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Paracomplex Projective Models and Harmonic Maps into Them

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Abstract. Paracomplex projective models are para-Kaehler space forms (and models of constant paraholomorphic sectional curvature of dimension greater than 2) introduced in [9]. We consider these models with their usual para-Kaehler structure and also introduce another almost para-Kaehler structure on their underlying manifolds. Next the harmonicity of maps into the two given types of manifolds is discussed and some harmonic maps are constructed from certain paraholomorphic ones.

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Introduction

Since the first papers written on paracomplex geometry (a geometry related to the ring of para-complex numbers) by Rashevskij [23], Libermann [15] and Patterson [22] in 1948, 1952 and 1954 respectively, many other mathematicians and physicists have devoloped interests into it and used it (see [3], [4] and the references therein). The physicists have mobilized this geometry in quite significant ways: The quantum field theory of gravitation based on a non-symmetric tensor (such as a para-Hermitian form) as developed by Kunstatter, Moffat and Malzan [14] is known to have no inconsistency problem concerning unitarity. Also Moffat [19] has interpreted the paracomplex number ε, ($\varepsilon^2 = 1$) as *fermion* number. Consequently this interpretation has led to fundamental explanation of stability of fermionic matter. In [16], Lambert and Kibler used the paracomplex geometry to generate non-bijective canonical transformation.

Part of this paper is devoted to describe a paracomplex projective model $\mathbb{P}_m(\mathbb{A})$, a paracomplex analogue of complex projective space, as a differentiable manifold. And then two different extra structures are introduced so that $(\mathbb{P}_m(\mathbb{A}), J, g)$ becomes a para-Kaehler space form and $(\mathbb{P}_m(\mathbb{A}), P, G^c)$ becomes an almost para-Kaehler manifold.

The rest of the paper is devoted to describe and construct harmonic maps of manifolds with definite or neutral metrics into paracomplex projective models, examples of non-compact manifolds. One should note that harmonic maps of a manifold with neutral metric, e.g. a Lorentz surface, are the solutions of a system of wave equations while those of a Riemannian manifold are solutions of a system of elliptic ones. Harmonic maps between semi-Riemannian manifolds are connected to the hyperbolic systems. This connection is particularly significant in the areas of quantum field theory, solitons and scattering theory and general relativity, see [8]. Our work also demonstrates all along how beautifully the ring of paracomplex numbers fits to construct non-compact manifolds with neutral metrics and to attack problems of harmonic maps between them as the complex numbers do in the case of harmonic maps of Riemannian manifolds, see also [17] and [18].

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0. Preliminaries

Let θ denote ε with $\varepsilon^2 = 1$ or i with $i^2 = -1$ and then set $C_\theta = \{a + \theta b : a, b \in \mathbb{R}\}$. Clearly $C_i = \mathbb{C}$, the field of complex numbers. But C_ε, which we will denote by $\mathbb{A}$, forms a ring with zero divisors $k(1+\varepsilon)$ and $k(1-\varepsilon)$, $k \in \mathbb{R}$. The set $\mathbb{A}$ is called *paracomplex (or double or hyperbolic complex) numbers.*

Notations. We frequently use one of the following notations where it is convenient.

$$\begin{aligned} Z = (z_k) &= (z_1, \ldots, z_N) \\ &= (x_1 + \theta y_1, \ldots, x_N + \theta y_N) = (x_k) + \theta(y_k) = ((x_k), (y_k)) = (x + \theta y) \in C_\theta^N \end{aligned}$$

$\overline{Z} = (\overline{z}_k)$ and $\overline{z} = x - \theta y$, θ-complex conjugate of z. For $Z = (z_k)$, $W = (w_k) = (u_k) + \theta(v_k)$, $r = 0, 1, \ldots, N-1$, let

$$\langle Z, W \rangle_r^\theta = -\sum_{k=1}^{r} z_k w_k + \sum_{k=r+1}^{N} z_k w_k$$

the symmetric C_θ-bilinear form and

$$\langle Z, W \rangle_r^{\overline{\theta}} = -\sum_{k=1}^{r} z_k \overline{w}_k + \sum_{k=r+1}^{N} z_k \overline{w}_k$$

the C_θ-Hermitian form. They are both nondegenerate on C_θ^N. For $\theta = \varepsilon$ we write $\langle , \rangle_r^A$ and $\langle , \rangle_r^{\overline{A}}$ for symmetric $\mathbb{A}$-bilinear (or paracomplex bilinear) and para-Hermitian forms respectively. For $Z, W \in \mathbb{R}^N$ we write

$$G_r(Z, W) = \langle Z, W \rangle_r = \langle Z, W \rangle_r^R = \langle Z, W \rangle_r^\theta = \langle Z, W \rangle_r^{\overline{\theta}}.$$

Let V be a real vector space of dimension n and $J_\theta : V \to V$ be an endomorphism with $J_\varepsilon^2 = I$ and $J_i^2 = -I$, where I is the identity. J_i is called complex structure on V. Let k_1, k_2 be the dimensions of the eigenspaces of J_ε corresponding to eigenvalues $1, -1$ respectively. Then J_ε is called *product structure* if $k_1, k_2 \geq 1$. A *paracomplex structure* on V is a product structure with $k_1 = k_2$. A (semi-) Riemannian metric g on V is said to be *Hermitian* (resp. *para-Hermitian*) if $g(X,Y) = mg(J_\theta X, J_\theta Y) \;\; X, Y \in V$, where $m = 1$ when $J_\theta = J_i$ (resp. $m = -1$ when $J_\theta = J_\varepsilon$). Note that if g is para-Hermitian then it has to be an indefinite metric of signature (s,s) on V in which case g is called *neutral metric*, where $dimV = n = 2s$. (Almost) paracomplex, (almost) para-Hermitian, (almost) para-Kaehler manifolds and paracomplex, paraholomorphic, para-Hermitian vector bundles are defined in the usual way with the difference that the defining elements are replaced with the corresponding "para" ones where necessary.

Let $f : (M, g, J) \to (M', g', J')$ be a differentiable map between (almost) para-Hermitian manifolds. The second fundamental form $\triangledown df$ of f is given by

$$\triangledown df(u,v) = \tilde{\mathbf{D}}_u(df(v)) - df(D_u^M v)$$

where D^M and $\tilde{\mathbf{D}}$ are the metric connection on the indicated manifold M and the pull-back connection of $D^{M'}$ under f. Also the tension field $\tau(f)$ of f is given by

$$\tau(f) = \sum_{i,j=1}^{n} g^{ij} \triangledown df(e_i, e_j) \in TM'$$

where g^{ij} is the (i,j)-th entry of the matrix $(g_{ij})^{-1}$, with $g_{ij} = g(e_i, e_j)$ and $\{e_j, \ldots, e_m\}$ is a basis for TM.

