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Rational 1- and 2-cuspidal Plane Curves

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Abstract. We prove the existence of series of plane rational 1- and 2-cuspidal curves by constructing suitable Cremona transformations which transform the curves $V(xy^{d-1} - z^d)$ resp. $V(xy^{d-1} - z^d - z^{d-1}y)$ into the desired ones. Further, we obtain a complete list of existing types of rational cuspidal plane curves having one or two cusps with maximal multiplicity $m_{\max} = \deg C - 2$ resp. $m_{\max} = \deg C - 3$.

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1. Introduction

In this paper we assume the ground field to be the field of complex numbers. Let $C \subset \mathbb{P}^2$ be a rational curve with only cusps as singularities. By a cusp we mean an analytic irreducible singular point. Let $(C,p) \subset (\mathbb{C}^2, 0)$ an irreducible analytic plane curve germ and let

$$V_n \xrightarrow{\sigma_n} V_{n-1} \longrightarrow \cdots \longrightarrow V_1 \xrightarrow{\sigma_1} V_0 = \mathbb{C}^2 \tag{1}$$

be the minimal embedded resolution of the singularity of C at p. The complete preimage of C is a simple normal crossing (SNC) divisor $D = E + C_n$ where C_n is the proper preimage of C and E is the exceptional divisor. Let $C_i \subset V_i$ resp. $E_i \subset V_i$ denote the proper transform of C resp. the exceptional divisor under the blowing up σ_i. Then the multiplicities of C_{i-1} at $p_{i-1} = \sigma_i(E_i)$ form a sequence $m_0 \geq m_1 \geq \cdots \geq m_n = 1$ of decreasing integers called the multiplicity sequence of C at p (see [3]). Since the number of times the multiplicity 1 occurs in the sequence equals the last multiplicity greater than 1 in the sequence (see Proposition 3.1) we omit the multiplicity 1 and denote such a sequence by $\overline{m}_p = [m_0, \ldots, m_l]$ where

$$m_0 \geq m_1 \geq \cdots \geq m_l > m_{l+1} = 1$$

for a suitable $l \geq 0$. We sometimes write $[\hat{m}_{0_{r_0}}, \dots, \hat{m}_{k_{r_k}}]$ for a multiplicity sequence which means that the multiplicity $\hat{m}_i$ occurs r_i times for $i = 0, \dots, k$. For example, $[4_2, 2_3]$ means $[4, 4, 2, 2, 2]$.

Then we ask whether there exists a rational cuspidal curve of given degree and given multiplicity sequences. In his classification of plane curves up to degree 5, Namba has given explicit equations for all possible rational cuspidal curves of degree 3, 4 and 5 ([10]). Only a few examples are known for the case $\deg C \geq 6$.

Recently, Flenner and Zaidenberg classified rational cuspidal plane curves with at least three cusps and maximal multiplicity $\deg C - 2$ [3] and $\deg C - 3$ [4]. They showed that those curves cannot have more than three cusps. They further showed that the only existing curves with maximal multiplicity $\deg C - 2$ are of type (i): $[a+b]$, $[2_a]$, $[2_b]$ with $a + b = \deg C - 2$ and of type (ii): $[3a, 2_a]$, $[3_a]$, $[2_1]$ if the maximal multiplicity is $\deg C - 3$, a and b being positive integers. They proved the existence of these curves by two different methods. One method was to compute the rational parameterization and the other was to perform suitable Cremona transformations of $\mathbb{P}^2$ transforming the rational plane quadric $C_2 = V(xy - z^2)$ into curves of type (i) and the rational plane cubic $C_3 = V(xy^2 - z^3)$ into curves of type (ii).

Following their methods, we obtain rational 1- and 2-cuspidal plane curves by suitable Cremona transformations of $\mathbb{P}^2$ transforming the rational plane curve $C_{d,t} = V(xy^{d-1} - z^d - t\, z^{d-1} y)$ which are 1-cuspidal rational curves with a cusp of type $[d-1]$ and a d-fold point if $t = 0$ and a $(d-1)$-fold point if $t \neq 0$.

We obtain the following

Theorem 1.1. *Let $d \geq 2$ and $0 \leq b < a$ be integers. The following types of rational cuspidal plane curves exist.*

	$\deg C_{a,d}$	*type of cusps*	$\chi(\Theta_V\langle D\rangle)$
1	$da + d$	$[da, d_{a+b}, d-1]$, $[d_{a-b}]$	$1 - a$
1a	$da + d$	$[da, d_{2a}, d-1]$	$3 - d - a$ *if* $d > 2$ $1 - a$ *if* $d = 2$
2	$da + d$	$[da, d_{a+b}]$, $[d_{a-b}, d-1]$	$a - 1$
2a	$da + d$	$[da, d_{2a}]$, $[d-1]$	$-a$ *if* $d > 2$ $1 - a$ *if* $d = 2$
3	$da + d + 1$	$[da + 1, d_a]$, $[d_{a+1}]$	$1 - a$
4	$da + d + 1$	$[da, d_{a+1}]$, $[(d+1)_a]$	0
5	$da + d + 1$	$[da, d_a]$, $[(d+1)_a, d]$	0
6	$da + d + 2$	$[da + 1, d_a]$, $[(d+1)_{a+1}]$	0
7	$da + 2d - 1$	$[da + d - 1, d_a, d-1]$, $[d_{a+1}, d-1]$	$1 - a$
8	$a + 2$	$[a]$, $[2_a]$	0

Here, $\chi(\Theta_V\langle D\rangle)$ denotes the Euler-characteristic of the sheaf $\Theta_V\langle D\rangle$ of germs of vector fields on V tangent along D where the pair (V, D) is an SNC-pair consisting of a complete surface V and an SNC-divisor D obtained by a minimal embedded resolution $\phi : V \longrightarrow \mathbb{P}^2$ of the singularities of C with $D = \phi^{-1}(C)_{red}$ such that D has only simple normal crossings as singularities.

It turns out that all existing types of rational 1- and 2-cuspidal plane curves with maximal multiplicity $m_{\max} = d-2$ and $m_{\max} = d-3$ are already contained in the table of Theorem 1.1. We complete the results of Flenner and Zaidenberg and give the complete list of existing types of rational cuspidal plane curves with at most two cusps and maximal multiplicity $\deg C - 2$ resp. $\deg C - 3$.

Theorem 1.2. *Let $a \geq 1$ and $0 \leq b < a$ be integers.*

(1) *The only existing type of a rational cuspidal plane curve with one cusp and maximal multiplicity* $\deg C - 2$ *is of type* $[2a, 2_{2a}]$ *of degree* $2a+2$ *with* $\chi(\Theta_V\langle D\rangle) = 1-a$.

(2) *The only existing types of rational cuspidal plane curves with two cusps and maximal multiplicity* $\deg C - 2$ *are* (1) $[2a+1, 2_a]$, $[2_{a+1}]$ *of degree* $2a+3$ *with* $\chi(\Theta_V\langle D\rangle) = 1-a$, (2) $[a]$,$[2_a]$ *of degree* $a+2$ *with* $\chi(\Theta_V\langle D\rangle) = 0$ *and* (3) $[2a, 2_{a+b}]$, $[2_{a-b}]$ *of degree* $2a+2$ *with* $\chi(\Theta_V\langle D\rangle) = 1-a$.

For $m_{\max} = \deg C - 3$ we obtain

Theorem 1.3. *Let $a \geq 1$ and $0 \leq b < a$ be integers.*

(1) *The only existing types of rational cuspidal plane curves with one cusp and maximal multiplicity* $\deg C - 3$ *are* (1) $[3a, 3_{2a}, 2]$ *of degree* $3a+3$ *with* $\chi(\Theta_V\langle D\rangle) = -a$ *and* (2) $[2_6]$ *of degree* 5 *with* $\chi(\Theta_V\langle D\rangle) = 0$.

(2) *The only existing types of rational cuspidal plane curves with two cusps and maximal multiplicity* $\deg C - 3$ *are the following*

	$\deg C_i$	*type of cusps*	$\chi(\Theta_V\langle D\rangle)$
C_1	7	$[4]$, $[3_3]$	0
C_2	6	$[3]$, $[3_2, 2]$	0
C_3	5	$[2_4]$, $[2_2]$	0
C_4	$2a+3$	$[2a, 2_a]$, $[3_a, 2]$	0
C_5	$2a+4$	$[2a+1, 2_a]$, $[3_{a+1}]$	0
C_6	$2a+3$	$[2a, 2_{a+1}]$, $[3_a]$	0
C_7	$3a+3$	$[3a, 3_{2a}]$, $[2]$	$-a$
C_8	$3a+4$	$[3a+1, 3_a]$, $[3_{a+1}]$	0
C_9	$3a+3$	$[3a, 3_{a+b}, 2]$, $[3_{a-b}]$	$1-a$
C_{10}	$3a+3$	$[3a, 3_{a+b}]$, $[3_{a-b}, 2]$	$1-a$
C_{11}	$3a+5$	$[3a+2, 3_a, 2]$, $[3_{a+1}, 2]$	$1-a$

Remark 1.4. The case $m_{\max} = \deg C - 2$ was independently done by F. Sakai and K. Tono [11].

