G_{0,1,2} = WL = Z \times WL'$ and centralizing $Z$. As $WL'$ is the derived group $(WL)'$ of $WL$, the involution $i$ also normalizes $WL'$ and its action on $WL'$ is uniquely determined via the isomorphisms $WL' \cong 2^4{:}A_5$ and $(WL)\langle i\rangle/Z \cong 2^4{:}S_5$. Thus,

$$G_{1,2} = Z \times WL'\langle i\rangle = 3 \times 2^4{:}S_5.$$

To finish the proof in case (i.a) we only need to prove that $G_1$ is also uniquely determined by the above. We have $G_1 = \langle G_{1,2}, v\rangle$ and $\langle G_{0,1,2}, v\rangle = G_{0,1}$, for an involution $v \in V$. Thus, $G_1 = \langle G_{0,1,2}, v, i\rangle$. As $i$ permutes the two points of the line stabilized by $G_1$ and $G_{0,1}$ stabilizes each of those two points, $i$ normalizes $G_{0,1} = (V \times W)L$. Furthermore, as noticed above, $i$ centralizes $Z$ and normalizes $WL'$. Therefore, as $VZ$ is the centralizer of $WL'$ in $G_{0,1}$, the involution $i$ also normalizes $V$. As seen above, the action of $i$ on $WL$ is uniquely determined. Thus, we only need to determine the action of $i$ on $V$. As $i$ is an involution, it fixes an element $w$ of $V \setminus \{1\}$. Thus, $w^i = w$. On the other hand, $w, w^z$ and $w^{z^2}$ (with $z$ a generator of $Z$) are the non-trivial elements of $V$. However, as noticed above, $i$ centralizes $Z$. Hence $w^{zi} = w^{iz} = w^z$ and $w^{z^2 i} = w^{iz^2} = w^{z^2}$. That is, $i$ centralizes $V$. Thus, $G_1$ is uniquely determined:

$$G_1 = ((V \times W){:}L)\langle i\rangle$$

with $WL\langle i\rangle = Z \times WL'\langle i\rangle = 3 \times 2^4{:}S_5$ and $VL\langle i\rangle = VZ \times L'\langle i\rangle = GL_1(4) \times S_5$.

The above holds in (i.b), too, but for replacing $L = L_2(4)$ with $LF = \Sigma L_2(4)$, where the factor $F$ has order 2 and is contributed by the non-trivial automorphism of $GF(4)$. Accordingly, $F$ permutes the two non-trivial elements of $Z$ and acts non-trivially on both $V$ and $W$. Thus,

$$G_1 = ((V \times W){:}LF)\langle i\rangle$$

where

$$\begin{array}{ll} WLF = (Z \times WL')F & \text{with } ZF = S_3 \text{ and } WL'F = A\Sigma L_2(4), \\ VLF = (VZ \times L')F & \text{with } VZF = A\Gamma L_1(4) \text{ and } (Z \times L')F = \Gamma L_2(4). \end{array}$$

Furthermore, $i$ centralizes $VZF$. (Indeed, $i$ centralizes $V$ and $Z$, as above, and normalizes $ZF \cong S_3$.) Thus,

$$\begin{array}{l} G_{0,1} = (V \times W){:}LF = (2^2 \times 2^4)\Gamma L_2(4), \\ G_{1,2}/Z \cong WL'F\langle i\rangle = 2^5{:}S_5 \end{array}$$

(note also that $G_{1,2}/Z$ can be recognized inside $Aut(M_{22})$ as the stabilizer of a duad; see [11]). □

**Corollary 3.7.** *Let $\Gamma$ be a flag-transitive $c \cdot AG^*$-geometry of order* 4 *and residual dimension* 3*. Then* (i.a) *of Theorem* 3.5 *holds for some flag-transitive subgroup $G$ of $Aut(\Gamma)$.*

*Proof.* Let $\widetilde{\Gamma}$ be the unique flag-transitive simply connected $c \cdot AG^*$-geometry of order 4 and residual dimension 3 (Theorem 3.6) and let $\Gamma(\mathcal{S})$ be as in Section 2. The group $Aut(\Gamma(\mathcal{S}))$ satisfies (i.b) of Theorem 3.5 whereas (i.a) is satisfied by the commutator subgroup of $Aut(\Gamma(\mathcal{S}))$. Furthermore, as $\widetilde{\Gamma}$ is the universal cover of $\Gamma(\mathcal{S})$, $Aut(\Gamma(\mathcal{S}))$ lifts to a subgroup $\widetilde{G}$ of $Aut(\widetilde{\Gamma})$ and, as $Aut(\Gamma(\mathcal{S}))$ satisfies (i.b) of Theorem 3.5, that condition holds in $\widetilde{G}$, too. Hence, according to Theorem 3.5(i), $\widetilde{G} = Aut(\widetilde{\Gamma})$.

On the other hand, the commutator subgroup of $Aut(\Gamma(\mathcal{S}))$ lifts to a flag-transitive proper normal subgroup $\widetilde{G}'$ of $\widetilde{G}$ satisfying (i.a). (It is not difficult to see that $\widetilde{G}'$ is indeed the commutator subgroup of $\widetilde{G}$, but we will not exploit this fact here.) By Theorem 3.5(i), $\widetilde{G}'$ is the unique flag-transitive proper subgroup of $\widetilde{G}$.

Let $\Gamma$ be a flag-transitive $c \cdot AG^*$-geometry of order 4 and residual dimension 3. By Theorem 3.6, we have $\Gamma = \widetilde{\Gamma}/Z$ for some subgroup $Z$ of $\widetilde{G}$. As $\Gamma$ is flag-transitive, the normalizer of $Z$ in $\widetilde{G}$ is also flag-transitive. Hence it contains $\widetilde{G}'$. Therefore the group $G = Z\widetilde{G}'/Z$ is flag-transitive on $\Gamma$. As $\widetilde{G}'$ satisfies (i.a) of Theorem 3.5, that condition also holds for $G$. □

### 3.4. The case of $(n, q) = (3, 4)$. Properties (LL) and (T)

In the sequel $\Gamma$ is a flag-transitive $c \cdot AG^*$-geometry of order 4 and residual dimension 3. By Corollary 3.7, $Aut(\Gamma)$ admits a flag-transitive subgroup $G$ with $G_0 = ASL_3(4)$ and $G_2 = 3^{\cdot}M_{22}$. Henceforth $G$ will stand for such a subgroup of $Aut(\Gamma)$.

The reader is referred to Subsection 1.5 for the statement of the properties (LL) and (T). According to the conventions stated in that subsection, the symbol $\sim$ denotes the collinearity relation of $\Gamma$.

**Proposition 3.8.** *The property* (LL) *holds in* $\Gamma$.

*Proof.* The relation ‘having the same points’ is an equivalence relation on the set of lines of $\Gamma$. Let us denote that relation by $\Theta$. Clearly, no two lines in the same class of $\Theta$ are incident to the same plane. Furthermore, by the flag-transitivity of $G$, all classes of $\Theta$ have the same size, say $\nu$. We have $\nu = 1$ if and only if (LL) holds.

Assume $\nu > 1$. Given a point $a$ of $\Gamma$, let $\Theta_a$ be the relation induced by $\Theta$ on the set of lines incident to $a$. On the one hand, $\Theta_a$ is not the trivial relation (where any two lines correspond). Indeed no two lines in the same class of $\Theta$ are incident to the same plane. On the other hand, $\Theta_a$ is not even the identity relation, as $\nu > 1$ by assumption. Furthermore, $G_a$ $(= ASL_3(4))$ permutes the classes of $\Theta_a$. However, the partition in parallel classes is the unique non-trivial partition of the set of lines of $AG(3,4)$, different from the identity partition and preserved by $ASL_3(4)$. Consequently, $\Theta_a$ is the parallelism of the affine space $\Gamma^*_a$.