Definitions. *f is said to be*

i) *paraholomorphic if $df \circ J = J' \circ df$,*
ii) *harmonic if $\tau(f) = 0$,*
iii) *totally geodesic if $\triangledown df = 0$.*

We call

i) a paracomplex manifold of dimension two a *para-Riemann surface,*
ii) an oriented, paracompact, connected surface with a neutral metric (and therefore of signature (1,1)) a *Lorentz surface,*
iii) a free module over the ring $\mathbb{A}$ a *paracomplex vector space.*

On a surface (M,g) with its metric g, local coordinates (x,y) are said to be *g-isothermal* if $g(\partial_x, \partial_x) = kg(\partial_y, \partial_y)$ and $g(\partial_x, \partial_y) = 0$ where $\partial_x = \dfrac{\partial}{\partial x}$ etc. and $k = 1$ if g is positive definite and $k = -1$ if it is indefinite. Note that every Lorentz surface $(\mathbb{L}, g)$ possesses g-isothermal coordinates [7], [26]. Thus one obtains [7], that every Lorentz surface is a para-Riemann surface. The $\mathbb{A}^N$ is an obvious example of a paracomplex vector space with a canonical paracomplex structure J, $\;J(v) = \varepsilon v \in \mathbb{A}^N = \mathbb{R}^{2N}$.

1. Paracomplex projective models

We set $\langle , \rangle = \langle , \rangle_0^{\overline{A}}$, $g = Re(\langle , \rangle)$ which is the real part of $\langle , \rangle$ and $\mathbb{A}_0^N = \{Z \in \mathbb{A}^N : \langle Z, Z\rangle > 0\}$.

Note that g is the neutral metric (of signature (N, N)) on $\mathbb{R}^{2N}$ and $\langle Z, Z\rangle = g(Z, Z)$. Let $\mathbb{P}_n(\mathbb{A})$ denote the quotient of $\mathbb{A}_0^{n+1}$ under the equivalence relation: $Z \simeq W$ if $W = zZ$ for some $z \in \mathbb{A}_+$, where $\mathbb{A}_+ = \{z = x + \varepsilon y \in \mathbb{A} : z\overline{z} > 0, \quad x > 0\}$. Then the set $\mathbb{P}_n(\mathbb{A})$ may be endowed with a differentiable manifold structure ([4], [9], [10]). We shall endow this differentiable manifold with two different metrics and paracomplex structures in order to obtain two different (almost) para-Kaehler manifolds:

A) (cf. [9], [10]) Consider the principal bundle $\tilde{\pi} : \mathbf{S} \to \mathbb{P}_n(\mathbb{A})$ with its structure group $\mathbf{R}_+ = \{e^{\varepsilon t} = \cosh t + \varepsilon \sinh t : t \in \mathbb{R}\} \subseteq \mathbb{A}_+$ obtained from the bundle $\pi : \mathbb{A}_0^{n+1} \to \mathbb{P}_n(\mathbb{A})$ with its structure group $\mathbb{A}_+$ by restricting π to $\mathbf{S}$, where

$$\begin{aligned}\mathbf{S} = S_{n+1}^{2n+1} &= \{Z = (x + \varepsilon y) \in \mathbb{A}_0^{n+1} : \langle Z, Z\rangle = 1\}\\ &= \{Z \in \mathbb{A}_0^{n+1} : \langle x, x\rangle - \langle y, y\rangle = g(z, z) = 1\}.\end{aligned}$$

Then $\tilde{\pi}$ induces a metric g and an almost paracomplex structure J on $\mathbb{P}_n(\mathbb{A})$ (from that on $\mathbb{A}_0^{n+1}$) which coincides with the ones described in [9]. Thus $(\mathbb{P}_n(\mathbb{A}), g, J)$ becomes a para-Kaehler manifold of (nonzero) constant paraholomorphic sectional curvature. So it is a para-Kaehler space form. The letters g and J will be reserved to denote the metric and the paracomplex structure on $\mathbb{P}_n(\mathbb{A})$ respectively obtained in the way just described.

The para-Kaehler space form $(\mathbb{P}_n(\mathbb{A}), g, J)$ has the following properties ([4], [9], [10]): It is

i) complete since $(\mathbf{S}, g)$ is complete and $\tilde{\pi}$ is totally geodesic,

ii) simply connected for $n > 1$ and connected for $n = 1$,

iii) a symmetric (hence homogeneous) space,

iv) Einstein, (see [4], Corollary 4.13, p.25).

B) We shall now put other structures on $\mathbb{P}_n(\mathbb{A})$: Let (M, β) be a (semi-) Riemannian manifold of dimension m and the signature of its metric β be $(k, m-k)$, $k = 1, 2, \ldots, m$. We have a splitting $T(TM) = V(TM) \oplus H(TM)$ of $T(TM)$ into *vertical* and *horizontal* subbundles. Define, on the manifold TM:

i) A neutral metric β^c (i.e. a metric of signature (m, m)) by

$$\beta^c(X^V, Y^V) = \beta^c(X^H, Y^H) = 0 \quad \text{and} \quad \beta^c(X^H, Y^V) = \beta^c(X^V, Y^H) = \beta(X, Y), \tag{1.1}$$

where X^V and X^H are the vertical and horizontal lifts of the vector fields X and Y on M respectively. The neutral metric β^c will be called the *complete lift* of β.

ii) An almost paracomplex structure P by

$$P(X^V) = X^V, \quad P(X^H) = -X^H. \tag{1.2}$$

Then (TM, β^c, P) becomes an almost para-Hermitian manifold (see [2], [20], [27] for details).

Let us now take $M = \mathbb{S}^n$, the unit sphere in $(\mathbb{R}^{n+1}, G)$ with its standard Riemannian metric which will be denoted by the same letter G. Viewing the tangent bundle $T\mathbb{S}^n$ as a set

$$T\mathbb{S}^n = \{(x,y) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : G(x,x) = 1, \quad G(x,y) = 0\},$$

we define a map $\varphi : T\mathbb{S}^n \xrightarrow{\psi} \mathbf{S} \xrightarrow{\tilde{\pi}} \mathbb{P}_n(\mathbb{A})$ where

$$\begin{aligned} \psi(x,y) &= \frac{1}{2}(2x+y,\ y) \cong \frac{1}{2}\Big((2x+y) + \varepsilon y\Big) \\ &= \frac{1}{2}\Big((2x_1+y_1) + \varepsilon y_1, (2x_2+y_2) + \varepsilon y_2, \ldots, (2x_{n+1}+y_{n+1}) + \varepsilon y_{n+1}\Big). \end{aligned}$$