In [7], Matsuoka and Sakai proved that for the maximal multiplicity $m_{\max}$ of a rational cuspidal curve of degree d the inequality $d < 3\,m_{\max}$ must hold. Thus, by Theorems 1.2 and 1.3 we obtain a complete list of rational cuspidal plane curves of degree 6.

Corollary 1.5. ([1]) *The only rational cuspidal plane curves of degree* 6 *with* s *singular points are the ones in the following list.*

(1) $s = 1$

	type of cusp	$\chi(\Theta_V\langle D\rangle)$
C_1	$[5]$	-2
C_2	$[4, 2_4]$	-1
C_3	$[3_3, 2]$	-1

(2) $s = 2$

	type of cusps	$\chi(\Theta_V\langle D\rangle)$
C_4	$[3_3]$, $[2]$	-1
C_5	$[3_2, 2]$, $[3]$	0
C_6	$[3_2]$, $[3, 2]$	0
C_7	$[4, 2_3]$, $[2]$	-1
C_8	$[4, 2_2]$, $[2_2]$	-1
C_9	$[4]$, $[2_4]$	0

(3) $s = 3$

	type of cusps	$\chi(\Theta_V\langle D\rangle)$
C_{10}	$[4]$, $[2_3]$, $[2]$	0
C_{11}	$[4]$, $[2_2]$, $[2_2]$	0

In particular, there is no rational cuspidal plane sextic with more than three cusps.

In Section 2, we prove Theorem 1.1. We do this by constructing Cremona transformations transforming the curves $C_{d,t}$ into curves of desired type. In Section 3, we sum up some results which impose strong restrictions on the multiplicity sequences for a curve to exist. Section 3 contains the proofs of Theorem 1.2 and 1.3. We apply the results of Section 3 to all possible types of rational plane curves having one or two cusps and maximal multiplicity $m_{\max} = \deg C - 2$ resp. $m_{\max} = \deg C - 3$. It turns out that the remaining types of curves appear in the table of Theorem 1.1. In Section 5, we give concrete parameterizations and defining equations for the rational cuspidal plane curves of degree 6 which we obtained by explicit computations of the parameterization with the computer algebra system Maple and by elimination of variables with the program Singular [5].

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2. Proof of Theorem 1.1

Let X be a surface being isomorphic to a Hirzebruch surface $\Sigma_n = Proj\,(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ for some $n > 0$, F a fibre of X and E the horizontal section of X with $E^2 = -n$. We define an *elementary transformation* of F at p to be a blowing up of X at p followed by a blowing down of the proper transform of the fibre F. Then we obtain a new Hirzebruch surface X' where the fibre F is replaced by a new fibre F' being the exceptional divisor of the blowing up. Let E' be the proper transform of the horizontal section E. If $p \in E$ then $E'^2 = E^2 - 1$ otherwise we have $E'^2 = E^2 + 1$.

Let $\sigma_0 : X_1 \longrightarrow \mathbb{P}^2$ be the blowing up of $\mathbb{P}^2$ at r_0. Then $X_1 \cong \Sigma_1$ is a Hirzebruch surface with a ruling $\pi_1 : X_1 \longrightarrow \mathbb{P}^1$ given by the lines through r_0 in $\mathbb{P}^2$. We start with the curve $C_{d,i} = V(xy^{d-1} - z^d - i \cdot z^{d-1}y) \subset \mathbb{P}^2$ with $i = 0, 1$. It is easily checked that the curve $C_{d,0}$ has a cusp of type $[d-1]$ at $p_0 = (1:0:0) \in \mathbb{P}^2$, a d-fold point at $q_0 = (0:1:0) \in \mathbb{P}^2$ if $i = 0$ and a $(d-1)$-fold point at q_0 if $i = 1$. The strategy is to construct a sequence

$$C'_{d,i} \subset \mathbb{P}^2 \xleftarrow{\sigma_n} X_n \xrightarrow{\sigma_{n-1}} X_{n-1} \xrightarrow{\sigma_{n-2}} \cdots \xrightarrow{\sigma_1} X_1 \xrightarrow{\sigma_0} X_0 = \mathbb{P}^2 \supset C_{d,i} \tag{2}$$

of birational transformations of Hirzebruch surfaces beginning with the blowing up σ_0 followed by $(n-1)$ elementary transformations σ_i for $i = 1, \dots, n-1$ of a fibre F_i at a point $r_i \in F_i$ such that $E_n^2 = -1$ followed by a blowing down τ of E_n. The composition of these maps defines a Cremona transformation.

Before constructing the Cremona transformations we fix some notations. F_{i+1} denotes the new fibre of the ruling $\pi_{i+1} : X_{i+1} \longrightarrow \mathbb{P}^1$ replacing the fibre F_i under σ_i for $i = 1, \dots, n-1$. Let $\sigma_i : X_{i+1} \longrightarrow X_i$ be a birational transformation of Hirzebruch surfaces. Let E_i be the horizontal section with negative self-intersection in X_i, C_i be any curve on X_i and $p_i \in C_i$. Then E_{i+1} resp. C_{i+1} always denotes the proper transforms of E_i resp. C_i under σ_i and p_{i+1} the point of C_{i+1} being infinitesimally near to p_i for $i = 0, \dots, n$.

Cases 1, 1a. Let $C_0 = C_{d,0}$. We begin with the blowing-up σ_0 at $T_{p_0}C \cap T_{q_0}C = \{r_0\}$. Let F_1 resp. F'_1 be the fibre being the proper transform of the tangent line $T_{p_0}C$ resp. $T_{q_0}C$. Then by Bézout's theorem, we have $C_1.F_1 = d \cdot p_1$ and $C_1.F'_1 = d \cdot q_1$. For $i = 1, \dots, b$ we perform elementary transformations σ_i at $r_i \in E_i \cap F_i$ followed by elementary transformations σ_i at $r'_i \in E_i \cap F'_i$ for $i = b+1, \dots, a$. Since we performed the elementary transformations at a point lying on E_i we arrive at a Hirzebruch surface X_{a+1} with $E_{a+1}^2 = -a-1$, $C_{a+1}.F_{a+1} = d \cdot p_{a+1}$, $C_{a+1}.F'_{a+1} = d \cdot q_{a+1}$. p_{a+1} resp. q_{a+1} is a cusp of type $[d_b, d-1]$ resp. $[d_{a-b}]$. Now, for $i = a+1, \dots, 2a$ we perform elementary transformations at $s_i \in F_i \backslash E_i$. Thus, X_{2a+1} is a Hirzebruch surface with $E_{2a+1}^2 = -1$. So, we can blow down E_{2a+1} and arrive at $\mathbb{P}^2$. Let C' be the proper transform under 2. Then, C' is a rational cuspidal plane curve of type 1 resp. $1a$ of the table.

Cases 2, 2a. This sequence is constructed as the sequence in Case 1 with the following difference: For $i = 1, \dots, b$ the elementary transformations are performed at $r'_i \in E_i \cap F'_i$, for $i = b+1, \dots, a$ at $r_i \in E_i \cap F_i$ and for $i = a+1, \dots, 2a$ at any point $s'_i \in F'_i \backslash E_i$.

Case 3. We begin with $C_0 = C_{d+1,0}$. Then we perform the blowing up σ_0 at the $(d+1)$-fold point q_0. Let F_1 resp. F'_1 be the proper transform of $L = \overline{p_0 q_0}$ resp. $T_{q_0}C$. Then $C_1.F_1 = d \cdot p_1$, $C_1.F'_1 = d \cdot q_1$ and $C_1.E_1 = 1 \cdot q_1$. Now, for $i = 1, \dots, a$ we perform elementary transformations σ_i at $r_i \in E_i \cap F_i$ followed by elementary transformations σ_i at $r'_i \in F'_i \backslash E_i$ for $i = a+1, \dots, 2a$. Hence, X_{2a+1} is a Hirzebruch surface with $E_{2a+1}^2 = -1$ and $C_{2a+1}.E_{2a+1} = (da+1) \cdot q_{2a+1}$. Thus, after blowing down E_{2a+1} we arrive at $\mathbb{P}^2$ and the proper transform C' of C_d is a curve of type 3 in the table.

Case 4. Let $C_0 = C_{d+1,0}$. Let L be any line through p_0 which is neither the tangent line nor the line through q_0. Then by Bezout's theorem, L intersects C transversally in exactly one further point, say r_0. Let $s \in T_{q_0}C \cap L$. We perform the first blowing up σ_0 at s. Let F_1 resp. F'_1 be the proper transform of L resp. $T_{q_0}C$. Then $C_1.F_1 = d \cdot p_1 + 1 \cdot r_1$, $C_1.F'_1 = (d+1) \cdot q_1$ and $C_1 \cap E_1 = \emptyset$. Now, for $i = 1, \dots, a$ perform elementary transformations σ_i at r_i followed by elementary transformations σ_i at $r'_i \in E_i \cap F'_i$ for $i = a+1, \dots, 2a$. Hence, X_{2a+1} is a Hirzebruch surface $E^2_{2a+1} = -1$ with $C_{2a+1}.E_{2a+1} = da \cdot p_{2a+1}$. Thus, after blowing down E_{2a+1} we arrive at $\mathbb{P}^2$ and the proper transform C' of C is a curve of type 4 in the table.