Thus, $\nu = 16$ and the points of $a^\sim \setminus \{a\}$ bijectively correspond to the parallel classes of $\Gamma^*_a$. Hence $|a^\sim \setminus \{a\}| = 21$. Furthermore, for any parallel class $X$ of $\Gamma^*_a$ and any line $l$ of $\Gamma^*_a$ not in $X$, there are lines of $X$ meeting $l$ in a point of $\Gamma^*_a$, namely coplanar with $l$ in $\Gamma$. Therefore, any two points of $a^\sim$ belong to a common plane of $\Gamma$, whence they are collinear. It is now clear that $\Gamma$ has exactly 22 points. As 22 is the number of points in a plane, $\Gamma$ is flat. Accordingly, $|G : G_0| = 22$ and $|G : G_2| = 64$.

Let $K$ be the kernel of the action of $G$ on the set of points of $\Gamma$. Then $K$ is a normal subgroup of $G_a$, stabilizing all parallel classes of $\Gamma^*_a$. On the other hand, if an element $g \in G_a$ stabilizes all parallel classes of $\Gamma^*_a$, then $g$ also stabilizes all points of $\Gamma$. Consequently, $g \in K$. Therefore, $K = 2^6.3$. Hence $|G : G_2| = |K|/3$. It is also clear that $G_2 \cap K = K_2 = 3$. Thus, $G = K.\overline{G}_2 = 2^6.3.M_{22}$. Also, $G = O_2(K){:}G_2 = 2^6{:}3^{\cdot}M_{22}$.

Let $X$ be the elementwise stabilizer in $G$ of the set of planes of $\Gamma$. As noticed in Subsection 1.1, we have $K_0 = 1$. Hence $X$ acts semi-regularly on the set of points of $\Gamma$. Therefore, $X$ is a normal subgroup of $\overline{G}_2$ acting semi-regularly on the 22 points of $\Gamma$. However, $\overline{G}_2 = M_{22}$. This makes it clear that $X = 1$. Namely, $G$ acts faithfully on the set of planes of $\Gamma$. Furthermore, $G_0$ acts 2-transitively as $ASL_3(4)$ on that set of planes. Hence $G$ is also 2-transitive on the 64 planes of $\Gamma$. As a consequence, the stabilizer $G_u$ in $G$ of a plane $u$ permutes transitively the 63 planes of $\Gamma$ other than $u$, whence it admits a subgroup of index 63. However, $G_u = 3^{\cdot}M_{22}$ and the group $3^{\cdot}M_{22}$ has no subgroups of index 63, since $M_{22}$ has no subgroup of index 63 or 27 (see [11]). We have reached a final contradiction. □

**More notation.** As (LL) holds in $\Gamma$, the lines of $\Gamma$ are just the pairs of distinct collinear points. Given two distinct collinear points $a, b$, we denote the line $\{a, b\}$ by $ab$.

As $\Gamma$ satisfies (LL), it also satisfies the Intersection Property. In particular, distinct planes of $\Gamma$ are incident to distinct sets of points and a line $ab$ is incident to a plane $u$ precisely when both $a$ and $b$ are incident to $u$. Thus, planes may be viewed as sets of points. Accordingly, given a point $a$ and a plane $u$, we write $a \in u$ if $a$ is incident to $u$; if $a \notin u$, we denote the set of points of $u$ collinear with $a$ by $a^\sim \cap u$; and so on.

**Lemma 3.9.** *The group $G$ is transitive on the set of triples $(a, b, u)$ with $a, b$ distinct points of $\Gamma$, $u$ a plane of $\Gamma$ at distance 1 from $a$ and $b \in a^\sim \cap u$.*

*Proof.* Given a point $b$, its stabilizer $G_b$ $(\cong ASL_3(4))$ is transitive on the non-incident point-line pairs of the affine space $\Gamma^*_b$. The lemma follows from this and from the transitivity of $G$ on the set of points of $\Gamma$. □

As a consequence, $G$ is also transitive on the set of point-plane pairs $(a, u)$ of $\Gamma$, with $a$

at distance 1 from $u$. (Note that such pairs exist, otherwise $\Gamma$ would be flat, contrary to Proposition 3.8.)

According to the above, given a point-plane pair $(a, u)$ with $a$ at distance 1 from $u$, the number $|a^{\sim} \cap u|$ does not depend on the particular choice of $(a, u)$. We shall denote that number by $\gamma$. Clearly, $\gamma \leq 22$.

**Lemma 3.10.** *We have* $\gamma < 22$.

*Proof.* Assume the contrary. Given three points $a, b, c$ with $a \sim b \sim c$, let $u$ be a plane on the line $bc$. As $\gamma = 22$, $a$ is collinear with all points of $u$. In particular, $a \sim c$. Consequently, any two points of $\Gamma$ are collinear. Hence $\Gamma$ has $1 + 16 \cdot 21 = 337$ points. On the other hand, every point is in 64 planes and every plane has 22 points. Thus, the number of planes is $337 \cdot 64/22$. However, this number is not an integer; contradiction. □

**Proposition 3.11.** *The property* (T) *holds in* $\Gamma$.

*Proof.* It is easily seen that (T) holds if and only if $\gamma = 5$. Thus, we need to prove that $\gamma = 5$.

Given a plane $u$ and a point $a$ at distance 1 from $u$, let $b \in a^{\sim} \cap u$. The plane $u$, viewed as a point of the affine space $\Gamma^*_b$, does not belong to the line $L = ab$ of $\Gamma^*_b$. The points of $a^{\sim} \cap (u \setminus \{b\})$ bijectively correspond to certain lines of $\Gamma^*_b$ through $u$. Let $L(b)$ be the set of those lines. Then $\gamma = |L(b)| + 1$.

The stabilizer $G_{a,b,u}$ of $a$ and $u$ in $G_b$ is the stabilizer of the pair $(u, L)$ of $\Gamma^*_b$. However, $G_b \cong ASL_3(4)$ in its natural action on $AG(3, 4) \cong \Gamma^*_b$. The stabilizer of $(u, L)$ in $ASL_3(4)$ has three orbits on the set of lines of $AG(3, 4)$ through $u$. One of those orbits, say $L_1(b)$, contains the four lines joining $u$ with points of $L$. Another orbit, say $L_2(b)$, contains just one line, namely the line through $u$ parallel to $L$. Finally, the third orbit, say $L_3(b)$, contains all lines through $u$ skew with $L$. Note that the stabilizer of $(u, L)$ in $ASL_3(4)$ acts 2-transitively on $L_1(b)$.

Clearly, $L_1(b) \subseteq L(b)$. Also, $L(b) \neq L_1(b) \cup L_2(b) \cup L_3(b)$, by Lemma 3.10. Therefore, $L(b)$ is equal to one of $L_1(b)$, $L_1(b) \cup L_2(b)$ or $L_1(b) \cup L_3(b)$. Accordingly, $\gamma = 5, 6$ or 21. To finish, we only need to show that each of the hypotheses $\gamma = 6$ and $\gamma = 21$ yields a contradiction.

Let $\gamma = 6$. Given $b \in a^{\sim} \cap u$, let $U(b)$ be the set of points different from $b$ and incident to lines of $L_1(b)$ and let $u(b)$ be the point of the line of $L_2(b)$ different from $b$. Clearly, $|U(b)| = 4$ and $\{b\} \cup \{u(b)\} \cup U(b) = a^{\sim} \cap u$. Furthermore, $G_{a,b,u}$ stabilizes $u(b)$ and acts 2-transitively on $U(b)$. Note also that $\{b, u(b)\}$ is the unique line of $u$ on $b$ contained in $a^{\sim}$ but not coplanar with $a$. Therefore, if $b' = u(b)$, then $u(b') = b$. Consequently, the group $G_{a,u}$, which transitively permutes the six points of $a^{\sim} \cap u$, preserves a partition of that set in the three pairs. However, for every point $b \in a^{\sim} \cap u$, $G_{a,b,u}$ is 2-transitive on the four points of $a^{\sim} \cap u$ not in the pair containing $b$. Hence $G_{a,u}$ cannot preserve a partition of $a^{\sim} \cap u$ in three pairs; contradiction.