Note that $\psi(x,y) \in \mathbf{S} = S^{2n+1}_{n+1} \subseteq \mathbb{R}^{2n+2}$ since $g(\psi(x,y), \psi(x,y)) = 1$. (Recall that $g = Re(\langle, \rangle^{\overline{A}}_0)$ which is a neutral metric on $\mathbb{R}^{2n+2}$.) It is not difficult to see that (cf. [9]) the map $\varphi = \psi \circ \tilde{\pi}$ is a diffeomorphism of $T\mathbb{S}^n$ onto $\mathbb{P}_n(\mathbb{A})$. Thus $\mathbb{P}_n(\mathbb{A})$ and $T\mathbb{S}^n$ are diffeomorphic as differentiable manifolds. We now give $\mathbb{P}_n(\mathbb{A})$ the metric G^c and paracomplex structure P on $T\mathbb{S}^n$ described by (1.1) and (1.2) via the diffeomorphism φ so that $(\mathbb{P}_n(\mathbb{A}), G^c, P)$ becomes an almost para-Hermitian manifold. Since $\mathbb{S}^n$ is a Lagrange manifold, $(\mathbb{P}_n(\mathbb{A}), G^c, P)$ is in fact almost para-Kaehler [20]. The letters G^c and P will be reserved to denote the metric and almost paracomplex structure on $\mathbb{P}_n(\mathbb{A})$ respectively obtained in the way just described.

This manifold $(\mathbb{P}(\mathbb{A}), G^c, P)$ has the following properties:

i) It is a locally conformally flat, locally symmetric space since $\mathbb{S}^n$ is a symmetric, connected space of constant curvature [5].

ii) Unlike $(\mathbb{P}_n(\mathbb{A}), g, J)$, this is not a space of constant paraholomorphic sectional curvature since $\mathbb{S}^n$ is not locally flat. Nevertheless it has a vanishing scalar curvature [5]. Furthermore [11] it is null isotropic, that is, for every null vector $u \in T_p = T_p(\mathbb{P}_n(\mathbb{A}))$ the function

$$c_u = T_p \to \mathbb{R}, \quad c_u(X) = \frac{R(X,u,u,X)}{G^c(X,X)}$$

is constant for all non-null X orthogonal to u where R is the Riemannian curvature tensor field. In fact when $n \geq 2$, the null isotropy is equivalent to one of the following:

a°) $R(u,v,u,v) = 0$ for every orthogonal pair of null vectors u, v,

b°) local conformal flatness.

C) Some totally geodesic submanifolds of $(\mathbb{P}_n(\mathbb{A}), g)$

For the pseudo-sphere $S^n_r = \{u = (u_1, \ldots, u_{n+1}) \in \mathbb{R}^{n+1} : G_r(u,u) = 1\}$ where $G_r(u,u) = -\sum_{i=1}^{r} u_i^2 + \sum_{i=r+1}^{n+1} u_i^2$ consider the map

$$\overline{\varphi}_r : S^n_r \overset{\overline{\psi}_{n+1-r}}{\longrightarrow} \mathbf{S} = S^{2n+1}_{n+1} \xrightarrow{\tilde{\pi}} \mathbb{P}_n(\mathbb{A}) \tag{1.3}$$

given by

$$\begin{aligned} \overline{\psi}_t(u) &= (u_1, \ldots, u_t, 0, \ldots, 0; 0, \ldots, 0, \quad u_{t+1}, \ldots, u_{n+1}) \in \mathbf{S} \\ &\cong (u_1, \ldots, u_t, \varepsilon u_{t+1}, \ldots, \varepsilon u_{n+1}) \in \mathbb{A}^{n+1}. \end{aligned}$$

One can show (cf. [10], Theorem 5.3) that for each $r = 0, 1, \ldots, n$ the map $\overline{\varphi}_r$ is an isometric imbedding and $\overline{\varphi}_r(S^n_r)$ is a totally geodesic submanifold of $(\mathbb{P}_n(\mathbb{A}), g)$. In particular, when $r = 0$, we have

$$\overline{\varphi} = \overline{\varphi}_0 : \mathbb{S}^n = S^n_0 \overset{\overline{\psi}_{n+1}}{\to} \mathbf{S} \overset{\tilde{\pi}}{\to} \mathbb{P}_n(\mathbb{A})$$

with $\overline{\varphi}(u) = \tilde{\pi}(u, \overline{0}) \cong span_{\mathbb{A}_+}\{(u, \overline{0})\}$ which may be identified with $span_{\mathbb{R}_+}\{u, \overline{0}\}$ where $(u, \overline{0}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \cong \mathbb{A}^{n+1}$, $\overline{0} = (0, \ldots, 0) \in \mathbb{R}^{n+1}$. Thus $\overline{\varphi}(\mathbb{S}^n)$ constitutes the set of all real rays in $\mathbb{R}^{n+1}$ and it is also a totally geodesic submanifold of $(\mathbb{P}_n(\mathbb{A}), g)$. This is an analogous result of the fact that the real projective space $\mathbb{P}_n(\mathbb{R})$ is a totally geodesic submanifold of the complex projective space $\mathbb{P}_n(\mathbb{C})$ with their standard metrics.

We say that *an almost para-Hermitian manifold M with its almost paracomplex structure Q is* $(1,2)$*-symplectic if*

$$\nabla_{\overline{Z}} W \in \Gamma(T^{1,0}M) \quad \forall Z, W \in \Gamma(T^{1,0}M),$$

where $T^{1,0}M = \{(1 + \varepsilon Q)X : X \in \Gamma(TM)\}$ and $T^{0,1}M = \overline{T^{1,0}M} = \{(1 - \varepsilon Q)X : X \in \Gamma(TM)\}$.

Lemma 1.1. *Let (M, h) be a (semi-) Riemannian manifold. Then (TM, h^c, P) is* $(1,2)$*-symplectic. In particular, $(T\mathbb{S}^n, G^c, P) \cong (\mathbb{P}_n(\mathbb{A}), g, J)$ is* $(1,2)$*-symplectic. Moreover any para-Kaehler manifold is* $(1,2)$*-symplectic. Here h^c and P are the structures given by* (1.1) *and* (1.2) *respectively.*

Proof. For any (semi-) Riemannian manifold (M, h) the manifold (TM, h^c, P) is $(1,2)$-symplectic (cf. [2], Proposition 4.1). Then the result follows. □

2. Construction of harmonic maps into $\mathbb{P}_n(\mathbb{A})$

Let $\phi_r : (M, h) \to (S^n_r, G_r)$ be a smooth map of a (semi-) Riemannian manifold M of any signature into the unit pseudo-sphere with its standard metric G_r (which is of index $(r, n-r)$) and $r = 0, 1, \ldots, n$. Then we have:

Proposition 2.1. *If ϕ_r is harmonic then:*

(i) *The composition $\overline{\varphi}_r \circ \phi_r : (M, h) \to (\mathbb{P}_n(\mathbb{A}), g)$ is harmonic.*