Case 5. We start with $C_0 = C_{d+1,1}$. By Bezout's theorem $T_{q_0}C$ intersects C transversally in a further point r'_0. Now, we perform the blowing up σ_0 at $T_{p_0}C \cap T_{q_0}C = \{r_0\}$. Let F_1 resp. F'_1 be the proper transforms of $T_{p_0}C$ resp. $T_{q_0}C$. Then $C_1.F_1 = (d+1) \cdot p_1$, $C_1.F'_1 = d \cdot q_1 + 1 \cdot r'_1$ and $C_1 \cap E_1 = \emptyset$. Now, for $i = 1, \dots, a$ we perform elementary transformations σ_i at $r_i \in E_i \cap F_i$ followed by elementary transformations σ_i at r'_i for $i = a+1, \dots, 2a$. Then $E^2_{2a+1} = -1$ and $C_{2a+1}.E_{2a+1} = da \cdot q_{2a+1}$. Thus, after blowing down E_{2a+1} we arrive at $\mathbb{P}^2$ and the proper transform C' of C is a curve of type 5 in the table.

Case 6. Now, let $C = C_{d+2,1}$. By Bezout's theorem, $T_{q_0}C$ intersects C in one further point r'_0. We perform the blowing up σ_0 at q_0. Let F_1 resp. F'_1 be the proper transforms of $\overline{p_0 q_0}$ resp. $T_{q_0}C$. Then $C_1.F_1 = (d+1) \cdot p_1$, $C_1.F'_1 = d \cdot q_1 + 1 \cdot r'_1$ and $C_1.E_1 = 1 \cdot q_1$. For $i = 1, \dots, a$ we perform elementary transformations σ_i at $r_i \in E_i \cap F_i$ followed by elementary transformations at r'_i for $i = a+1, \dots, 2a$. We have $E^2_{2a+1} = -1$ and $C_{2a+1}.E_{2a+1} = (da+1) \cdot q_{2a+1}$. After blowing down E_{2a+1} we arrive at $\mathbb{P}^2$ and the proper transform C' of C is a rational curve of type 6 in the table.

Case 7. We start with $C = C_{d,0}$ and perform the blowing up σ_0 at $T_{p_0}C \cap T_{q_0}C = \{r_0\}$. Let F_1 resp. F'_1 be the proper transform of $T_{p_0}C$ resp. $T_{q_0}C$. Then $C_1.F_1 = d \cdot p_1$, $C_1.F'_1 = d \cdot q_1$ and $C_1 \cap E_1 = \emptyset$. We perform an elementary transformation σ_1 at q_1 followed by an elementary transformation σ_2 at $r_2 \in E_2 \cap F_2$. We obtain a Hirzebruch surface X_3 with $E^2_3 = -1$ and $C_3.F_3 = d \cdot p_3$, $C_3.F'_3 = d \cdot q_3$ and $C_3.E_3 = (d-1) \cdot q_3$. The point p_3 is a cusp of type $[d, d-1]$ and the point q_3 a cusp of type $[d-1]$. For $i = 3, \dots, a+2$ we perform elementary transformations σ_i at $r_i \in E_i \cap F_i$ followed by elementary transformations σ_i at $r'_i \in F'_i \setminus E_i$ for $i = a+3, \dots, 2a+2$. Hence $E^2_{2a+3} = -1$ and $C_{2a+3}.E_{2a+3} = (da+d-1) \cdot q_{2a+3}$. After blowing down E_{2a+3} we arrive at $\mathbb{P}^2$ and the proper transform C' is a curve of type 7 in the table.

Case 8. We start with $C = C_{3,0}$. Let $q_0 \in C$ be any point which is not a 3-fold point. Let $L = \overline{p_0 q_0}$. By Bezout's theorem, $T_{q_0}C$ intersects C in a further point, say r'_0. Then we perform σ_0 at q_0. Let F_1 resp. F'_1 be the proper transform of L resp. $T_{q_0}C$. For $i = 1, \dots, a-1$, we perform elementary transformations σ_i at r'_i followed by elementary transformations σ_i at $r_i \in E_i \cap F_i$ for $i = a, \dots, 2a-2$. Then $C_{2a-1}.F_{2a-1} = 2 \cdot p_{2a-1}$, $C_{2a-1}.F'_{2a-1} = 1 \cdot r'_{2a-1} + 1 \cdot q_{2a-1}$ and $C_{2a-1}.E_{2a-1} = a \cdot q_{2a-1}$. C_{2a-1} has a cusp of type $[2_a]$ at p_{2a-1}. After blowing down E_{2a-1} we arrive at $\mathbb{P}^2$ and the proper transform C' is a curve of type 8 in the table.

Let (V, D) be an (SNC)-pair constructed from a minimal embedded resolution of a rational cuspidal curve $C \subset \mathbb{P}^2$. In order to compute the Euler-characteristic of $\Theta_V\langle D\rangle$ we use the following formula ([3], Lemma 2.3):

$$\chi(\Theta_V\langle D\rangle) = K_V(K_V + D) = -3(\deg C - 3) + \sum_{p \in \operatorname{Sing} C} \chi_p$$

where K_V denotes the canonical divisor of V, $\chi_p = M_p + \omega_p - 1$ and

$$M_p = \sum_{i=0}^{r}(m_i - 1) \qquad\qquad \omega_p = \sum_{i=1}^{r}(\lceil \frac{m_{i-1}}{m_i} \rceil - 1)$$

for a singular point $p \in \operatorname{Sing} C$ with multiplicity sequence $[m_0, \dots, m_r]$. By $\lceil a \rceil$, we denote the smallest integer $\geq a$.

ω_{p_i} denotes the number of inner blowing up in the resolution of p_i. The blowing up σ_{i+1} of V_i at p_i in Case 1 is called inner, if $p_i \in E_i \cap E_{i-k}^{(k)}$ for some $k > 0$ where $E_{i-k}^{(k)}$ is the proper transform of the exceptional divisor E_{i-k} of σ_{i-k} in V_i. σ_{i+1} is called outer in the opposite. (cf. [3] or [7])

Now, for the case 1 we get

$$\sum_{p \in \operatorname{Sing} C} M_p = da - 1 + (d-1)(a+b) + d - 2 + (d-1)(a-b) = 3da + d - 2a - 3$$

and

$$\sum_{p \in \operatorname{Sing} C} \omega_p = a - 1 + 1 + d - 2 + d - 1 = 2d + a - 3$$

and therefore

$$\chi(\Theta_V\langle D\rangle) = -3(da + d) + 9 + 3da + d - 2a - 3 + 2d + a - 3 - 2 = -a + 1.$$

By analogous computations we obtain the Euler-characteristic of the cases 1a to 7.

3. Restrictions on rational cuspidal plane curves

We begin with the characterization of multiplicity sequences as given in [3], Prop. 1.2. Let $(C, p) \subset (\mathbb{C}^2, 0)$ be a rational cuspidal curve germ with multiplicity sequence $m_0 \geq m_1 \geq \dots \geq m_n = 1$ coming from a minimal embedded (SNC) resolution.

Proposition 3.1. *The multiplicity sequence $\overline{m}_p$ has the following two properties:*

(1) *For each $i = 1, \dots, n$ there exists a $k \geq 0$ such that*

$$m_{i-1} = m_i + \dots + m_{i+k}$$

where $m_i = m_{i+1} = \dots = m_{i+k-1}$.

(2) *If $m_{n-r} > m_{n-r+1} = \dots = m_n = 1$ then $m_{n-r} = r - 1$.*

From the genus formula for plane projective curves we obtain a formula relating the multiplicity sequences to the degree of the curve. Let $C \subset \mathbb{P}^2$ be a rational cuspidal curve of degree d and let $p_1, \dots, p_s$ be the singular points and $\overline{m}_{p_i} = [m_{i,0}, \dots, m_{i,r_i}]$ the multiplicity sequences. The arithmetical genus of C is

$$p_a(C) = \binom{d-1}{2}$$

and also $p_a(C) = g_{\tilde{C}} + \sum_{i=1}^{s} \delta_{p_i}$ where $g_{\tilde{C}}$ is the genus of the normalization $\tilde{C}$ of C and δ_{p_i} the number of virtual double points at p_i ([8]). In our case, we can compute δ_{p_i} by the formula

$$\delta_{p_i} = \sum_{j=0}^{r_i} \frac{m_{i,j}(m_{i,j}-1)}{2}\,.$$

Hence, we get

$$\binom{d-1}{2} = \sum_{i=1}^{s} \sum_{j=0}^{r_i} \frac{m_{i,j}(m_{i,j}-1)}{2}\,. \tag{3}$$

Now by Proposition 3.1 and (3), we are able to write down a finite list of possible types of rational cuspidal plane curves of given degree.