Let $\gamma = 21$. With the above notation, $u(b)$ is the unique point of $u$ at distance 2 from $a$. Namely, given three points $a, b, c$ with $a \sim b \sim c$, we have $a \not\sim c$ if and only if the lines $ba$ and $bc$ are parallel in $\Gamma^*_b$. As a consequence, for every point $b \sim a$, there are precisely 15 points collinear with $b$ at distance 2 from $a$. Furthermore, given two points $b, c$ with $c \sim b \sim a$ and

$c \not\sim a$, let $v$ be a plane on $c$. There are four planes on $bc$ meeting $v$ in a line. Let $u$ be one of them and let $d$ be the point of $u \cap v$ different from $c$. The plane $u$ has distance 1 from $a$. Hence all its points but one are collinear with $a$. However, $c \not\sim a$. Hence $d \sim a$. Thus, $v$ has distance 1 from $a$. As a consequence, all planes of $\Gamma$ have distance at most 1 from $a$. Therefore, for every plane $u$ of $\Gamma$, at most one point of $u$ has distance 2 from $a$.

Given two collinear points $c, d$, let $u$ be a plane on the line $cd$. As at most one point of $u$ has distance 2 from $a$, at most one of $c$ and $d$ has distance 2 from $a$. Therefore, given a point $c$ at distance 2 from $a$, all points collinear with $c$ are also collinear with $a$. Furthermore, as noticed above, every point collinear with $a$ is collinear with exactly 15 points at distance 2 from $u$. Hence there are precisely 15 points at distance 2 from $a$. Each of them belongs to 64 planes and these have distance 1 from $a$. On the other hand, $G_a$ transitively permutes the planes at distance 1 from $a$ and every such plane contains just one point at distance 2 from $a$. Hence $G_a$ transitively permutes the 15 points at distance 2 from $a$. Consequently, $G_a$ contains a subgroup of index 15. However, $G_a \cong ASL_3(4)$ and it is easily seen that no subgroup of index 15 exists in $ASL_3(4)$. A contradiction has been reached. □

### 3.5. The case of $(n, q) = (3, 4)$. Local geometries

Let $\Gamma$ and $G$ be as in the previous subsection.

**Definition.** *Given two points $a, b$ of $\Gamma$ at distance 2, an $(a, b)$-local subspace is a connected component of the graph induced by the point-line system of $\Gamma$ on the set $a^\sim \cap b^\sim$. If $X$ is an $(a, b)$-local subspace, then the structure formed by the graph $X$ and by the planes of $\Gamma$ with at least three points in $X$ will be called an $(a, b)$-local geometry of $a$ and $b$, also the local geometry of $X$.*

**The second neighbourhood of a point.** Given a point $a$, by $P(a)$ (respectively, $S(a)$) we mean the set of points at distance 2 from $a$ such that for some point $c \in a^\sim \cap b^\sim$ the lines $ac$ and $bc$ are parallel (skew) in the affine space $\Gamma^*_c$. Given a point $b$ at distance 2 from $a$, by $P(a, b)$ (respectively, $S(a, b)$) we mean the set of points $c \in a^\sim \cap b^\sim$ such that the lines $ac$ and $bc$ are parallel (skew) in $\Gamma^*_c$. Given a point $c \sim a$, the group $G_{a,c}$ has two orbits on the set of lines on $c$ non-coplanar with $ac$. Therefore,

**Lemma 3.12.** *Either $S(a) = P(a)$ or $S(a) \cap P(a) = \emptyset$.*

In the latter case, $|c^\sim \cap P(a)| = 15$ and $|c^\sim \cap S(a)| = 16 \cdot 15$. In any case, $P(a) \neq \emptyset$. The following is also clear.

**Lemma 3.13.** *Let $b \in P(a)$ and $c \in P(a, b)$. Let $\pi$ be the plane of $\Gamma^*_c$ spanned by the two parallel lines $ac$ and $ab$. Then the lines of $c$ coplanar with both $ac$ and $bc$ in $\Gamma$ are the* 16 *lines of $\pi$ not parallel to $ac$ and $ab$.*

**Terminology.** We recall that, given a Laguerre plane $\Xi$, a rank 3 geometry can be built over $\Xi$ by 'enrichment' [26, Chapter 2], namely by taking the 2-subsets of the circles as new elements, to be placed between the points and the circles of $\Xi$. The incidence relation is the natural one (symmetrized inclusion). Henceforth, when speaking of a *Laguerre plane*, we will always mean its rank 3 enrichment, constructed as above. Accordingly, the circles of a Laguerre plane will be called *planes* and the 2-subsets of the circles will be called *lines*.

**Lemma 3.14.** *Let $b \in P(a)$. Then $P(a,b)$ is a join of $(a,b)$-local subspaces. Each of them has size* 20 *and its local geometry is a Laguerre plane of order* 4.

*Proof.* Let $c \sim d$ with $c \in P(a,b)$ and $d \in S(a,b)$, if possible. By (T) (Proposition 3.11), the line $cd$ is coplanar with both $ca$ and $cb$ and, by Lemma 3.13, there are 12 lines in $\Gamma_c$ coplanar with $ac$, $bc$ and $dc$. Therefore, and by (T), there are 12 lines on $d$ coplanar with $ad$, $bd$ and $cd$. However, the lines $ad$ and $bd$ are skew in the affine space $\Gamma_d^*$ and $cd$, being a transversal of them in $\Gamma_d^*$, cannot meet 12 distinct transversals of $ad$ and $cd$; contradiction. Consequently, $P(a,b)$ is the disjoint union of $(a,b)$-local subspaces.

Let $\Xi$ be an $(a,b)$-local geometry contained in $P(a,b)$. By the above, the residues of the points of $\Xi$ are isomorphic to the geometry obtained from $AG(2,4)$ by removing a bundle of parallel lines. Hence $\Xi$ has 20 points and it is a Laguerre plane. □

**Lemma 3.15.** *The number $|P(a)|$ divides* 252.

*Proof.* For every point $c \in a^{\sim}$ there are precisely 15 points $b \in P(a)$ for which $c \in P(a,b)$. On the other hand, $|a^{\sim}| = 2^4 \cdot 3 \cdot 7$. The conclusion follows from Lemma 3.14. □

**Lemma 3.16.** *We have $P(a) \cap S(a) = \emptyset$.*

*Proof.* Assume the contrary. Then $S(a) = P(a)$ by Lemma 3.12. However, given $c \sim a$, there are precisely $17 \cdot 15 = 255$ points $b \sim c$ at distance 2 from $a$: too many! (Compare Lemma 3.15.) □

**Lemma 3.17.** *The group $G_a$ has two orbits on the set of points at distance* 2 *from $a$, namely $P(a)$ and $S(a)$. Furthermore, it is transitive on the set of paths $a \sim c \sim b$ with $b \in P(a)$ (respectively, $b \in S(a)$).*

*Proof.* Clear, by Lemma 3.16 and because, given $c \in a^{\sim}$, $G_{a,c}$ is transitive on $P(a) \cap c^{\sim}$ and $S(a) \cap c^{\sim}$. □

We recall that an extended grid of order $s$ is a $c.C_2$-geometry satisfying (LL) and where the residues of the points are grids of order $s$ (Cameron, Hughes and Pasini [10]).

**Lemma 3.18.** *Let $b \in S(a)$. Then the $(a,b)$-local geometries are extended grids of order* 3 *satisfying the property* (T).