(ii) *The map $d\phi_0 = \Phi : (TM, h^c, P') \to (\mathbb{P}_n(\mathbb{A}), G^c, P)$ is harmonic. Moreover, $\phi_\circ$ is totally geodesic if and only if Φ is totally geodesic if and only if Φ is paraholomorphic.*

Proof. Since $\overline{\varphi}_r$ a is totally geodesic map, (i) follows. For (ii), since $\phi_\circ$ is harmonic by the assumption, Φ is harmonic when h is Riemannian [24]. But also Φ remains harmonic when h is semi-Riemannian as one can see by combining the results in [24], pp.134–135; [20], p.247; [27], p.106, Proposition 6.3, p.100; and [28], Proposition 7.5. The last part of (ii) follows directly from the result in [2]. □

Corollary 2.2. *For (M, h) with $\dim M = 1$ and $r = 0, 1, \ldots, n$ we have*

i) *$\phi_r : (M, h) \to (S^n_r, G_r)$ is harmonic (and therefore it is totally geodesic) if and only if $\overline{\varphi}_r \circ \phi_r : (M, h) \to (\mathbb{P}_n(\mathbb{A}), g)$ is totally geodesic,*

ii) $\phi = \phi_\circ : (M,h) \to (\mathbb{S}^n, G)$ *is harmonic if and only if* Φ *is totally geodesic if and only if* Φ *is paraholomorphic.* $\square$

Note that for $M = \mathbb{S}^1$, we have $(T\mathbb{S}^1, h^c, P') \cong (\mathbb{P}_1(\mathbb{A}), h^c, P')$ which is locally diffeomorphic to $S_1^2 = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : -x_1^2 + x_2^2 + x_3^2 = 1\} \cong \mathbb{S}^1 \times \mathbb{R}$. So any harmonic (and therefore totally geodesic) map $\phi : (\mathbb{S}^1, h) \to (\mathbb{S}^n, G)$ gives rise to a totally geodesic and therefore a paraholomorphic map $\Phi = d\phi : (\mathbb{P}_1(\mathbb{A}), h^c, P') \to (\mathbb{P}_n(\mathbb{A}), G^c, P)$.

Proposition 2.3. *Let* $\psi_1 : (TM, h^c, P') \to (\mathbb{P}_n(\mathbb{A}), G^c, P)$ *and* $\psi_2 : (TM, h^c, P') \to (\mathbb{P}_n(\mathbb{A}), g, J)$ *be paraholomorphic maps, where* (M,h) *is a (semi-) Riemannian manifold and* h *is of any signature. Then* ψ_1 *and* ψ_2 *are harmonic.*

Proof. Since the manifolds involved are all $(1,2)$-symplectic and any paraholomorphic map between $(1,2)$-symplectic almost para-Hermitian manifolds is harmonic [2], the harmonicity of ψ_1 and ψ_2 easily follows. $\square$

Notation. For $z = x + \varepsilon y \in \mathbb{A}$ set $\delta' = \delta_z = \frac{1}{2}(\partial_x + \varepsilon\partial_y)$, $\delta'' = \delta_{\overline{z}} = \frac{1}{2}(\partial_x - \varepsilon\partial_y)$ and for $z = x + iy \in \mathbb{C}$ set $\partial' = \partial_z = \frac{1}{2}(\partial_x - i\partial_y), \partial'' = \partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y)$, where $\partial_x = \frac{\partial}{\partial x}$ etc. Also we reserve the letters $\mathbb{L} = (\mathbb{L}, h, Q)$ for a Lorentz surface and $\mathbb{M} = (\mathbb{M}, h', Q')$ for a Riemann surface. Clearly h is an indefinite metric and Q is a paracomplex structure on $\mathbb{L}$ while h' is positive definite and Q' is a complex structure on $\mathbb{M}$.

For $\theta \in \{i, \varepsilon\}$ let $\xi_\theta : M_\theta \to C_\theta^{2n+1}$ be a smooth map, where $M_i = \mathbb{M}$ and $M_\varepsilon = \mathbb{L}$.

Definition 2.4. *For* $r = 0, 1, \ldots, 2n$

i) *the map* $\xi = \xi_\varepsilon$ *is said to be totally* $\langle,\rangle_r^A$*-paraisotropic if* $\langle \delta'^k\xi, \delta'^k\xi\rangle_r^A = 0$ *for* $0 \le k \le n-1$,

ii) *the map* $\xi = \xi_i$ *is said to be totally* $\langle,\rangle_r^C$*-isotropic if* $\langle \partial'^k\xi, \partial'^k\xi\rangle_r^C = 0$ *for* $0 \le k \le n-1$.

The total $\langle,\rangle_r^{\overline{A}}$-paraisotropy and $\langle,\rangle_r^{\overline{c}}$-isotropy are defined in the same way.

We say that a smooth map $f : M_\theta \to \mathbb{P}_{2n}(C_\theta)$ *is totally* $\langle,\rangle_r^{C_\theta}$*-isotropic (resp.* $\langle,\rangle_r^{\overline{C}_\theta}$*-isotropic) if* f *has a local lift* $\xi : U \subseteq M_\theta \to C_\theta^{2n+1}$ *which is* $\langle,\rangle_r^{C_\theta}$*-isotropic (resp.* $\langle,\rangle_r^{\overline{C}_\theta}$*-isotropic).*

By the same argument that was used for the complex case [25], one can show that a smooth (resp. a paraholomorphic) map $f : \mathbb{L} \to \mathbb{P}_{2n}(\mathbb{A})$ has a smooth (resp. paraholomorphic) local lift $\xi : U \subseteq \mathbb{L} \to \mathbb{A}^{2n+1}$ over a small enough open set U in $\mathbb{L}$. It is not difficult to see that the concept of paraisotropy is well-defined in all cases, i.e. it does not depend on the local lift and h-isothermal coordinates choosen.

Remark. For $r = 0$, there are no nonconstant maps $\xi : M_\theta \to C_\theta^{2n+1}$ with $\langle,\rangle_0^{\overline{C}}$-isotropy or $\langle,\rangle_0^A$-paraisotropy and $f : \mathbb{L} \to \mathbb{P}_{2n}(\mathbb{A})$ with $\langle,\rangle_0^{\overline{A}}$-paraisotropy, since $\langle\xi,\xi\rangle_0^{\overline{C}} = 0 = \langle\xi,\xi\rangle_0^A$ implies that $\xi = 0$, and $f(p) \in \mathbb{P}_{2n}(\mathbb{A})$ implies that $\langle f(p), f(p)\rangle_0^{\overline{A}} > 0$. Also note that for a para-Riemann surface (resp. Riemann surface) E, a local lift $\xi : U(\subseteq E) \to \mathbb{A}^N$ (resp. $\xi : U(\subseteq E) \to \mathbb{C}^N$) is paraholomorphic (resp. holomorphic) if and only if $\delta''\xi = 0$ (resp. $\partial''\xi = 0$).