If we intersect C with a line L which is either the tangent line T_pC of a singular point p or a line through two different singular points, Bézout's theorem yields

Proposition 3.2. *Let $C \subset \mathbb{P}^2$ be a rational cuspidal curve and $p_1, \dots, p_s$ its cusps with multiplicity sequences $\overline{m}_{p_i} = [m_{i,0}, \dots, m_{i,r_i}]$. Then*

(1) $m_{i,0} + m_{i,1} \leq d$

(2) $m_{i,0} + m_{j,0} \leq d$ *for $i \neq j$.*

We obtain another useful restriction if we apply the general Plücker formula

$$2 - 2g = 2d^* - d - \sum_{p \in \operatorname{Sing} C} (m_p - r_p) \tag{4}$$

where d^* is the degree of the dual curve C^*, g the genus of C, m_p the multiplicity of C at p and r_p the number of branches at $p \in C$.

Proposition 3.3. *Let C, p_i, $\overline{m}_{p_i}$ be as in Proposition 3.2. Then*

$$3d - 6 \geq 2\sum_{i=1}^{s}(m_{i,0}-1) + \sum_{i=1}^{s}(m_{i,1}-1)\,.$$

Proof. First, we apply the general Plücker formula to the dual curve.

$$2 - 2g = 2d - d^* - \sum_{p \in \operatorname{Sing} C^*} (m_p^* - r_p^*)\,. \tag{5}$$

Now, if we put (4) and (5) together, we get

$$3d - 6(1-g) = 2\sum_{p \in \operatorname{Sing} C} (m_p - r_p) + \sum_{p \in \operatorname{Sing} C^*} (m_p^* - r_p^*)\,.$$

We are left to prove the inequality $m_{p_i}^* \geq m_{i,1}$ for $i = 1, \dots, s$. Set $p = p_i$ and $\overline{m}_{p_i} = \overline{m} = [m_0, \dots, m_r]$ for some i.

Let $\Phi(t) = (1 + \cdots : t^{1+\alpha_1} + \cdots : t^{2+\alpha_1+\alpha_2} + \cdots)$ be the parameterization of C near the singular point p with $1 + \alpha_1 = m_0$ and $2 + \alpha_1 + \alpha_2 = (T_pC.C)_p = m_0 + \cdots + m_k$ with $m_0 = \cdots = m_{k-1}$. Moreover,

$$1 + \alpha_2 = m_0 + \cdots + m_k \geq m_1 \,.$$

The Plücker embedding for C near p yields the parameterization

$$\Phi^*(t) = \Big((1+\alpha_2)t^{2+\alpha_1+\alpha_2} + \cdots : -(2+\alpha_1+\alpha_2)t^{1+\alpha_2} + \cdots : (1+\alpha_2) + \cdots\Big)$$

of the dual curve. Hence $m_0^* \geq m_1$. □

In [2], Flenner and Zaidenberg proved the following: If an SNC-pair (V, D) admits a generic $\mathbb{P}^1$-fibration over some smooth curve C such that $D.F \leq 3$ for a generic fibre F, then the dimension of $\mathrm{H}^2(V; \Theta_V\langle D\rangle)$ is zero.

If $C \subset \mathbb{P}^2$ is a cuspidal curve with maximal multiplicity $m_{\max} \geq d-2$ at $p \in C$ then a blowing up of $\mathbb{P}^2$ at p yields a $\mathbb{P}^1$-fibration over $\mathbb{P}^1$ and by Bezout's theorem, the proper transform intersects the generic fibre in $d - m_{\max} \leq 2$ points. Hence, by resolving all singularities we obtain an SNC-pair (V, D) satisfying the assumptions of the proposition.

Proposition 3.4. *Let $C \subset \mathbb{P}^2$ be a rational cuspidal curve with maximal multiplicity $m_{\max} \geq d-2$. Then*

$$\mathrm{H}^2(V; \Theta_V\langle D\rangle) = 0$$ □

In [4], they extended this result as follows:

Proposition 3.5. *Let $\pi : S \longrightarrow \mathbb{P}^1$ be a Hirzebruch surface with a section $\mathbb{P}^1 \cong E \subset S$ of π and let $C \subset S$ be a rational cuspidal curve such that $\pi_{|C\cup E} : C \cup E \longrightarrow \mathbb{P}^1$ has degree 4. Assume that $q \in C \cup E$ is a branch point of $\pi_{|C\cup E}$ and that $F_{\pi(q)}$ meets $C \cup E$ in three different points. Then*

(1) $\pi_*(\Theta_S\langle C \cup E\rangle) = 0$

(2) *If (V, D) is the (SNC)-pair obtained by a minimal embedded resolution $V \longrightarrow S$ of $C \cup E$ then*

$$\mathrm{H}^2(V; \Theta_V\langle D\rangle) = 0$$

The proof is based on the following Lemma ([4] Lemma 2.3).

Lemma 3.6. *Let $\Delta \subseteq \mathbb{C}$ be an open connected subset, and let $B \subseteq S := \Delta \times \mathbb{P}^1$ be a closed curve such that the projection $\pi : B \longrightarrow \Delta$ is finite with $d \geq 3$ sheets. Assume that there is a vector field $\xi \neq 0$ on S which is tangent to B. Then either $B \longrightarrow \Delta$ is unramified over a point $a \in \Delta$, or the fibre $F_a := \{a\} \times \mathbb{P}^1$ intersects B in at most two different points.* □

Proof of Proposition 3.5. (2) follows from (1). Indeed, by Serre duality, we have

$$\mathrm{H}^2(V;\Theta_V\langle D\rangle) \cong \mathrm{H}^0(V;\Omega^1\langle D\rangle \otimes \omega_V)$$

The perfect pairing

$$\Omega^1_V\langle D\rangle \otimes \Omega^1_V\langle D\rangle \longrightarrow \omega_V(D)$$

induces an isomorphism $\Omega^1_V\langle D\rangle \cong \Theta_V\langle D\rangle \otimes \omega_V(D)$. Therefore

$$\mathrm{H}^2(\Theta_V\langle D\rangle) \cong \mathrm{H}^0(\Theta_V\langle D\rangle \otimes \omega^2_V(D)) \subseteq \mathrm{H}^0(S;\Theta_S\langle E\cup C\rangle \otimes \omega^2_S(E\cup C)) \qquad (*)$$

Since $E\cup C \longrightarrow \mathbb{P}^1$ has degree 4, there is a Zariski open subset U in $\mathbb{P}^1$ such that $\omega^2_S(E\cup C)_{|\pi^{-1}(U)} \cong \mathcal{O}_{|\pi^{-1}(U)}$. Therefore, (1) implies $\pi_*(\Theta_S\langle E\cup C\rangle \otimes \omega^2_S(E\cup C)) = 0$ saying that the last cohomology group in $(*)$ vanishes.

In order to prove (1) we assume that $\pi_*(\Theta_S\langle E\cup C\rangle) \neq 0$ and apply Lemma 3.6 to $B := E\cup C$, to the fibre $F_{\pi(q)}$ and to a small neighbourhood of $\pi(q)$ giving a contradiction which proves (1). $\square$

Corollary 3.7. *Let $C \subset \mathbb{P}^2$ be a rational* 1- *or* 2-*cuspidal curve of degree d having a cusp $p \in C$ with multiplicity sequence $\overline{m}_p = [d-3, m_{p,1}, \ldots, m_{p,r_p}]$. Assume that C satisfies at least one of the following conditions:*

(1) *C has a further cusp q with multiplicity sequence (2_b) and $b > 1$,*

(2) $m_{p,1} = 1$,

(3) $m_{p,1} = 2$ *and* $0 \equiv d-3 \bmod 2$.

Then

$$\mathrm{H}^2(V;\Theta_V\langle D\rangle) = 0$$

where (V,D) is the (SNC)-pair obtained by a minimal embedded resolution $V \longrightarrow \mathbb{P}^2$ of the singularities of C.

Proof. We perform a blowing up $\sigma_0 : X_1 \longrightarrow \mathbb{P}^2$ at p. Then X_1 is isomorphic to the Hirzebruch surface Σ_1 with a ruling $\pi_1 : X_1 \longrightarrow \mathbb{P}^1$ given by the lines of $\mathbb{P}^2$ through p. Let C_1 be the proper transform of C under σ_0 and let E_1 be the exceptional divisor and F_1 the fibre corresponding to the tangent line of C at p.

First, we assume that (1) holds. Since $b > 1$ it follows by Bezout's theorem that $F_1'.(C_1 \cup E_1) = 2\cdot q_1 + q_1' + q_1''$ where F_1' is the fibre corresponding to the line $\overline{pq}$ and where $q_1 = \sigma^{-1}(q)$ is a branch point of the degree 4 mapping $\pi_{1|C_1\cup E_1} : C_1 \cup E_1 \longrightarrow \mathbb{P}^1$. Now, the result follows directly from Proposition 3.4.