*Proof.* The lines of $AG(3,4)$ meeting two given skew lines form a dual grid of order 3. The conclusion follows from this, from Lemma 3.17 and from the fact that (T) holds in $\Gamma$. □

**Lemma 3.19.** *Distinct points of $S(a)$ give rise to distinct local geometries.*

*Proof.* Indeed, given $c \in a^{\sim}$ and $b, b' \in S(a) \cap c^{\sim}$, let $L_{ab}$ be the set of lines of $\Gamma_c^*$ meeting both $ca$ and $cb$. If all those lines also meet $cb'$, then $cb = cb'$. □

**Definition.** *Given $b$ at distance 2 from $a$, let $\Xi$ be an $(a,b)$-local geometry. We say that $\Xi$ is of Laguerre type (of grid type) if it is a Laguerre plane (an extended grid), namely if $b \in P(a)$ (if $b \in S(a)$).*

**Planes at distance 1 from a point.** Given a plane $u$ at distance 1 from $a$, let $c \in u \cap a^{\sim}$. As (T) holds in $\Gamma$, the other points of $u \cap a^{\sim}$ are the four lines on $c$ contained in $u$ and coplanar with $ac$. Thus,

**Lemma 3.20.** *We have $|u \cap a^{\sim}| = 5$.*

Given $u$ and $c$ as above, in $\Gamma_c^*$ we see that $G_{a,u}$ stabilizes a unique point $b$ of $u$ distinct from $c$, namely the point $b$ for which $cb$ is parallel to $ca$, and permutes the remaining 16 points of $u \backslash a^{\sim}$ transitively. We denote that point by $p_u(a)$ and we call it the *projection* of $a$ onto $u$. Clearly, $c \in P(a, p_u(a))$. Therefore,

**Lemma 3.21.** *We have $u \cap P(a) = \{p_u(a)\}$ and $u \backslash (a^{\sim} \cup \{p_u(a)\}) \subset S(a)$.*

**Lemma 3.22.** *There are $2^6 \cdot 3^2 \cdot 7$ planes at distance 1 from $a$.*

*Proof.* There are $16 \cdot 21$ points in $a^{\sim}$, each of them is in $4^3 - 4$ planes at distance 1 from $a$ and each such plane is counted 5 times in this way. Therefore, the number of planes at distance 1 from $a$ is $16 \cdot 21 \cdot 12$. □

**Lemma 3.23.** *Let $u$ be a plane at distance 1 from $a$ and let $c_1, c_2, c_3, c_4, c_5$ be the five points of $u$ collinear with $a$. Then the lines $ac_1, ac_2, ac_3, ac_4, ac_5$ belong to a plane of the affine space $\Gamma_a^*$ and form a non-degenerate dual conic in that plane.*

*Proof.* Any two of those lines meet in the affine space $\Gamma_a^*$. On the other hand, no plane of $\Gamma$ on $a$ contains three of them. Indeed, if $v$ were such a plane, then $v = u$ by (T) and this cannot be. The conclusion follows. □

**Local geometries of Laguerre type.** Let $\Xi$ be an $(a,b)$-local geometry of Laguerre type. For every point $c$ of $\Xi$, the line $ac$ belongs to 4 planes of $\Xi$ and each of these planes contains 5 points of $\Xi$. Therefore, precisely 16 of the 64 planes of $\Xi$ contain $a$.

**Lemma 3.24.** *The 16 planes of $\Xi$ on $a$ and the 20 lines of $\Xi$ on $a$ are the 16 points and the 20 lines of a plane $\pi_\Xi$ of the affine space $\Gamma_a^*$. The remaining planes of $\Xi$ are the $3 \cdot 16$ non-degenerate dual conics of $\pi_\Xi$ with the line at infinity of $\pi_\Xi$ as the nuclear line. These conics are shared in three orbits by the stabilizer $ASL_2(4)$ of $\pi_\Xi$ in $G_a$. One of these orbits is contributed by the 16 planes of $\Gamma$ on $b$ at distance 1 from $a$.*

*Proof.* Let $u$ be a plane of $\Xi$ on $a$. In the affine space $\Gamma_a^*$, let $L_1, L_2, \dots, L_5$ be the five lines of $\Xi \cap \Gamma_a^*$ through $u$. The points of $\bigcup_{i=1}^5 L_i$ in $\Gamma_a^*$ are the 16 points of $\Gamma_a^* \cap \Xi$. Similarly, if $v \in L_i \backslash \{u\}$, the five lines of $\Gamma_a^* \cap \Xi$ on $v$ contribute 15 points of $\Gamma_a^* \cap \Xi$ other than $v$. However, the 5 lines on $u$ already give us all points of $\Gamma_a^* \cap \Xi$. Therefore, the 5 lines of $\Gamma_a^* \cap \Xi$ on $v$ meet each of $L_1, L_2, \dots, L_5$. The conclusion is clear (see also Lemma 3.23). □

**Local geometries of grid type.** By Lemma 3.17, all local geometries of grid type have the same isomorphism type.

Let $b \in S(a)$ and let $\Xi$ be an $(a,b)$-local geometry. We denote by $\Pi_a$ (by $\Pi_b$) the set of planes of $\Xi$ containing $a$ (respectively, $b$) and $\Xi_a$ will denote the configuration of points and lines of the affine space $\Gamma_a^*$ formed by the planes of $\Pi_a$ and the lines joining $a$ with points of $\Xi$.

**Lemma 3.25.** *The pair $\{\Pi_a, \Pi_b\}$ is a partition of the set of planes of $\Xi$ with the property that no two planes in the same class have a line in common.*

*Proof.* If $n$ is the number of points of $\Xi$, then $|\Pi_a| = |\Pi_b| = 4n/5$. However, $\Xi$ has $8n/5$ planes in total and no two planes on $a$ (or on $b$) meet in a line of $\Xi$. The statement follows.□

By Lemma 3.18, $\Xi$ is an extended grid of order 3 satisfying (T). Furthermore, by Lemma 3.17, $\Xi$ admits a point-transitive automorphism group and, by Lemma 3.25, its planes are partitioned in two classes in such a way that no two planes of the same class meet in a line. By Fisher [13] (also Blokhuis and Brouwer [4]), $\Xi$ is one of the following:

(I) The geometry obtained from the Coxeter complex of type $A_7$ by truncating the wings of the diagram but saving three nodes in the middle, then forming the Grassmann geometry [26, Chapter 5] with respect to the central node. This extended grid is flag-transitive, it has 70 points and 112 planes, partitioned in two classes of size 56.

(II) The extended grid called $2.CF^-(4)$ in [25]. This extended grid is flag-transitive, it has 40 points and 64 planes, partitioned in two classes of size 32.

**Lemma 3.26.** *The extended grid $\Xi$ is of type* (II).

*Proof.* Let $\Xi$ be of type (I), if possible. Then $\Xi_a$ contains 56 points of the affine space $\Gamma_a^*$. Thus, 8 out of the 64 points of $\Gamma_a^*$ are missing in $\Xi_a$. $G_{a,b}$ acts transitively on the set of lines of $\Xi_a$. Given a line $ac \in \Xi_a$, $G_{a,c,b}$ has order $2^4 \cdot 3$. As there are 70 lines in $\Xi_a$, the stabilizer $H$ of $\Xi$ in $G_a$ has order $70 \cdot 2^4 \cdot 3 = 2^5 \cdot 3 \cdot 5 \cdot 7$. The group $H$ acts on the 8 points of $\Gamma_a^*$ missing in $\Xi_a$. Let $u$ be one of them. Its stabilizer $H_u$ in $H$ has order $2^2 \cdot 3 \cdot 5 \cdot 7$ or a multiple of this number. It is a subgroup of $SL_3(4)$ $(= G_{a,u})$. Thus, its index in $SL_3(4)$ is a divisor of $9 \cdot 16$. However, no maximal subgroup of $SL_3(4)$ has index a divisor of $9 \cdot 16$ (see [11]). Therefore $H_u = SL_3(4)$. But this is clearly impossible: if $H_u = SL_3(4)$ then $H$ cannot have an orbit of size 70 on the set of lines of $\Gamma_a^*$. □

As $\Xi$ is of type (II), $\Xi_a$ contains 32 points and 40 lines of the affine space $\Gamma_a^*$. Furthermore, $G_{a,b}$ acts transitively on the set of lines of $\Xi_a$. Given a line $ac \in \Xi_a$, we have $|G_{a,c,b}| = 2^4 \cdot 3$. As there are 40 lines in $\Xi_a$, the stabilizer $H$ of $\Xi$ in $G_a$ has order $40 \cdot 2^4 \cdot 3 = 2^7 \cdot 3 \cdot 5$. (Note that $H$ also stabilizes $b$, by Lemma 3.19.) The group $H$ acts on the 32 points of $\Xi_a$. Let $u$ be one of them.