Now for a fixed value of $r = 0, 1, \dots, 2n$ suppose $f : (\mathbb{L}, h, Q) \to (\mathbb{P}_{2n}(\mathbb{A}), g, J)$ is a totally $\langle , \rangle_r^{\overline{A}}$-paraisotropic (resp. totally $\langle , \rangle_r^{A}$-paraisotropic) paraholomorphic map such that for every $p \in \mathbb{L}$ there is a local paraholomorphic lift ξ and h-isothermal coordinates (x, y) on an open set U containing p on which the set $V = V(\xi) = \{\xi, \delta'\xi, \dots, \delta'^{n-1}\xi\}$ is linearly independent and the symmetric paracomplex form $\langle , \rangle_r^{A}$ (resp. the para-Hermitian form $\langle , \rangle_r^{\overline{A}}$) is nondegenerate on $span_{\mathbb{A}} V \subseteq \mathbb{A}^{2n+1}$.

For $\sigma(p) = (\xi \wedge \delta'\xi \wedge \dots \wedge \delta'^{n-1}\xi)(p) \in \bigwedge^n \mathbb{A}^{2n+1}$, the n-th exterior power of $\mathbb{A}^{2n+1}$, set $\alpha(p) = \sigma(p) \wedge \overline{\sigma(p)} = (\xi \wedge \dots \wedge \delta'^{n-1}\xi \wedge \overline{\xi} \wedge \dots \wedge \delta''^{n-1}\overline{\xi})(p)$ and $\tilde{\alpha} = \sim \circ \alpha$, where

$$\sim \Big(\sum_{k=1}^{2n+1} \lambda_k \big(\bigwedge_{j \in \Omega_k} e_j \big) \Big) = \sum_{k=1}^{2n+1} (-1)^{k+1} \lambda_k e_k$$

and $\{e_j\}$ is the standard basis for $\mathbb{R}^{2n+1} \subseteq \mathbb{A}^{2n+1}$ and $\Omega_k = \{1, 2, \dots, 2n+1\} \setminus \{k\}$.

Lemma 2.5. *The map $\tilde{\alpha} : U \subseteq \mathbb{L} \to \mathbb{A}^{2n+1}$ never vanishes, i.e. $\tilde{\alpha}(q) \neq 0$, $\forall q \in U$.*

Proof. For $V = V(\xi) = \{\xi, \dots, \delta'^{n-1}\xi\}$, note that the set $V \cup \overline{V}$ forms a linearly independent set since $V \cap \overline{V}$ is empty. Indeed, let $v \in V \cap \overline{V}$, then $\langle v, \overline{w} \rangle_r^{\overline{A}} = \langle v, w \rangle_r^{A} = 0$, $\forall w \in V$ by the $\langle , \rangle_r^{\overline{A}}$-paraisotropy. But then the nondegeneracy of $\langle , \rangle_r^{A}$ on V gives that $V = 0$. (The arguments for the case where $\langle , \rangle_r^{\overline{A}}$-paraisotropy and $\langle , \rangle_r^{A}$-nondegeneracy are involved are still valid when the $\langle , \rangle_r^{A}$-paraisotropy and $\langle , \rangle_r^{\overline{A}}$-nondegeneracy involve instead.) So $span_{\mathbb{A}}(V(q) \cup \overline{V(q)})$ is a paracomplex subspace (i.e. a free submodule) of $\mathbb{A}^{2n+1}$. By Theorem 7 in [13], p.171, we deduce that $\alpha(q) \neq 0$, and therefore $\tilde{\alpha}(q) \neq 0$, $\forall q \in U$. □

Observe that

i) $\overline{\tilde{\alpha}} = (-1)^n \tilde{\alpha}$, so $\tilde{\alpha}$ is pure imaginary for an odd n and real for an even n. Thus we write $\tilde{\alpha}$ for $\varepsilon\tilde{\alpha}$ when n is even so that $\tilde{\alpha}$ denotes the real section of $\underline{\mathbb{A}}^{2n+1} = U \times A^{2n+1}$ for even and odd n.

ii) $\langle \tilde{\alpha}, \delta'^k \xi \rangle_0^{\overline{A}} = 0 = \langle \tilde{\alpha}, \overline{\delta'^k \xi} \rangle_0^{\overline{A}}$, $\forall k = 0, \dots, n-1$.

Put $F = \dfrac{\tilde{\alpha}}{\mu}$ at points where $\mu = |\langle \tilde{\alpha}, \tilde{\alpha} \rangle_r^{\overline{A}}|^{1/2} \neq 0$. Then

Lemma 2.6. *The map F does not depend on the particular choices of a local lift and h-isothermal coordinates.*

Proof. Let $z = x + \varepsilon y \cong (x, y)$ and $w = u + \varepsilon v$ be two local h-isothermal coordinates on $U \subseteq \mathbb{L}$. Then $\delta_w^k \xi = (\delta_z^k \xi)(\delta_w z)^k +$ terms in $\delta_z^s \xi$ with $s < k$, so $\alpha(w) = \alpha(z) ||\delta_w z||^{2(1+2+\dots+(n-1))}$, where $||\delta_w z|| = (\delta_w z)\overline{(\delta_w z)}$.

Claim. $||\delta_w z|| \neq 0$ on U.

Indeed, note that $z = z(w)$ is paraholomorphic, so for $z = x(u, v) + \varepsilon y(u, v)$ with $w = u + \varepsilon v$ we have $x_u = \dfrac{\partial x}{\partial u} = y_v$ and $x_v = y_u$. But then $\delta_w z = (x_u + y_v) + \varepsilon(x_v + y_u) = 2(x_u + \varepsilon x_v)$, so $||\delta_w z||^2 = 4(x_u^2 - x_v^2)$. On the other hand $\dfrac{\partial}{\partial u} = x_u \dfrac{\partial}{\partial x} + y_u \dfrac{\partial}{\partial y} = x_u \dfrac{\partial}{\partial x} + x_v \dfrac{\partial}{\partial y}$ and then $h(\dfrac{\partial}{\partial u}, \dfrac{\partial}{\partial u}) = (x_u^2 - x_v^2) h(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial x}) \neq 0$. This gives $x_u^2 - x_v^2 \neq 0$ which proves the claim.