Let us assume that (2) holds. By Bezout's theorem, $C_1.E_1 = (d-3)\cdot p_1$ and either (a) $F_1.(C_1\cup E_1) = 2\cdot p_1 + p_1' + p_1''$ or (b) $F_1.(C_1\cup E_1) = 2\cdot p_1 + 2\cdot p_1'$. Hence, p_1 is a branch point of $\pi_{|C_1\cup E_1}$ and in case (b) p_1' is a branch point $\pi_{|C_1\cup E_1}$, too. Now, for (a) the result follows directly from Proposition 3.4.

In order to prove the statement for case (b) we perform elementary transformations of the fibre F_1. We use the same notation as in Section 2. For $i = 1, \ldots, d-3$ we perform elementary transformations of F_i at $p_i \in E_i \cap F_i$. Then for $i < d-3$, $E_i.C_i = (d-3-i)\cdot p_i$

and $F_i.(C_i \cup E_i) = 2 \cdot p_i + 2 \cdot p_i^{'}$. If $i = d-3$ then we arrive at a Hirzebruch surface X_{d-2} with a ruling $\pi_{d-2} : X_{d-2} \longrightarrow \mathbb{P}^1$ such that $C_{d-2}.E_{d-2} = 0$ and $F_{d-2}.(C_{d-2} \cup E_{d-2}) = p_{d-2} + p_{d-2}^{''} + 2 \cdot p_{d-2}^{'}$. Hence $p_{d-2}^{'}$ is a branch point of $\pi_{d-2|C_{d-2} \cup E_{d-2}}$. From Proposition 3.4 it follows $\pi_*(\Theta_{X_{d-2}}\langle C_{d-2} \cup E_{d-2}\rangle) = 0$. Since $\pi_*(\Theta_{X_1}\langle C_1 \cup E_1\rangle)$ is torsion free this sheaf is also zero. Now, the same argument as in the proof of Proposition 3.4 (1) proves the statement for case (b).

Now, we assume that (3) holds. Let $n := \frac{d-3}{2}$. We have $(C_1 \cup E_1).F_1 = 3 \cdot p_1 + p_1^{'}$ and $E_1.C_1 = (d-3) \cdot p_1$. Now, for $i = 1, \ldots, n$ we perform elementary transformations σ_i at $p_i \in E_i \cap F_i$. Then, for $i < n$, $E_i.C_i = (d-3-2i) \cdot p_1$ and $(C_i \cup E_i).F_i = 3 \cdot p_i + p_i^{'}$. Since we always performed the elementary transformation at a point lying on the horizontal section we arrive at a Hirzebruch surface X_{n+1} with $E_{n+1}^2 = -(n+1)$, $C_{n+1}.E_{n+1} = 0$ and $F_{n+1}.(C_{n+1} \cup E_{n+1}) = 2 \cdot p_{n+1} + p_{n+1}^{'} + p_{n+1}^{''}$ where p_{n+1} is a branch point of the ruling $\pi_{n+1} : X_{n+1} \longrightarrow \mathbb{P}^1$. By Proposition 3.4 it follows $\pi_*(\Theta_{X_{n+1}}\langle C_{n+1} \cup E_{n+1}\rangle) = 0$. Hence $\pi_*(\Theta_{X_1}\langle C_1 \cup E_1\rangle) = 0$. $\square$

Remark 3.8. (1) If C has at least three cusps and maximal multiplicity $\deg C - 3$ then it follows from Proposition 3.5 that $\mathrm{H}^2(V; \Theta_V\langle D\rangle) = 0$ [4]. By [12] the complement $\mathbb{P}^2 \backslash C$ has logarithmic Kodaira-dimension 2 and therefore by [2], $\mathrm{H}^0(V; \Theta_V\langle D\rangle) = 0$. Hence, we have $\chi(\Theta_V\langle D\rangle) \leq 0$.

(2) In order to prove a similar result for curves satisfying the conditions of Corollary 3.7 we need to check that $\mathrm{H}^0(V; \Theta_V\langle D\rangle) = 0$. It is sufficient to show that $\mathrm{H}^0(\mathbb{P}^2; \Theta_{\mathbb{P}^2}\langle C\rangle) = 0$, i.e. that there are no infinitesimal automorphisms. Therefore, C must not be an integral curve of a vector field of $\mathbb{P}^2$. In [2], Flenner and Zaidenberg computed all vector fields and integral curves of $\mathbb{P}^2$. It turned out that the only rational cuspidal curves which occur as integral curves of vector fields of $\mathbb{P}^2$ are those with affine defining equation $u^r - cv^s = 0$, $c \in \mathbb{C}$ and $(r, s) = 1$ for suitable affine charts.

Matsuoka and Sakai [7] proved a lower bound for the maximal multiplicity of a rational cuspidal plane curve. The main ingredient in their proof was the well-known log-Miyaoka-Yau inequality [9].

Proposition 3.9. *Let $C \subset \mathbb{P}^2$ be a rational cuspidal curve of degree d. Let $m_{\max}$ denote the maximum of the multiplicities of its cusps. Then the inequality $d < 3m_{\max}$ holds.*

4. Proof of Theorem 1.2 and 1.3

Now, we apply the restrictions and formulas for the existence of rational cuspidal plane curves to the case where C has maximal multiplicity $m_{\max} = \deg C - 2$ resp. $m_{\max} = \deg C - 3$. The pair (V, D) always denotes the SNC-pair coming from the minimal embedded resolution of singularities of $C \subset \mathbb{P}^2$ as described in Section 1. Remember that the Euler characteristic of $\Theta_V\langle D\rangle$ is given by the formula

$$\chi(\Theta_V\langle D\rangle) = -3(\deg C - 3) + \sum_{p \in \operatorname{Sing} C} \chi_p$$

where $\chi_p = M_p + \omega_p - 1$ (see Section 2).

Proof of Theorem 1.2. Let $C \subset \mathbb{P}^2$ be a rational cuspidal curve of degree d and let $[d-2, m_1, \dots, m_r]$ be the multiplicity sequence of a cusp p of C. Then by Bezout's theorem, $m_i \leq 2$ for $i = 2, \dots, r$. Hence, the only possible multiplicity sequences for a cusp with multiplicity $d-2$ are $[d-2]$ and $[d-2, 2_a]$. If $p \in \operatorname{Sing} C$ has multiplicity sequence $[d-2]$ then

$$M_p = d-3, \quad \omega_p = d-3 \quad \text{and} \quad \chi_p = 2d-7$$

If p has multiplicity sequence $[d-2, 2_a]$ then by Proposition 3.1

$$2 | d-2 \quad \text{and} \quad d-2 \leq 2a \qquad \text{or} \qquad d-2 = 2a+1 \tag{$\dagger$}$$

and

$$M_p = d-3+a, \quad \omega_p = \lceil \frac{d-2}{2} \rceil \quad \text{and} \quad \chi_p = d-4+a+\lceil \frac{d-2}{2} \rceil .$$

If $q \in \operatorname{Sing} C$ with multiplicity sequence $[n_0, \dots, n_s]$ is another cusp then again by Bezout's theorem, $n_j \leq 2$ for $j = 0, \dots, s$. Hence, the only possible multiplicity sequence is $[2_b]$ where

$$M_q = b, \quad \omega_q = 1 \quad \text{and} \quad \chi_\mathrm{q} = \mathrm{b}\,.$$

By the genus formula, we obtain

$$d-2 = \binom{d-1}{2} - \binom{d-2}{2} = \sum_{i=1}^{r} \binom{m_i}{2} + \sum_{i=0}^{s} \binom{n_i}{2} = a+b\,.$$

The only possible types of 1- and 2-cuspidal curves with $m_{\max} = d-2$ are

$$\{[d-2, 2_a]\} \quad \text{where} \quad a = d-2 \tag{1}$$
$$\{[d-2], [2_b]\} \quad \text{where} \quad b = d-2 \tag{2}$$
$$\{[d-2, 2_a], [2_b]\} \quad \text{where} \quad a+b = d-2\,. \tag{3}$$

Case 1. If $1 \equiv d-2 \bmod 2$ then by ($\dagger$) $d-2 = 2a+1$. On the other hand, $a = d-2$ which is a contradiction. If $0 \equiv d-2 \bmod 2$ then C must be of type

$$[2a, 2_{2a}] \quad \text{with} \quad 2a = d-2\,.$$

By Theorem 1.1 ($1a$), such a curve exists.

Case 2. A curve of type

$$[a], [2_a] \quad \text{with} \quad a = d-2$$

always exists by Theorem 1.1 (8).

Case 3. If $1 \equiv d-2 \bmod 2$ then ($\dagger$) yields $d-3 = 2a$. It follows $d-1 = 2b$. Hence, this is a curve of type

$$[2a+1, 2_a], [2_{a+1}] \quad \text{with} \quad 2a+1 = d-2$$

which exists by Theorem 1.1 (3). If $0 \equiv d-2 \bmod 2$ then by (†) $2a \geq d-2$. Hence, this is a curve of type

$$[2a', 2_{a'+b'}], [2_{a'-b'}] \quad \text{with} \quad 2a' = d-2 \quad \text{and} \quad a'+b' = a$$

which exists by Theorem 1.1 (1). This completes the proof. □

Proof of Theorem 1.3. First we determine those types of rational cuspidal curves which are integral curves of $\mathbb{P}^2$. By Remark 3.8, their affine defining equation must be of the form $u^{d-3} - cv^d = 0$, $c \in \mathbb{C}$ for suitable affine charts. It defines a rational cuspidal curve of type

$$\{[3a+1, 3_a], [3_{a+1}]\} \quad \text{where} \quad 3a+4 = d$$

or

$$\{[3a+2, 3_a, 2], [3_{a+1}, 2]\} \quad \text{where} \quad 3a+5 = d\,.$$

For all other rational cuspidal curves with $m_{\max} = d-3$ we have $\mathrm{H}^0(V; \Theta_V\langle D\rangle) = 0$.