**Lemma 3.27.** *The stabilizer $H_u$ of $u$ in $H$ is isomorphic to the alternating group $Alt(5)$ of degree* 5 *and the five lines of $u$ through $a$ belonging to $\Xi_a$ form a non-degenerate quadratic cone of $\Gamma_a^*$ with vertex $u$.*

*Proof.* Given a line $ac$ in $\Xi_a$, $G_{a,b,c}$ is on the four points of $\Gamma_c^*$ belonging to the line $ac$. Hence $H$ is transitive on the 32 points of $\Gamma_a^*$ belonging to $\Xi_a$ and the stabilizer $H_u$ of $u$ in $H$ has order $2^2 \cdot 3 \cdot 5$. $H_u$ is a subgroup of $G_{a,u} = SL_3(4)$ of index $63 \cdot 16$. There are only two maximal subgroups of $SL_3(4)$ that might fit with this situation (see [11]), namely $3.Alt(6)$ and $3.(2^4{:}Alt(5))$. Assume $H_u \leq 3.Alt(6)$. As $H_u$ has index 18 in $3.Alt(6)$, we have $H_u = 3.Alt(5)$ $(= 3 \times Alt(5))$ (see [11]), acting as $Alt(5)$ on the set of lines of $\Gamma_a^*$ on $u$. On the other hand, if $H_u$ is a subgroup of $3.(2^4{:}Alt(5))$, it has index $3 \cdot 16$ in it. In this case, comparing the information we have on stabilizers of planes of $\Gamma$, we get $H_u = Alt(5)$. In any case, $H_u$ acts as $Alt(5)$ on the star of $u$ in $\Gamma_a^*$. In the first case, $H_{u,c} = 3 \times Alt(4)$. In the second case, $H_{u,c} = Alt(4)$. However, it is easy to check that $H_{u,c} = Alt(4)$. Hence $H_u = Alt(5)$.

Each subgroup of $L_3(4)$ of type $Alt(5)$ is the stabilizer of a non-degenerate conic (and its nucleus) or of a non-degenerate dual conic (and its nuclear line). Therefore, $H_u$ is either the stabilizer of a set of 5 planes of $\Gamma_a^*$ on $u$ forming a non-degenerate dual conic in the star of $u$ or the stabilizer of a non-degenerate quadratic cone of $\Gamma_a^*$ of vertex $u$.

Assume the first case and let $C$ be the set of five planes on $u$ stabilized by $H_u$. Then $C$ is a non-degenerate conic in the projective plane dual of the star of $u$ in $\Gamma_a^*$. Let $\pi$ be its nucleus. The group $H_u$ has three orbits on the set of lines of $\Gamma_a^*$ on $u$, of size 5, 6 and 10 and the plane $\pi$ gives us the orbit of size 5. Comparing the information we have given on stabilizers of planes of $\Gamma$, the 5 lines of $\Xi_a$ on $u$ are the lines of $\pi$ on $u$, say them $L_1, L_2, L_3, L_4, L_5$. Let $v \in L_1 \setminus \{u\}$. If the five lines of $\Xi_a$ on $v$ still belong to $\pi$, then, by the transitivity of $H$ on the set of point-line pairs of $\Xi_a$ and by the connectivity of $\Xi_a$, all lines of $\Xi_a$ belong to $\pi$. This is impossible (there are 40 lines in $\Xi_a$, but only 20 of them can stay in $\pi$). Therefore, the plane of $\Gamma_a^*$ containing the five lines on $v$ is different from $\pi$. Thus we get at least $16 + 3 \cdot 12$ points for $\Xi_a$, namely 16 points in $\pi$, and 12 more points in the plane of $\Gamma_a^*$ containing the five lines of $\Xi_a$ on $v$, for each of the three points $v \in L_1 \setminus \{u\}$. However, $16 + 3 \cdot 12 > 32$; contradiction. Therefore, $H_u$ stabilizes a non-degenerate quadratic cone of $\Gamma_a^*$ with vertex $u$. □

Let $C_0$ be the cone of $\Gamma_a^*$ formed by the five lines of $u$ belonging to $\Xi_a$ and let $L_0$ be the nuclear line of $C_0$ in $\Gamma_a^*$. Let $u_0$, $u_1$, $u_2$, $u_3$ be the four points of $L_0$, with $u_0 = u$, and for $i = 1, 2, 3$, let $C_i$ be the image of $C_0$ by the translation of $\Gamma_a^*$ sending $u_0$ to $u_i$ and let $\overline{C}_i$ be the set of points of $\Gamma_a^*$ belonging to lines of $C_i$ but different from $u_i$. The following is straightforward:

**Lemma 3.28.** *The sets $\overline{C}_0, \overline{C}_1, \overline{C}_2, \overline{C}_3, \{u_0\}, \{u_1\}, \{u_2\}, \{u_3\}$ are the orbits of $H_u$ on the point-set of the affine space $\Gamma_a^*$.*

Clearly, the set of points of $\Gamma_a^*$ belonging to $\Xi_a$ is a union of orbits of $H$ on $\Xi_a$. Hence, it is a union of orbits of $H_u$. As $\Xi_a$ has 32 points, it is the union of $C_0$ with one of the cones $C_1, C_2$ or $C_3$. Needless to say, we may assume to have chosen indices in such a way that $\overline{C}_0 \cup \overline{C}_1 \cup \{u_0, u_1\}$ is the point-set of $\Xi_a$.

**Lemma 3.29.** *Every line of $\Gamma_a^*$ not contained in $\Xi_a$ meets $\Xi_a$ in either 0 or 2 points.*

*Proof.* Let $L$ be a line of $\Gamma_a^*$ not contained in $\Xi_a$ but meeting $\Xi_a$ in a point $u$. By Lemma 3.28, $\Xi_a$ is the union of two parallel cones $C_0$ and $C_1$ with vertices $u_0 = u$ and $u_1$, respectively, the latter lying on a line $L_0$, which is the nuclear line of both $C_0$ and $C_1$. As $L$ is not contained in $\Xi_a$, it is not a line of $C_0$. If $L = L_0$, then $L \cap \Xi_a = \{u_0, u_1\}$. In this case we are done.

Otherwise, let $\pi$ be the plane of $\Gamma_a^*$ spanned by $L$ and $L_0$. As $L_0$ is the nuclear line of both $C_0$ and $C_1$, the plane $\pi$ meets $C_i$ (for $i = 0, 1$) in a line $M_i$ through $u_i$, and the lines $M_0, M_1$ are parallel, the translation sending $u_0$ to $u_1$ also maps $C_0$ onto $C_1$. The points of $L$ in $\Xi_a$ are the meet-points of $L$ with $M_0$ and $M_1$. □

### 3.6. The case of $(n, q) = (3, 4)$. The diameter of $\Gamma$

With $\Gamma$ as in the previous subsection, we have the following:

**Proposition 3.30.** *The geometry $\Gamma$ has diameter $\delta \leq 4$.*

*Proof.* Let $b, b'$ be points of $\Gamma$ at distance 5, if possible, and let $b \sim c \sim a \sim a' \sim c' \sim b'$ be a path from $b$ to $b'$. Let $u$ be a plane of $\Gamma$ on the line $aa'$. As $b, b'$ have distance 5, the plane $u$ contains no points of either $a^{\sim} \cap b^{\sim}$ or $a'^{\sim} \cap b'^{\sim}$.