By the claim we see that $F_w = \dfrac{\tilde{\alpha}(w)}{\mu_w} = \dfrac{\tilde{\alpha}(z)}{\mu_z} = F_z$. Next, let ξ and η be two local paraholomorphic lifts of f, then $\xi = \lambda\eta$ for some $\lambda : \mathbb{L} \to \mathbb{A}$ with $\lambda\overline{\lambda} > 0$. So $\alpha_\xi = (\lambda\overline{\lambda})^n \alpha_\eta$ and consequently $F_\xi = F_\eta$. □

Now observe that for $V = V(\xi)(q) = \{\xi(q), \ldots, \delta'^{n-1}\xi(q)\}$ the paracomplex subspace $W(q) = span_{\mathbb{A}}(V(q) \cup \overline{V(q)})$ has a basis $\{w_1, \ldots, w_{2n}\}$ with $w_t \in \mathbb{R}^{2n+1}, t = 1, 2, \ldots, 2n$ since $\overline{W} = W$. Set $W^R(q) = span_{\mathbb{R}}\{w_1, \ldots, w_{2n}\}$ and let $\langle , \rangle_r$ denote the metric which is the restriction of $\langle , \rangle_r^A$ (or equivalently $\langle , \rangle_r^{\overline{A}}$) on $\mathbb{R}^{2n+1} \subseteq \mathbb{A}^{2n+1}$.

Further suppose that f is such that $\langle , \rangle_r$ is of signature $(r, 2n - r)$ on W^R. As this signature does not depend on the particular choices of charts and local lifts, one can see that $\mu \neq 0$ on the whole of $\mathbb{L}$ and therefore $F = \frac{\tilde{\alpha}}{\mu}$ defines a map

$$F : \mathbb{L} \to S_r^{2n} \subseteq \mathbb{R}^{2n+1}.$$

from the whole of the Lorentz surface into a pseudo-sphere. Define an endomorphism $\rho_r : \mathbb{A}^N \to \mathbb{A}^N$ by $\rho(Z) = \rho_r(Z) = (-z_1, \ldots, -z_r, z_{r+1}, \ldots, z_N)$ for $Z = (z_k) \in \mathbb{A}^N$.
Then we have for $r = 0, 1, \ldots, N - 1$:

i) $\langle Z, W\rangle_r^A = \langle \rho_r(Z), W\rangle_0^A = \langle Z, \rho_r(W)\rangle_0^A$.

ii) $\langle \rho_r(Z), \rho_r(W)\rangle_r^A = \langle Z, W\rangle_r^A$.

iii) For a smooth map $f : M_\theta \to \mathbb{C}_\theta^N, \quad \theta = i, \varepsilon$

$$\rho_r(\partial'^k f) = \partial'^k \rho_r(f) \text{ and } \rho_r(\delta'^k f) = \delta'^k(\rho_r(f))$$

where $M_i = \mathbb{M}$ and $M_\varepsilon = \mathbb{L}$. These relations are valid also for the operators δ''^k and ∂''^k.

Note that $\rho_r(F) \notin W = span_{\mathbb{A}}(V \cup \overline{V})$, so one can write $\delta'^n\xi = v + w + s(\rho_r(F))$ where $v \in span_{\mathbb{A}}(V)$, $w \in span_{\mathbb{A}}(\overline{V})$ and $s(q) \in \mathbb{A}$. We have

$$\langle \delta'^n\xi, \delta'^k\xi\rangle_r^{\overline{A}} = \langle v, \delta'^k\xi\rangle_r^{\overline{A}} + \langle w, \delta'^k\xi\rangle_r^{\overline{A}} + s\langle \rho_r(F), \delta'^k\xi\rangle_r^{\overline{A}}.$$

Since $\langle v, \delta'^k\xi\rangle_r^{\overline{A}} = \langle \delta'^n\xi, \delta'^k\xi\rangle_r^{\overline{A}} = 0, \ \forall k = 0, \ldots, n - 1$ by the $\langle , \rangle_r^{\overline{A}}$-paraisotropy and $\langle \rho_r(F), \delta'^k\xi\rangle_r^{\overline{A}} = \langle F, \delta'^k\xi\rangle_0^{\overline{A}} = 0$ we have $\langle w, \delta'^k\xi\rangle_r^{\overline{A}} = \langle w, \delta''^k\overline{\xi}\rangle_r^A = 0$. But then by the nondegeneracy of $\langle , \rangle_r^A$ on V and therefore on $\overline{V}$, this gives that $w = 0$ so that

$$\delta'^n\xi = v + \langle \delta'^n\xi, \rho_r(F)\rangle_r^A \rho_r(F). \tag{2.1}$$

Remark. For the case where $\langle , \rangle_r^A$-paraisotropy and $\langle , \rangle_r^{\overline{A}}$-nondegeneracy involve instead, we still get the same formula (2.1).

As we sum up: Let $f : (\mathbb{L}, h, Q) \to (\mathbb{P}_{2n}(\mathbb{A}), g, J)$ be a paraholomorphic totally $\langle , \rangle_r^{\overline{A}}$-paraisotropic (resp. totally $\langle , \rangle_r^A$-paraisotropic) map such that for every $p \in \mathbb{L}$ there is a local paraholomorphic lift ξ and h-isothermal coordinates (x, y) on an open set U containing p on which the set $V = V(\xi) = \{\xi, \ldots, \delta'^{n-1}\xi\}$ is linearly independent and the symmetric paracomplex form $\langle , \rangle_r^A$ (resp. para-Hermitian form $\langle , \rangle_r^{\overline{A}}$) is nondegenerate on $span_{\mathbb{A}} V \subseteq \mathbb{A}^{2n+1}$; and is of signature $(r, 2n-r)$ on $W^R \subseteq W = span_{\mathbb{A}}(V \cup \overline{V})$.

Remark. The symmetric paracomplex form $\langle , \rangle_0^A$ is always nondegenerate on any subspace of $\mathbb{A}^{2n+1}$ and it is always of signature $(0, 2n)$ on W^R. Therefore we drop the assumptions of nondegeneracy of $\langle , \rangle_0^A$ on $span_{\mathbb{A}} V$ and the signature of being $(0, 2n)$.

The map f determines the maps

$$F : (\mathbb{L}, h, Q) \to S_r^{2n} \subseteq \mathbb{R}^{2n+1}$$

into the pseudo-sphere with its standard semi-Riemannian metric of signature $(r, 2n-r)$ and

$$K : U \subseteq (\mathbb{L}, h, Q) \to \mathbb{A}$$

given by

$$K = \langle \delta'^n \xi, F \rangle_0^A = \langle \delta'^n \xi, \rho_r(F) \rangle_r^A = det[\xi, \delta'\xi, \ldots, \delta'^n \xi, \overline{\xi}, \ldots, \overline{\delta'^{(n-1)}\xi}],$$

the determinant of a matrix whose columns formed by the indicated sections $\delta'^k \xi$ and $\overline{\delta'^k \xi}$ of $\underline{\mathbb{A}}^{2n+1}$. Note that unlike F, the map K is defined locally and depends on the local lift ξ and h-isothermal coordinates choosen. Nevertheless its paraholomorphicity does not depend on the particular choice of ξ and h.