By Proposition 3.1, the only possible multiplicity sequences for a cusp with $m_{\max} = d-3$ are $[d-3]$, $[d-3, 2_a]$, $[d-3, 3_a]$ or $[d-3, 3_a, 2]$. If $p \in \operatorname{Sing} C$ has multiplicity sequence $\overline{m}_p = [d-3]$ then

$$M_p = d-4, \quad \omega_p = d-4 \quad \text{and} \quad \chi_p = 2d-9\,.$$

If $\overline{m}_p = [d-3, 2_a]$ then again by the characteristic property of 3.1

$$2 | d-3 \quad \text{and} \quad d-3 \leq 2a \qquad \text{or} \qquad d-3 = 2a+1 \tag{$*$}$$

and

$$M_p = d-4+a, \quad \omega_p = \lceil \frac{d-3}{2} \rceil \quad \text{and} \quad \chi_p = d-5+a+\lceil \frac{d-3}{2} \rceil\,.$$

If $\overline{m}_p = [d-3, 3_a]$ then

$$3 | d-3 \quad \text{and} \quad d-3 \leq 3a \qquad \text{or} \qquad d-3 = 3a+1 \tag{$**$}$$

and

$$M_p = d-4+2a, \quad \omega_p = \lceil \frac{d-3}{3} \rceil + 1 \quad \text{and} \quad \chi_p = d-4+2a+\lceil \frac{d-3}{3} \rceil\,.$$

If $\overline{m}_p = [d-3, 3_a, 2]$ then

$$3 | d-3 \quad \text{and} \quad d-3 \leq 3a \qquad \text{or} \qquad d-3 = 3a+2 \tag{$***$}$$

and

$$M_p = d-3+2a, \quad \omega_p = \lceil \frac{d-3}{3} \rceil + 1 \quad \text{and} \quad \chi_p = d-3+2a+\lceil \frac{d-3}{3} \rceil\,.$$

Let $C \subset \mathbb{P}^2$ be a rational cuspidal curve of degree d with one cusp of type $[d-3, m_1, \ldots, m_r]$. By the genus formula

$$2d-5 = \binom{d-1}{2} - \binom{d-3}{2} = \sum_{i=1}^{r} \binom{m_i}{2}$$

and by Bézout's theorem, it follows $m_i \leq 3$. Thus, the following multiplicity sequences are possible.

$$[d-3, 2_a] \quad \text{where} \quad \mathrm{a} = 2\mathrm{d} - 5 \tag{1}$$
$$[d-3, 3_a] \quad \text{where} \quad 3\mathrm{a} = 2\mathrm{d} - 5 \tag{2}$$
$$[d-3, 3_a, 2] \quad \text{where} \quad 3\mathrm{a} = 2\mathrm{d} - 6\,. \tag{3}$$

Case 1. A curve with a multiplicity sequence as in (1) satisfies condition (1) of Corollary 3.7. Therefore, $\chi(\Theta_V\langle D\rangle) \leq 0$. Since $a = 2d-5$ and $0 \equiv d-3 \bmod 2$ we have

$$\chi(\Theta_V\langle D\rangle) = -3d + 9 + d - 5 + 2d - 5 + \frac{d-3}{2} = \frac{d-3}{2} - 1 \leq 0$$

Hence, only $d = 5$ is possible. By Namba [10], a 1-cuspidal rational curve of degree 5 of type $[2_6]$ exists.

Case 2. If $0 \equiv d-3 \bmod 3$ then $3|d$. Hence $3 \not| \, 2d-5$. Thus, such a curve cannot exist. If $1 \equiv d-3 \bmod 3$ then $a = 2d-5$ and by $(**)$ $2d-5 = d-4$. But this is impossible.

Case 3. If $0 \equiv d-3 \bmod 3$ then this is exactly Theorem 1.1 $(1a)$. Hence, such a curve exists. If $2 \equiv d-3 \bmod 3$ then $3a = 2d-6$ and because of $(***)$ $d-3 = 2d-4$ which is impossible for $d \neq 1$. This completes the proof for 1-cuspidal curves.

In order to prove (2) we list all possible types of 2-cuspidal curves of degree d with maximal multiplicity $d-3$. If $q \in \operatorname{Sing} C$ with multiplicity sequence $\overline{n}_q = [n_0, \ldots, n_s]$ is another cusp then again by Bezout's theorem, $n_i \leq 3$ for $i = 0, \ldots, s$. Hence, by Proposition 3.1, the only multiplicity sequences being possible are either $\overline{n}_q = [2_b]$ where

$$M_q = b, \quad \omega_q = 1 \quad \text{and} \quad \chi_q = b$$

or $\overline{n}_q = [3_b]$ where

$$M_q = 2b, \quad \omega_q = 2 \quad \text{and} \quad \chi_q = 2b+1$$

or $\overline{n}_q = [3_b, 2]$ where

$$M_q = 2b+1, \quad \omega_q = 2 \quad \text{and} \quad \chi_q = 2b+2\,.$$

We obtain the following list of possible types of 1- and 2-cuspidal curves with maximal multiplicity $m_{\max} = d-3$.

$$\{[d-3],[2_b]\} \quad \text{where} \quad b = 2d-5 \tag{1}$$
$$\{[d-3],[3_b]\} \quad \text{where} \quad 3b = 2d-5 \tag{2}$$
$$\{[d-3],[3_b,2]\} \quad \text{where} \quad 3b = 2d-6 \tag{3}$$
$$\{[d-3,2_a],[2_b]\} \quad \text{where} \quad a+b = 2d-5 \tag{4}$$
$$\{[d-3,2_a],[3_b,2]\} \quad \text{where} \quad a+3b = 2d-6 \tag{5}$$
$$\{[d-3,2_a],[3_b]\} \quad \text{where} \quad a+3b = 2d-5 \tag{6}$$

$$\{[d-3, 3_a], [2_b]\} \quad \text{where} \quad 3a+b = 2d-5 \tag{7}$$
$$\{[d-3, 3_a, 2], [2_b]\} \quad \text{where} \quad 3a+b = 2d-6 \tag{8}$$
$$\{[d-3, 3_a], [3_b]\} \quad \text{where} \quad 3a+3b = 2d-5 \tag{9}$$
$$\{[d-3, 3_a, 2], [3_b]\} \quad \text{where} \quad 3a+3b = 2d-6 \tag{10}$$
$$\{[d-3, 3_a], [3_b, 2]\} \quad \text{where} \quad 3a+3b = 2d-6 \tag{11}$$
$$\{[d-3, 3_a, 2], [3_b, 2]\} \quad \text{where} \quad 3a+3b = 2d-7\,. \tag{12}$$

Curves with multiplicity sequences of type (1), (2) or (3) satisfy condition (2) of Corollary 3.7. In these cases, $\chi(\Theta_V\langle D\rangle) \leq 0$.

Case 1. We have

$$\chi(\Theta_V\langle D\rangle) = -3d + 9 + (2d-9) + (2d-5) = d - 5 \leq 0$$

It follows that for $d > 5$ such a curve cannot exist. But for $d = 5$, such a curve does not exist either ([10]).

Case 2. Since $3a = 2d-5$ we have $3|2d-5$, i.e. $1 \equiv d \bmod 3$. The Euler-characteristic is

$$\chi(\Theta_V\langle D\rangle) = -3d + 9 + (2d-9) + (2\frac{2d-5}{3} + 1) = \frac{d-7}{3} \leq 0$$

For $d > 7$ such a curve cannot exist. For $d = 7$ it is a curve with two cusps of type

$$[4]\,, \quad [3_3]\,.$$

This is a curve of type $[da, d_{a+1}]$, $[(d+1)_a]$ with $d = 3$ and $a = 1$. By Theorem 1.1 (4) such a curve exists.

Case 3. We have $3a = 2d-6$. Hence $3|2d-6$, i.e. $0 \equiv d \bmod 3$. The Euler-characteristic is

$$\chi(\Theta_V\langle D\rangle) = -3d + 9 + (2d-9) + (2\frac{2d-6}{3} + 2) = -d + \frac{4d-12}{3} + 2 = \frac{d-6}{3} \leq 0$$

For $d > 6$ such a curve cannot exist. For $d = 6$ such a curve exists by Theorem 1.1. Indeed, their multiplicity sequences are

$$[3]\,, \quad [3_2, 2]$$

which is a curve of type $[da, d_{2a-1}, d-1]$, $[d]$ for $d = 3$ and $a = 1$ which exists by Theorem 1.1 (1).