Let $\Xi$ be an $(a, b)$-local geometry. Assume that $b \in P(a)$. By Lemma 3.24, 16 out of the 21 lines of $\Gamma_a^*$ through $u$ are incident to a plane of $\Xi$. Hence $u$ contains 16 points of $\Gamma_a^*$ different from $a$ and at distance 2 from $b$.

On the other hand, let $b \in S(a)$. By Lemma 3.29, every line of $\Gamma_a^*$ through $u$ meets $\Xi_a$ in either 0 or 2 points. Furthermore, $\Xi_a$ has 32 points. Hence $u$ contains precisely 16 points of $\Gamma$ at distance 2 from $b$ and different from $a$.

Thus, in any case, $u$ contains 17 points at distance 2 from $b$, including $a$ among them. Similarly, $u$ contains 17 points at distance 2 from $b'$. However, $17 + 17 > 22$, which is the total number of points of $u$. Hence $u$ contains some points at distance 2 from both $b$ and $b'$, contrary to the hypothesis that $b$ and $b'$ have distance 5. □

### 3.7. The case of $(n, q) = (3, 4)$. Quotients

Let $\Gamma$ and $G$ be as in Subsection 3.4 and 3.5, but now we also assume that $\Gamma$ admits a flag-transitive proper quotient $\overline{\Gamma}$. Let $f : \Gamma \to \overline{\Gamma}$ be a covering from $\Gamma$ to $\overline{\Gamma}$.

**Lemma 3.31.** *The geometry $\Gamma$ has diameter $\delta = 4$ and any two distinct points in the same fiber of $f$ have mutual distance 4.*

*Proof.* We have $\delta \leq 4$ by Proposition 3.30. On the other hand, both properties (LL) and (T) also hold in $\overline{\Gamma}$, by Propositions 3.8 and 3.11. If $f(a) = f(a')$ for distinct points $a, a'$ of $\Gamma$ at distance $\leq 3$, then either (T) or (LL) would fail in $\overline{\Gamma}$. The conclusion follows. □

**Lemma 3.32.** *Let $a, b$ be distinct points of $\Gamma$ at distance 2. Then $f(a)$, $f(b)$ have distance 2 in $\overline{\Gamma}$ and we have $f(b) \in S(f(a))$ (or $f(b) \in P(f(a))$) if and only if $b \in S(a)$ (resp. $b \in P(a)$).*

*Proof.* The points $f(a), f(b)$ are distinct by Lemma 3.31. Clearly, they have distance 2. Furthermore, $f$ isomorphically maps every $(a, b)$-local geometry onto an $(f(a), f(b))$-local geometry. The conclusion is obvious. □

**Lemma 3.33.** *Let $a, a'$ be distinct points of $\Gamma$ in the same fiber of $f$. Then there is a point $b$ at distance 2 from both $a$ and $a'$ and such that $b \in S(a) \cap S(a')$.*

*Proof.* The points $a$, $a'$ have distance 4 by Lemma 3.31. Let $b$ be a point at distance 2 from both $a$ and $a'$. By Lemma 3.32, $a \in S(b)$ if and only if $f(a) \in S(f(b))$. Similarly, $a' \in S(b)$ if and only if $f(a') \in S(f(b))$. Hence $a \in S(b)$ if and only if $a' \in S(b)$.

If $b \in S(a)$, then we are done. Otherwise, let $b \in P(a)$. Hence $b \in P(a')$, by the above. Let $c \in a^{\sim} \cap b^{\sim}$ and $c' \in a'^{\sim} \cap b^{\sim}$. Let $L = bb'$ be a line of $\Gamma^*_b$ meeting each of the lines $bc$ and $bc'$ in the affine space $\Gamma^*_b$. That is, $bb'$ is coplanar with each of $bc$ and $bc'$ in $\Gamma$. As the points $b, b', c$ belong to the same plane of $\Gamma$, the lines $cb$ and $cb'$ meet in a point of the affine space $\Gamma^*_c$. Hence $cb'$ is not parallel to $ca$, as $ca$ is parallel to $cb$. Therefore, $b' \in S(a)$. Similarly, $b' \in S(a')$. □

**Lemma 3.34.** *Let $a, b$ be points of $\Gamma$ at distance 2 with $b \in S(a)$. Then we have $|a^{\sim} \cap b^{\sim}| = 40$ and $|f(a)^{\sim} \cap f(b)^{\sim}| = 80$.*

*Proof.* Let $a, a'$ be distinct points of $\Gamma$ in the same fiber of $f$. Then $a$, $a'$ have distance 4 by Lemma 3.31 and, by Lemma 3.33, there is a point $b$ at distance 2 from both $a$ and $a'$ and such that $b \in S(a) \cap S(a')$. By Lemma 3.26, the connected components of $a^{\sim} \cap b^{\sim}$ have size 40. More precisely, either $a^{\sim} \cap b^{\sim}$ has size 40 and is connected, or it has two connected components of size 40.

On the other hand, as every point of $a^{\sim} \cap b^{\sim}$ has distance 2 from every point of $a'^{\sim} \cap b^{\sim}$, the restriction of $f$ to $(a^{\sim} \cap b^{\sim}) \cup (a'^{\sim} \cap b^{\sim})$ is injective. Furthermore, each of the sets $a^{\sim} \cap b^{\sim}$ and $a'^{\sim} \cap b^{\sim}$ is mapped by $f$ onto a connected component of $f(a)^{\sim} \cap f(b)^{\sim}$ ($= f(a')^{\sim} \cap f(b)^{\sim}$, as $f(a) = f(a')$). Therefore, $f(a)^{\sim} \cap f(b)^{\sim}$ has two connected components, which are the images by $f$ of $a^{\sim} \cap b^{\sim}$ and $a'^{\sim} \cap b^{\sim}$, and we have $|a^{\sim} \cap b^{\sim}| = |a'^{\sim} \cap b^{\sim}| = 40$. The conclusion follows from the transitivity of $G$ on the set of ordered pairs $(a, b)$ of points at distance 2 with $b \in S(a)$ (Lemma 3.17) and from the analogous property of $Aut(\overline{\Gamma})$. □

**Proposition 3.35.** *The geometry $\Gamma$ is simply connected.*

*Proof.* Let $\widetilde{\Gamma}$ be a flag-transitive proper cover of $\Gamma$. By replacing the pair $(\Gamma, \overline{\Gamma})$ with $(\widetilde{\Gamma}, \Gamma)$ in Lemmas 3.32 and 3.34, we obtain that $|a^{\sim} \cap b^{\sim}| = 80$ for any two points $a, b$ of $\Gamma$ at distance 2 with $b \in S(a)$, whereas $|a^{\sim} \cap b^{\sim}| = 40$ by the statement of Lemma 3.34 for the pair $(\Gamma, \overline{\Gamma})$; contradiction. Therefore, $\Gamma$ does not admit any flag-transitive proper cover. However, the universal cover of a flag-transitive geometry is flag-transitive [26, Theorem 12.13]. Hence $\Gamma$ is its own universal cover. □

**Proposition 3.36.** *The geometry $\overline{\Gamma}$ has at least 1408 points. It has precisely that number of points if and only if its collinearity graph is strongly regular with parameters $(k, \lambda, \mu) = (336, 80, 80)$.*

*Proof.* The number of points of $\overline{\Gamma}$ is minimal when $\overline{\Gamma}$ has diameter 2 and, for any two points $\bar{a}, \bar{b}$ of $\Gamma$ at distance 2, the number $|\bar{a}^{\sim} \cap \bar{b}^{\sim}|$ is maximal, namely equal to 80. If that is the case, then the collinearity graph of $\overline{\Gamma}$ is strongly regular with parameters $(336, 80, 80)$. A strongly regular graph with these parameters has 1408 vertices. □

### 3.8. End of the proof of Theorem 1

The geometry $\Gamma(\mathcal{S})$ of Section 2 admits a flag-transitive proper quotient, namely $\Gamma(\mathcal{S})/2$. Hence $\Gamma(\mathcal{S})$ is simply connected, by Proposition 3.35. According to Theorem 3.6, $\Gamma(\mathcal{S})$ is the unique simply connected flag-transitive $c \cdot AG^*$-geometry of order 4 and residual dimension 3.