Proposition 2.7. *Let $f : (\mathbb{L}, h, Q) \to (\mathbb{P}_{2n}(\mathbb{A}), g, J)$, F and K be the maps described as above. Then F is harmonic if K is paraholomorphic.*

Proof. One can show that F is harmonic if and only if $\delta'\delta'' F = \lambda F$ for some smooth map $\lambda : \mathbb{L} \to \mathbb{R}$ (cf. [6], [7]). In doing that it is enough to show

$$\langle \delta'\delta'' F, \delta'^k \xi \rangle_0^A = 0, \ \forall k = 0, 1, \ldots, n-1.$$

Note that

$$\langle \delta'\delta'' F, \delta'^k \xi \rangle_0^A = \delta''(\delta' \langle F, \delta'^k \xi \rangle_0^A - \langle F, \delta'^{k+1} \xi \rangle_0^A) = 0, \ \forall k = 0, \ldots, n-2.$$

Thus $0 = \langle \delta'\delta'' F, \delta'^{n-1} \xi \rangle_0^A = -\delta'' \langle F, \delta'^n \xi \rangle_0^A = -\delta'' K$ which completes the proof. □

Corollary 2.8. *When K is paraholomorphic then the maps*
1°) $\overline{\varphi}_r \circ F : (\mathbb{L}, h) \xrightarrow{F} S_r^{2n} \xrightarrow{\overline{\varphi}_r} (\mathbb{P}_{2n}(\mathbb{A}), g)$,
2°) $dF : (T\mathbb{L}, h^c) \to (TS_0^{2n}, G^c) \simeq (\mathbb{P}_{2n}(\mathbb{A}), G^c)$
are harmonic.

Lemma 2.9. *Let f be as above with $\langle , \rangle_r^A$ paraisotropy and for $K = a + \varepsilon b$, set $u = \dfrac{\partial a}{\partial x} + \dfrac{\partial b}{\partial y}$, $v = \dfrac{\partial a}{\partial y} + \dfrac{\partial b}{\partial x}$ so that $\delta'' K = u + \varepsilon v$. Then either $u = v$ or $u = -v$. Further if $u(q) \neq 0$ for some q then $K(q)$ is either zero or a zero divisor.*

Proof. By the remark before Proposition 2.7 we have $\delta'^n\xi = v + K\rho_r(F)$. So

$$\langle \delta'^n\xi, \delta'^n\xi\rangle_r^A = \langle v, v\rangle_r^A + 2K\langle v, \rho_r(F)\rangle_r^A + K^2\langle \rho_r(F), \rho_r(F)\rangle_r^A = K^2$$

since $\langle v, v\rangle_r^A = 0$ by the $\langle,\rangle_r^A$-paraisotropy and $\langle v, \rho_r(F)\rangle_r^A = \langle v, F\rangle_r^A = 0$ by the construction and $\langle \rho_r(F), \rho_r(F)\rangle_r^A = \langle F, F\rangle_r^A = 1$. Thus $\delta''\langle \delta'^n\xi, \delta'^n\xi\rangle_r^A = \delta'' K^2 = 2K(\delta'' K) = 0$. Hence the result follows. □

Corollary 2.10. *Let f, K, u be as in Lemma 2.9. If $K(q)$ is not a zero divisor for all q or $u(q) = 0$ whenever $K(q)$ is a zero divisor then $\delta'' K = 0$, that is, K is paraholomorphic.*

Proposition 2.11. *Let f, $\overline{\varphi_r} \circ F$, K, u be the maps described earlier with f being totally $\langle,\rangle_r^A$-paraisotropic and $K(q)$ not be a zero divisor for all q or $u(q) = 0$ whenever $K(q)$ is a zero divisor for $r = 1, 2, \ldots, 2n$. Then the map*

$$\overline{\varphi}_r \circ F : (\mathbb{L}, h) \to (\mathbb{P}_{2n}(\mathbb{A}), g)$$

is harmonic.

Proof. It follows from Proposition 2.7 and Corollary 2.10. □

Remark. For a map $f : \mathbb{L} \to \mathbb{P}_{2n}(\mathbb{A})$, as there is no local lift $\xi_f : U \subseteq \mathbb{L} \to \mathbb{A}^{2n+1}$ of f with $\langle,\rangle_0^{\overline{A}}$-paraisotropy, we start with any paraholomorphic map $\xi : \mathbb{L} \to \mathbb{A}^{2n+1}$ with $\langle,\rangle_0^{\overline{A}}$-paraisotropy in order to construct $F : \mathbb{L} \to S_r^{2n}$, a candidate for a harmonic one.

Let $f : \mathbb{M} \to (\mathbb{P}_{2n}(\mathbb{C}), g)$ be a map of a Riemann surface into complex projective spaces with the properties that f is totally $\langle,\rangle_r^C$-isotropic, holomorphic and the Hermitian form $\langle,\rangle_r^{\overline{c}}$ is nondegenerate on $span_{\mathbb{C}} V$ and is of signature $(r, 2n-r)$ on $W^R \equiv W = span_{\mathbb{C}}(V \cup \overline{V})$. We see that the map $K : U(\subseteq M) \to \mathbb{C}$ is holomorphic and therefore $F : \mathbb{M} \to S_r^{2n}$ is harmonic [6].

Proposition 2.12. *Let f be just as above. Then the maps*

$$\overline{\varphi}_r \circ F : (\mathbb{M}, h) \to (\mathbb{P}_{2n}(\mathbb{A}), g), \;\; r = 0, 1, \ldots, 2n; \quad \textit{and}$$

$$dF : (T\mathbb{M}, h^c) \to (TS_0^{2n}, G^c) \cong (\mathbb{P}_{2n}, G^c)$$

are harmonic. □

Examples. I) For the symmetric bilinear form $\langle Z, W\rangle_1^A = -z_1w_1 + z_2w_2 + z_3w_3 = \ell(Z, W)$, $Z = (z_i)$, $W = (w_i) \in \mathbb{A}^3$, let $\{u_1, u_2, u\} \subseteq \mathbb{R}^3 \subseteq \mathbb{A}^3$ be an ℓ-orthonormal basis for $\mathbb{R}^3$ with $\ell(u_1, u_1) = -1$ and $\ell(u_2, u_2) = \ell(u, u) = 1$. Then set $E = k(u_2 + \varepsilon u_1)$ so that $\overline{E} = k(u_2 - \varepsilon u_1)$, $k = \frac{1}{\sqrt{2}}$. We see that $\ell(E, E) = \ell(\overline{E}, \overline{E}) = 0$ and $\ell(E, \overline{E}) = 1$. Now define a map $\xi : \mathbb{L} \to \mathbb{A}^3$ by

$$\xi(z) = -\frac{1}{2}E + e^{mz}u + e^{2mz}\overline{E},$$

where m is a nonzero real number. Note that

1°) $e^z = e^{x+\varepsilon y} = e^x(\cosh y + \varepsilon \sinh y)$.