Case 4. From the genus formula we have $a + b = 2d - 5$. From Corollary 3.7 (1) it follows $\chi(\Theta_V\langle D\rangle) \leq 0$ if $b > 1$. If $b = 1$ then $a = 2d-6$ and hence $0 \equiv d-3 \bmod 2$. Therefore $\chi(\Theta_V\langle D\rangle) \leq 0$ by Corollary 3.7 (3). We compute the Euler-characteristic:

$$\chi(\Theta_V\langle D\rangle) = -3d + 9 + (d - 5 + a + \lceil\frac{d-3}{2}\rceil) + (b) = \lceil\frac{d-3}{2}\rceil - 1 \leq 0\,.$$

Let $0 \equiv d-3 \bmod 2$. Then $\chi(\Theta_V\langle D\rangle) = \frac{d-3}{2} - 1 > 0$ for $d \geq 7$.

If $1 \equiv d-3 \bmod 2$ then $\chi(\Theta_V\langle D\rangle) = \frac{d-2}{2} - 1 > 0$ for $d \geq 6$. By [10], for $d = 5$ there is a curve with two cusps of type

$$[2_4]\,, \quad [2_2]\,.$$

For curves with multiplicity sequences of type (5) or (6) we have $\chi(\Theta_V\langle D\rangle) \leq 0$ if $0 \equiv d-3 \bmod 2$ (Corollary 3.7 (3)).

Case 5. First, let $1 \equiv d-3 \bmod 2$. Then, since $(*)$ and $a + 3b = 2d - 6$ holds we have

$$2a = d-4 \qquad \text{and} \qquad 6b = 3d-8$$

But $3 \nmid 3d-8$. A contradiction. Hence, such a curve does not exist.

Now, let $0 \equiv d-3 \bmod 2$. Taking into account that $a + 3b = 2d - 6$ we obtain

$$\chi(\Theta_V\langle D\rangle) = -3d + 9 + (d - 5 + \frac{d-3}{2} + a) + (2b+2) = \frac{d-3}{2} - b \leq 0\,.$$

On the other hand, $(*)$ holds. We obtain the inequality

$$3b = 2d - 6 - a \leq \frac{4d-12}{2} - \frac{d-3}{2} = \frac{3d-9}{2}\,.$$

Putting these inequalities together we get

$$3d - 9 \leq 6b \leq 3d - 9\,.$$

Hence, only $2b = d - 3$ is possible and therefore $2a = d - 3$. This is a rational cuspidal curve with two cusps of type

$$[2a, 2_a]\,, \quad [3_a, 2] \quad \text{for} \quad a \geq 1$$

By Theorem 1.1 (5), such a curve exists.

Case 6. Let $1 \equiv d-3 \bmod 2$. Then $2a = d-4$ and $2b = d-2$. A curve with cusps of type

$$[2a+1, 2_a]\,, \quad [3_{a+1}] \quad \text{for} \quad a \geq 1$$

exists by Theorem 1.1 (6).

Now let $0 \equiv d-3 \bmod 2$. Since $2a \geq d-3$ and $a + 3b = 2d - 5$ we have

$$3b = 2d - 5 - a \leq \frac{4d-10}{2} - \frac{d-3}{2} = \frac{3d-7}{2}\,.$$

The Euler-characteristic is

$$\chi(\Theta_V\langle D\rangle) = -3d + 9 + (d - 5 + 2d - 5 - 3b + \frac{d-3}{2}) + (2b+1) = \frac{d-3}{2} - b \leq 0\,.$$

Therefore

$$3d - 9 \leq 6b \leq 3d - 7\,.$$

Only $2b = d-3$ is possible. Then $2a = d-1$. Hence, the multiplicity sequences are

$$[2a, 2_{a+1}]\,, \quad [3_a] \quad \text{for} \quad a \geq 1\,.$$

By Theorem 1.1 (4) such a curve exists.

Case 7. First, let $1 \equiv d-3 \bmod 3$. Then $3a = d-4$ and $b = d-1$. We have

$$\chi(\Theta_V\langle D\rangle) = -3d + 9 + (d - 4 + 2\frac{d-4}{3} + \frac{d-1}{3}) + (d-1) = 1\,.$$

Since $b > 1$ it follows from Corollary 3.7 (1) that $\chi(\Theta_V\langle D\rangle) \leq 0$. A contradiction. Hence, such a curve does not exist. The case $0 \equiv d-3 \bmod 3$ is treated in the next Lemma.

Case 8. If $2 \equiv d-3 \bmod 3$ then $3a = d-5$ and $b = d-1$. Hence

$$\chi(\Theta_V\langle D\rangle) = -3d + 9 + (d - 3 + 2\frac{d-5}{3} + \frac{d-2}{3}) + (d-1) = 1\,.$$

As in Case 7 $b > 1$ implies $\chi(\Theta_V\langle D\rangle) \leq 0$ contradicting the upper computation. The case $0 \equiv d-3 \bmod 3$ is treated in the next Lemma.

Case 9. Let $0 \equiv d-3 \bmod 3$. From the genus formula it follows $3(a+b) = 2d-5$, i.e. $3|2d-5$ which implies $1 \equiv d \bmod 3$. A contradiction.

If $1 \equiv d-3 \bmod 3$ then $3a = d-4$ and $3b = d-1$. We obtain a curve with cusps of type

$$[3a+1, 3_a]\,, \quad [3_{a+1}] \quad \text{for} \quad a \geq 1$$

of degree $3a+4$ which clearly exists. By Theorem 1.1 (3) such a curve exists.

Case 10. If $2 \equiv d-3 \bmod 3$ then $3a = d-5$, i.e. $3|d-5$ and from the genus formula, it follows $3b = 2d-6-d+5 = d-1$, i.e. $3|d-1$ which is a contradiction.

If $0 \equiv d-3 \bmod 3$ then because of $(**)$ $3a \geq d-3$. These are curves of type

$$[3a', 3_{a'+b'}, 2]\,, \quad [3_{a'-b'}] \quad \text{with} \quad 3a' = d-3 \quad \text{and} \quad a'+b' = a\,.$$

By Theorem 1.1 (1) those curves exist.

Case 11. If $1 \equiv d-3 \bmod 3$ then $3a = d-4$, i.e. $3|d-4$ and from the genus formula, it follows $3b = 2d-6-d+4 = d-2$, i.e. $3|d-2$ which is a contradiction.

If $0 \equiv d-3 \bmod 3$ then because of $(**)$ $3a \geq d-3$. These are curves of type

$$[3a', 3_{a'+b'}]\,, \quad [3_{a'-b'}, 2] \quad \text{with} \quad 3a' = d-3 \quad \text{and} \quad a'+b' = a\,.$$

By Theorem 1.1 (2) those curves exist.

Case 12. Let $0 \equiv d-3 \bmod 3$. We have $3a+3b = 2d-7$ by the genus formula. Hence, $3|2d-7$ which contradicts the assumption.

Otherwise, if $2 \equiv d-3 \bmod 3$ then $3a = d-5$ and $3b = d-2$: a curve with two cusps of type

$$[3a+2, 3_a, 2]\ , \quad [3_{a+1}, 2]\ .$$

By Theorem 1.1 (7) those curves exist.

We complete the proof with the following lemma.

Lemma 4.1. *Let $d \equiv 0 \bmod 3$ and $0 < b \leq a$ be positive integers.*

(1) *A curve of degree d and of type $[d-3, 3_a]$, $[2_b]$ exists if and only if $b = 1$.*

(2) *A curve of degree d of type $[d-3, 3_a, 2]$, $[2_b]$ does not exist.*

Proof. As in the proof of Theorem 3.1 we transform the curves by a suitable Cremona transformation. We use the notation of Section 2.

(1) Suppose, C is a curve of type $[d-3, 3_a]$, $[2_b]$. Let $p_0 \in C$ be the cusp of type $[d-3, 3_a]$ and $q_0 \in C$ be the cusp of type $[2_b]$.

(a) If $a = \frac{d-3}{3}$, then $b = d-2$. Perform the blowing up σ_0 at p_0. Let F_1 resp. F_1' be the proper transform of T_pC resp. $L = \overline{pq}$. Then by Bezout's theorem, $C_1.F_1 = 3 \cdot p_1, C_1.F_1' = 2 \cdot q_1 + q_1'$ and $C_1.E_1 = (d-3) \cdot p_1$. C_1 has two cusps, one at p_1 of type $[3_a]$ and one at q_1 of type $[2_b]$. For $i = 1, \ldots, a+2$, we perform elementary transformations σ_i at $\mathsf{p}_i \in E_i \cap F_i$ followed by elementary transformations σ_i at $q_i \in C_i$ for $i = a+3, \ldots, 2a+4$. We arrive at X_{2a+5} with $E_{2a+5}^2 = -1$. Then $C_{2a+5}.F_{2a+5} = 3 \cdot p_{2a+5}$, $C_{2a+5}.F_{2a+5}' = 2 \cdot q_{2a+5} + q_{2a+5}'$ and $C_{2a+5}.E_{2a+5} = (a+2) \cdot q_{2a+5}'$. C_{2a+5} has a cusp of type $[3_2]$ at p_{2a+5} and a cusp of type $[2_{b-a-2}]$ at q_{2a+5}. Note that $b-a-2 > 0$ if and only if $d \geq 5$. Hence, after blowing down E_{2a+5} we arrive at $\mathbb{P}^2$ and the proper transform C' is a rational cuspidal curve of type $[a+2]$, $[3_2]$ and $[2_{d-a-4}]$ of degree $a+5$. By [4], such a curve cannot exist.