In order to finish the proof of Theorem 1, we only need to show that $\Gamma(\mathcal{S})/2$ is the unique flag-transitive proper quotient of $\Gamma(\mathcal{S})$. Let $\Gamma$ be such a quotient. As $\Gamma(\mathcal{S})$ is simply connected, we have $\Gamma = \Gamma(\mathcal{S})/X$ for some subgroup $X$ of $Aut(\Gamma(\mathcal{S}))$ acting semi-regularly on the set of elements of $\Gamma(\mathcal{S})$ and normalized by a flag-transitive subgroup $G$ of $Aut(\Gamma(\mathcal{S}))$ ([26, Theorem 12.13]). As $X$ is semi-regular on the set of points of $\Gamma(\mathcal{S})$, the order of $X$ divides 2816, which is the number of points of $\Gamma(\mathcal{S})$, and $2816/|X|$ is the number of points of $\Gamma$. However, $\Gamma$ has at least 1408 points, by Proposition 3.36, and 1408 is a half of 2816. Hence $|X| = 2$ and $X$, being normal in $G$, is also central in $G$. The commutator subgroup of $Aut(\Gamma(\mathcal{S}))$ is the minimal flag-transitive automorphism group of $\Gamma(\mathcal{S})$ (compare Theorem 3.5(i)). Hence $G$ contains the commutator subgroup of $Aut(\Gamma(\mathcal{S}))$. This forces $X$ to be the center of $Aut(\Gamma(\mathcal{S}))$. That is, $\Gamma = \Gamma(\mathcal{S})/2$.

## 4. Two 1-dimensional examples

### 4.1. Constructing $c \cdot AG^*$-geometries as gluings

Given a prime power $q$ and an integer $n > 1$, let $\mathcal{K}$ be a complete graph on $v = v(n,q)$ vertices, with

$$\text{(i)} \qquad v(n,q) = 1 + \frac{q^n - 1}{q - 1}$$

(namely, $v(n,q) - 1$ is the number of points of $PG(n-1,q)$) and let $\mathcal{A}$ be the $n$-dimensional affine space of order $q$. If $v$ is even (which, in view of (i), means that either $q$ is even or both $q$ and $n$ are odd), then $\mathcal{K}$ admits a 1-factorization. Hence, given a 1-factorization $\chi$ of $\mathcal{K}$ and a bijection $\alpha$ from the set $\chi^\infty$ of classes of $\chi$ to the set $\mathcal{A}^\infty$ of points at infinity of $\mathcal{A}$, we can glue $\mathcal{K}$ with $\mathcal{A}$ via $\alpha$ (Buekenhout, Huybrechts and Pasini [7]). Let $Gl_\alpha(\mathcal{K},\mathcal{A})$ be that glued geometry. We recall that its elements are the points of $\mathcal{A}$ (which we call *planes*), the vertices of $\mathcal{K}$ (called *points*) and the pairs $(e, L)$ with $e$ and $L$ an edge of $\mathcal{K}$ and a line of $\mathcal{A}$ such that $\alpha(\chi(e)) = L^\infty$, where $L^\infty$ stands for the point at infinity of $L$. These pairs are called *lines*. A line $(e, L)$ of $Gl_\alpha(\mathcal{K},\mathcal{A})$ is incident with precisely the vertices of $e$ and the points of $L$, whereas all vertices of $\mathcal{K}$ are declared to be incident to all points of $\mathcal{A}$ (namely, $Gl_\alpha(\mathcal{K},\mathcal{A})$ is flat).

$Gl_\alpha(\mathcal{K},\mathcal{A})$ is $c \cdot AG^*$-geometry of order $q$ and residual dimension $n$, but it need not be flag-transitive. According to Theorem 1, if $Gl_\alpha(\mathcal{K},\mathcal{A})$ is flag-transitive, then $v = 2^m$ for some positive integer $m$ and $\chi$ is the natural parallelism of $AG(m,2)$ (compare Cameron and Korchmaros [9]). Consequently, and by (i), the prime power $q$ and the integers $n$, $m$ satisfy the following equation

$$\text{(ii)} \qquad 2^m - 1 = \frac{q^n - 1}{q - 1}$$

(which is a special case of equation (E1) of Subsection 1.3, with $q' = 2^m$). For any solution $(q,n,m)$ of (ii), a flag-transitive gluing exists. Indeed, given Singer cycles $\theta$ and $\zeta$ in $L_m(2)$ and $PGL_n(q)$ respectively and chosen a class $c^\infty \in \chi^\infty$ and a point $p^\infty \in \mathcal{A}^\infty$, let $\alpha$ be the bijection from $\chi^\infty$ to $\mathcal{A}^\infty$ sending $c \in \chi^\infty$ to $\zeta^k(p^\infty)$, with $k$ the unique positive integer less than $v$ and such that $\theta^k(c^\infty) = c$. Then, according to Buekenhout, Huybrechts and Pasini [7], the geometry $Gl_\alpha(\mathcal{K},\mathcal{A})$ admits a flag-transitive automorphism group $G$ as follows:

(iii) $$G \;=\; (2^m \times p^{dm} Z).Z_s$$

where $p$ is prime, $d$ is a positive integer such that $q = p^d$, $s = (q^n-1)/(q-1)$ (which is the period of $\theta$ and $\zeta$) and $Z = \langle \hat{\zeta}^s \rangle$, where $\hat{\zeta}$ stands for a representative of $\zeta$ in $GL_n(q)$. (We warn that $G$ need not be the full automorphism group of $Gl_\alpha(\mathcal{K},\mathcal{A})$, even if the latter is never much larger than $G$.)

Clearly, once $\theta$ and $\zeta$ are given, the choice of $c$ and $p$ has no influence on the isomorphism type of $Gl_\alpha(\mathcal{K},\mathcal{A})$. Accordingly, we write $Gl(\theta,\zeta)$ for $Gl_\alpha(\mathcal{K},\mathcal{A})$.

### 4.2. The examples

As remarked in Subsection 1.3, the triple $(q,n,m) = (5,3,5)$ is the only known solution of the equation (ii) where $n,q > 2$. Chosen Singer cycles $\theta$ and $\zeta$ in $L_5(2)$ and $SL_3(5)$ respectively, for any pair $(r,k)$ with $r,k = 1,2,\ldots,30$ we can consider the gluing $Gl(\theta^r,\zeta^k)$. Clearly, $Gl(\theta^{rh},\zeta^{kh}) = Gl(\theta^r,\zeta^k)$ for all $h = 1,2,\ldots,30$. Thus, we may assume that $r=1$. Consequently, at most 30 geometries can be obtained by the above gluing construction.

**Lemma 4.1.** *We have $Gl(\theta,\zeta^k) \cong Gl(\theta,\zeta^h)$ if and only if there is some $r \in Z_{31}$ such that $f\theta f^{-1} = \theta^r$ and $g\zeta^k g^{-1} = \zeta^{hr}$ for some $f \in L_5(2)$ normalizing $\langle\theta\rangle$ and some $g \in GL_3(5)$ normalizing $\langle\zeta\rangle$.*

*Proof.* We have $Gl(\theta,\zeta^k) \cong Gl(\theta^r,\zeta^t)$ if and only if $\theta^r = f\theta f^{-1}$ and $\zeta^t = g\zeta^k g^{-1}$ for suitable elements $f \in L_5(2)$ and $g \in GL_3(5)$ normalizing $\langle\zeta\rangle$ and $\langle\theta\rangle$, respectively (Buekenhout, Huybrechts and Pasini [7, §3.4]). On the other hand, $Gl(\theta,\zeta^k) = Gl(\theta^r,\zeta^{kr})$. The conclusion follows. □

The following are the orbits on $\langle\theta\rangle \setminus \{1\}$ for the normalizer of $\langle\theta\rangle$ in $L_5(2)$. (In the rightmost column we record the minimal polynomial of the elements of each row, modulo a choice of one out of the six possibilities for the first row.)