2°) $\ell(\xi, \overline{\xi}) = \frac{1}{4} + e^{2mx} + e^{4mx} > 0$, so $\xi(z) \in \mathbb{A}_0^3$ and $\delta''\xi = 0$, i.e. ξ is paraholomorphic. So $f = \pi \circ \xi : (\mathbb{L}, h, Q) \to (\mathbb{P}_2(\mathbb{A}), g, J)$ is paraholomorphic and therefore harmonic.

3°) $\ell(\xi, \xi) = 0$, so ξ is totally ℓ-paraisotropic. Further $V = \{\xi\}$ is linearly independent and the para-Hermitian form $\overline{\ell} = \langle , \rangle_1^{\overline{A}}$ is nondegenerate on $span_{\mathbb{A}} V$.

4°) Set $w_1 = \xi + \overline{\xi}$, $w_2 = \varepsilon(\xi - \overline{\xi})$ and observe that $W^R = span_{\mathbb{R}}\{w_1, w_2\} \subseteq \mathbb{R}^3$. Further, since

$$\overline{\ell}(w_1, w_1) = \ell(\xi + \overline{\xi}, \xi + \overline{\xi}) = 2\ell(\xi, \overline{\xi}) = 2(\frac{1}{4} + e^{2mx} + e^{4mx}) > 0,$$

$$\overline{\ell}(w_2, w_2) = -2\overline{\ell}(\xi, \overline{\xi}) < 0 \quad \text{and} \quad \overline{\ell}(w_1, w_2) = 0,$$

we see that $\overline{\ell}$ is of signature $(1,1)$ on W^R.

5°) Set $\tilde{\alpha} =\sim (\xi \wedge \overline{\xi}) \in \mathbb{A}^3$ and $F = \dfrac{\tilde{\alpha}}{\mu}$ with $\mu = [\ell(\tilde{\alpha}, \overline{\tilde{\alpha}})]^{1/2}$. Then $F : (\mathbb{L}, h) \to S_1^2$ is defined on the whole of $\mathbb{L}$.

6°) $$K = det[\xi, \overline{\xi}, \xi'] = det \begin{bmatrix} -1/2 & e^{2mz} & e^{mz} \\ e^{2m\overline{z}} & -1/2 & e^{m\overline{z}} \\ 0 & 2me^{2mz} & me^{mz} \end{bmatrix} = me^{mz}(e^{2mx} + e^{4mx} + 1/4).$$

So

$$K\overline{K} = m^2(e^{2mx} + e^{4mx} + 1/4)^2 e^{2mx} \neq 0.$$

Thus $K(z)$ is not a zero divisor for all z and hence by Proposition 2.7 and Corollary 2.10, F is harmonic. By Proposition 2.11 the map $\overline{\varphi}_1 \circ F : (\mathbb{L}, h) \to (\mathbb{P}_2(\mathbb{A}), g)$ is harmonic.

II) Let $\xi = \xi(z) = ((1+\varepsilon)e^{m(x+y)}, e^{mz}, \varepsilon e^{mz})$, with $z = x + \varepsilon y$, where m is a nonzero real number. Then note that

1°) $\xi : (\mathbb{L}, h, Q) \to \mathbb{A}^3$ is a paraholomorphic map.

2°) Denoting $\vartheta = \langle , \rangle_0^{\overline{A}}$, the para-Hermitian form, we have

$$\vartheta(\xi, \xi) = \langle \xi, \xi \rangle_0^{\overline{A}} = \langle \xi, \overline{\xi} \rangle_0^{A} = (1 \pm \varepsilon)(1 \mp \varepsilon)e^{2k(x+y)} + e^{2mx} - e^{2mx} = 0,$$

so ξ is totally ϑ-paraisotropic. Further, $\{\xi, \overline{\xi}\}$ is a linearly independent set.

By the remark before Proposition 2.7 it is enough to check the paraholomorphicity of $K = \xi \wedge \overline{\xi} \wedge \delta'\xi$. But, $K \equiv 0$ since ξ and $\delta'\xi$ are linearly dependent. So the maps $F = \frac{\tilde{\alpha}}{\mu}$: $(\mathbb{L}, h) \to \mathbb{S}^2 = S_0^2$ and $\overline{\varphi}_0 \circ F : (\mathbb{L}, h) \to \mathbb{P}_2(\mathbb{A})$, (the real projective model), are harmonic. Thus the map $dF : (T\mathbb{L}, h^c) \to (\mathbb{P}_2(\mathbb{A}), G^c)$ is harmonic too.

III) See [1], p.101, for the examples of holomorphic maps $f : (\mathbb{M}, h) \to (\mathbb{P}_{2n}(\mathbb{C}), g, J)$ satisfying the properties stated just before Lemma 2.12, for $r = 0$. This f gives rise to following harmonic maps:

$$F : (\mathbb{M}, h) \to (\mathbb{P}_{2n}(\mathbb{R}), g) \quad \text{and} \quad dF : (T\mathbb{M}, h^c) \to (\mathbb{P}_{2n}(\mathbb{A}), G^c).$$

IV) Consider $Z = (z_i) \in \mathbb{A}^{m+1}$ as $(m+1)\times 1$ matrix and put $Z^{\#} = \overline{Z}^t$, the transpose of $\overline{Z} = (\overline{z}_i)$. We then set $p(Z) = ZZ^{\#}/||Z||^2$, where $||Z||^2 = \langle Z, Z\rangle_0^{\overline{A}} = \sum_{i=1}^{m+1} z_i\overline{z}_i$ for a non-isotropic vector Z, i.e. $||Z|| \neq 0$. Note that $p(Z)$ is a projector, i.e. $p(Z)^2 = p(Z)^{\#} = p(Z)$ with $1 = Tr(p(Z))$, the trace of $p(Z)$.

Lemma. (cf. [18]) *The set of all projectors $p(Z)$ can be identified with the nonisotropic part of the straight line $span_{\mathbb{A}}\{Z\}$ which is diffeomorphic to $T(\mathbb{P}_m(\mathbb{R}))$, the tangent bundle of the real projective space.*

For a nonisotropic paraholomorphic map $W : \mathbb{A} \to \mathbb{A}^m$, define a map

$$B(z) = \frac{1}{1+||W(z)||^2}\begin{bmatrix} 1 & W(z)^{\#} \\ W(z) & W(z)W(z)^{\#} \end{bmatrix}.$$

Then $B(z)$ is a projector and therefore, by the above Lemma, it may be viewed as a map into the paracomplex projective model $(\mathbb{P}_m(\mathbb{A}), g)$. This map $B : (\mathbb{A}, \langle,\rangle^{\overline{A}}) \to (\mathbb{P}_m(\mathbb{A}), g)$ is in fact harmonic (see [18]).

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