(b) Let $a > \frac{d-3}{3} =: \omega$. We start as in the case before. But then for $i = 1, \ldots, \omega$, σ_i is an elementary transformation of F_i at $\mathsf{p}_i \in E_i \cap F_i$ followed by an elementary transformation σ_i of F_i' at q_i' for $i = \omega+1, \ldots, 2\omega$. Then, we arrive at $X_{2\omega+1}$ and $C_{2\omega+1}.F_{2\omega+1} = 3p_{2\omega+1}$, $C_{2\omega+1}.F_{2\omega+1}' = 2q_{2\omega+1} + q_{2\omega+1}'$ and $C_{2\omega+1}.E_{2\omega+1} = 2\omega q_{2\omega+1}$. $C_{2\omega+1}$ is a curve with a cusp of type $[3_{a-\omega}]$ at $p_{\omega+1}$ and a cusp of type $[2_{b+\omega}]$. After blowing down $E_{\omega+1}$ we arrive at $\mathbb{P}^2$ and the proper transform C' is a rational cuspidal curve of type $[2\omega, 2_{b+\omega}]$ and $[3_{a-\omega}]$ and degree $2\omega+3$. But this is Case 6. Hence

$$a - \omega + 1 = b + \omega\,.$$

Moreover, the equality $3a + b = 2d - 5$ holds. $b = 1$ and $3a = 2d - 6$ is the unique solution. Therefore such a curve must be of type

$$[3a, 3_{2a}]\,, \quad [2] \qquad \text{with} \quad a \geq 1 \tag{13}$$

which exists by Theorem 1.1 (1a).

(2) Suppose such a curve exists. Let $\omega := \frac{d-3}{3}$. We perform the same transformations as described in (1) (b). We obtain the following rational cuspidal plane curves of degree $2\omega+3$.

$$[2]\,, \quad [2\omega, 2_{b+\omega}] \qquad \text{if} \quad a = \omega \tag{14}$$

and

$$[3_{a-\omega}, 2]\,, \quad [2\omega, 2_{b+\omega}] \qquad \text{if} \quad a > \omega\,. \tag{15}$$

(14) is Case 4 of the proof. Hence such a curve cannot exist. (15) is Case 5. Such a curve exists if and only if

$$b + \omega = a - \omega\,.$$

The equality $3a + b = 2d - 6$ also holds. The unique solution is $b = 0$ and $3a = 2d - 6$ which is a contradiction to $b > 0$. Hence, such a curve cannot exist. □

5. Parameterization and defining equations of the rational cuspidal plane curves of degree 6

Finally, we will give the parameterizations and defining equations of the curves of Corollary 1.5 up to projective equivalence.

1-cuspidal curves:

1. C_1 is projectively equivalent to one of the curves
   (a) $\Psi(s,t) = (st^5 : t^6 : s^6)$ resp.
   $$f = zy^5 - x^6$$
   (b) $\Psi(s,t) = (st^5 : t^6 : s^6 + s^5t + as^4t^2 + bs^3t^3)$ resp.
   $$f = x^6 + x^5y + ax^4y^2 + bx^3y^3 - y^5z$$
   (c) $\Psi(s,t) = (st^5 : t^6 : s^6 + as^5t + s^4t^2 + bs^3t^3)$ resp.
   $$f = x^6 + ax^5y + x^4y^2 + bx^3y^3 - y^5z$$
   (d) $\Psi(s,t) = (st^5 : t^6 : s^6 + as^5t + bs^4t^2 + s^3t^3)$ resp.
   $$f = x^6 + ax^5y + bx^4y^2 + x^3y^3 - y^5z$$
   with $a, b \in \mathbb{C}$.
2. C_2 is projectively equivalent to one of the curves
   (a) $\Psi(s,t) = (s^2t^4 + st^5 : t^6 : s^6 + 3s^5t + as^4t^2 + (2a-5)s^3t^3 + bst^5)$ resp.
   $$\begin{aligned} f = x^6 &+ (2a-6)x^5y + (a^2-8a+15)x^4y^2 + (-2a^2+11a-b-15)x^3y^3 \\ &+ (-ab+3b)x^2y^4 + (-ab-b^2+3b)xy^5 - 2x^3y^2z + (6-2a)x^2y^3z \\ &+ (2a-6)xy^4z + (a+b-3)y^5z + y^4z^2 \end{aligned}$$
   (b) $\Psi(s,t) = (s^2t^4 : t^6 : s^6 + as^4t^2 + st^5)$ resp.
   $$f = -x^6 - 2ax^5y - a^2x^4y^2 + xy^5 + 2x^3y^2z + 2ax^2y^3z - y^4z^2$$
   with $a, b \in \mathbb{C}$.
3. C_3 is projectively equivalent to one of the curves
   (a) $\Psi(s,t) = (s^3t^3 : t^6 : s^6 + bst^5)$ resp.
   $$f = -x^6 - b^3xy^5 + 3x^4yz - 3x^2y^2z^2 + y^3z^3$$
   (b) $\Psi(s,t) = (s^3t^3 + st^5 : t^6 : s^6 + 2s^4t^2 + s^2t^4 + bst^5)$ resp.
   $$f = x^6 + b^2x^2y^4 + b^3xy^5 - 3x^4yz - b^2y^5z + 3x^2y^2z^2 - y^3z^3$$

(c) $\Psi(s,t) = (s^3t^3 + s^2t^4 + ast^5 : t^6 : s^6 + (1+2a)s^4t^2 + (a^2-2a)s^2t^4 + bst^5)$ resp.

$$\begin{aligned} f = & -x^6 + 6ax^5y + (b-10a^2)x^4y^2 - 4abx^3y^3 \\ & + (8a^4 - 4a^5 - 4a^3b + 4a^2b - ab^2)x^2y^4 - (4a^4b + 4a^2b^2 + b^3)xy^5 \\ & + 3x^4yz - 12ax^3y^2z + (8a^2-2b)x^2y^3z + (8a^3+4ab)xy^4z \\ & + (4a^5 + 4a^3b + ab^2)y^5z - 3x^2y^2z^2 + 6axy^3z^2 + (2a^2+b)y^4z^2 + y^3z^3 \end{aligned}$$

with $a, b \in \mathbb{C}$.

2-cuspidal curves:

1. C_4 is projectively equivalent to one of the curves
   (a) $\Psi(s,t) = (t^6 : s^3t^3 + s^2t^4 : s^6 + 2s^5t + s^4t^2 + as^3t^3)$ with $a \neq 0$ resp.

   $$\begin{aligned} f = & -a^3x^4y^2 - 2a^2x^3y^3 - ax^2y^4 - y^6 + 2a^2x^4yz + 2ax^3y^2z \\ & + 3xy^4z - ax^4z^2 - 3x^2y^2z^2 + x^3z^3 \end{aligned}$$

   (b) $\Psi(s,t) = (t^6 : s^3t^3 : s^6 + s^2t^4)$ resp.

   $$f = x^4y^2 + y^6 - 3xy^4z + 3x^2y^2z^2 - x^3z^3$$

2. C_5 is projectively equivalent to
   $\Psi(s,t) = (t^6 : s^3t^3 : s^6 + s^4t^2)$ resp.

   $$f = -y^6 + 3xy^4z - 3x^2y^2z^2 + x^3z^3 - x^2y^4$$

3. C_6 is projectively equivalent to
   $\Psi(s,t) = (t^6 : s^3t^3 : s^6 + s^5t)$ resp.

   $$f = x^3z^3 - xy^5 - y^6 + 3xy^4z - 3x^2y^2z^2$$

4. C_7 is projectively equivalent to
   $\Psi(s,t) = (t^6 : s^2t^4 : s^6 + as^4t^2 + s^3t^3)$ with $a \neq 0$ resp.

   $$f = x^3y^3 - a^2x^2y^4 - 2axy^5 - y^6 + 2ax^3y^2z + 2x^2y^3z - x^4z^2$$

5. C_8 is projectively equivalent to
   $\Psi(s,t) = (t^6 : s^2t^4 : s^6 + s^5t + as^4t^2)$ with $a \neq \frac{1}{4}$ resp.

   $$f = (1-2a)xy^5 - y^6 + 2ax^3y^2z + 2x^2y^3z - x^4z^2 - a^2x^2y^4$$

6. C_9 is projectively equivalent to
   $\Psi(s,t) = (st^5 + t^6 : s^2t^4 : s^6 - 2s^5t + 2s^4t^2)$ resp.

   $$f = 4xy^5 - 4x^2y^4 - 5y^6 + 4x^3y^2z - 2x^2y^3z + 2xy^4z + y^5z - x^4z^2$$

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