TABLE 1

| | | | | | |
|---|---|---|---|---|---|
| $\theta$ | $\theta^2$ | $\theta^4$ | $\theta^8$ | $\theta^{16}$ | $x^5+x^2+1$ |
| $\theta^3$ | $\theta^6$ | $\theta^{24}$ | $\theta^{12}$ | $\theta^{17}$ | $x^5+x^4+x^3+x^2+1$ |
| $\theta^5$ | $\theta^{10}$ | $\theta^{20}$ | $\theta^9$ | $\theta^{18}$ | $x^5+x^4+x^2+x+1$ |
| $\theta^7$ | $\theta^{14}$ | $\theta^{28}$ | $\theta^{25}$ | $\theta^{19}$ | $x^5+x^3+x^2+x+1$ |
| $\theta^{11}$ | $\theta^{21}$ | $\theta^{22}$ | $\theta^{13}$ | $\theta^{26}$ | $x^5+x^4+x^3+x+1$ |
| $\theta^{15}$ | $\theta^{30}$ | $\theta^{27}$ | $\theta^{29}$ | $\theta^{23}$ | $x^5+x^3+1$ |

These are the orbits on $\langle\zeta\rangle \setminus \{1\}$ for the normalizer of $\langle\zeta\rangle$ in $GL_3(5)$.

TABLE 2

| | | | |
|---|---|---|---|
| $\zeta$ | $\zeta^5$ | $\zeta^{25}$ | $x^3+x-1$ |
| $\zeta^2$ | $\zeta^{10}$ | $\zeta^{19}$ | $x^3+2x^2+x-1$ |
| $\zeta^3$ | $\zeta^{15}$ | $\zeta^{13}$ | $x^3+2x^2-x-1$ |
| $\zeta^4$ | $\zeta^{20}$ | $\zeta^7$ | $x^3-2x^2-1$ |
| $\zeta^6$ | $\zeta^{30}$ | $\zeta^{26}$ | $x^3-x^2-1$ |
| $\zeta^8$ | $\zeta^9$ | $\zeta^{14}$ | $x^3+x^2+x-1$ |
| $\zeta^{11}$ | $\zeta^{24}$ | $\zeta^{27}$ | $x^3+2x-1$ |
| $\zeta^{12}$ | $\zeta^{29}$ | $\zeta^{21}$ | $x^3-x^2-2x-1$ |
| $\zeta^{16}$ | $\zeta^{18}$ | $\zeta^{28}$ | $x^3+x^2-2x-1$ |
| $\zeta^{17}$ | $\zeta^{23}$ | $\zeta^{22}$ | $x^3-x^2-x-1$ |

By comparing Tables 1 and 2 and by Lemma 4.1 we easily obtain the following:

**Proposition 4.2.** *Up to isomorphism, $Gl(\theta,\zeta)$ and $Gl(\theta,\zeta^3)$ are the unique flag-transitive gluings of the complete graph on 32 vertices with the affine space $AG(3,5)$.*

Writing $\Gamma$ for $Gl(\theta,\zeta)$ or $Gl(\theta,\zeta^3)$, let $G$ be the (flag-transitive) subgroup of $Aut(\Gamma)$ of this form: $G = (2^5\times 5^3).Z_{31}$ (as Subsection 4.1, (iii)). Let us consider the following set of relations

$$
\begin{array}{ll}
(1) & a^2 = 1 \\
(2) & b^5 = 1 \\
(3) & c^{31} = 1 \\
(4) & ab = ba \\
(5) & [a, a^{c^i}] = 1 \text{ (for } i = 1,2,3,4) \\
(6) & [b, b^c] = [b, b^{c^2}] = 1 \\
(7) & a^{c^5+c^2+1} = 1 \\
(8) & b^{p(c)} = 1
\end{array}
$$

with $p(c) = c^3+c-1$ or $c^3+2c^2-c-1$ in (8), according to whether $\Gamma$ is $Gl(\theta,\zeta)$ or $Gl(\theta,\zeta^3)$. The relations (1), (3), (5) and (7) give a presentation of $G_2$, the relations (2), (3), (6) and (8) form a presentation of $G_0$ and the relations (1)–(4) present $G_1$. By a computer aided calculation (we have done it by GAP [27]) one can see that the group presented by the relations (1)–(8) has size $2^5\cdot 5^3\cdot 31$. Hence, that group is $G$. Therefore [26, Corollary 12.30]:

**Proposition 4.3.** *Both $Gl(\theta,\zeta)$ and $Gl(\theta,\zeta^3)$ are simply connected.*

## 5. Proof of Theorem 2

Before to turn to the proof of Theorem 2, we make some remarks on $d$-dimensional dual hyperovals and $c\cdot AG^*$-geometries, to be exploited for that proof.

**Dimensional dual hyperovals and $c\cdot AG^*$-geometries.** Let $\mathcal{S}$ be a $d$-dimensional dual hyperoval of $PG(m,q)$ and $S_0$ be the set of points of $PG(m,q)$ belonging to some member of $\mathcal{S}$. Clearly, $(\mathcal{S},S_0)$, with the incidence relation inherited from $PG(m,q)$, is a circular space where every 'point' (namely, every member of $\mathcal{S}$) is incident to precisely $q^d+q^{d-1}+\cdots+q+1$ 'lines' (points of $S_0$). Therefore,

$$|\mathcal{S}| \; = \; q^d+q^{d-1}+\cdots+q+2.$$

Let $Af(\mathcal{S})$ be the affine expansion of the dual circular space $(S_0,\mathcal{S})$ to $AG(m+1,q)$ (Buekenhout, Huybrechts and Pasini [7, Section 4]). Clearly, the dual of $Af(\mathcal{S})$ is a $c\cdot AG^*$-geometry of order $q$ and residual dimension $n=d+1$.

Let $d=1$. Then, by definition of $\mathcal{S}$, the span of any two members of $\mathcal{S}$ contains all remaining members of $\mathcal{S}$. Therefore $m=2$ and $\mathcal{S}$ is just a dual hyperoval of a projective plane (whence, $q$ is even). Moreover, the dual of $Af(\mathcal{S})$ is a Laguerre structure.

Let $d=2$. The family of 22 planes of $PG(5,4)$ considered in Subsection 2.1 is a 2-dimensional dual hyperoval in $PG(5,4)$. If $\mathcal{S}$ is this family, then $Af(\mathcal{S})=\Gamma^*(\mathcal{S})/2$ (see Subsection 2.5).

If $q=2$, then $Af(\mathcal{S})$ is a semi-biplane. A lot of semi-biplanes can be obtained in this way since, as said in Subsection 1.4, a lot of $d$-dimensional dual hyperovals exist in $PG(2d,2)$ for any $d\geq 1$.

**Proof of Theorem 2.** Let $\mathcal{S}$ be a $d$-dimensional dual hyperoval of $PG(m,q)$. The automorphism group of $Af(\mathcal{S})$ contains the extension $V{:}Z.G_{\mathcal{S}}$ of the translation group $V$ of $AG(m+1,q)$ by $Z.G_{\mathcal{S}}$, with $Z$ the center of $GL_{m+1}(q)$ and $G_{\mathcal{S}}$ the stabilizer of $\mathcal{S}$ in $P\Gamma L_{m+1}(q)$.

Let $G_{\mathcal{S}}$ be 2-transitive. Then $Af(\mathcal{S})$ is flag-transitive and the conclusion of Theorem 2 follows from Theorem 1 and from the well-known fact that the only 2-transitive hyperovals are the (regular) hyperovals of $PG(2,2)$ and $PG(2,4)$. $\square$

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