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Systems of Stochastically Independent and Normally Distributed Random Points in the Euclidean Space E_3

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Abstract. We consider sets of three or four stochastically independent and normally distributed random points in E_3. We look at these points as the vertices of a triangle $\mathcal{T}_3$ or of a tetrahedron $\mathcal{T}_4$ and we study the random variables $\mathcal{A}$, area of $\mathcal{T}_3$, and $\mathcal{V}$, volume of $\mathcal{T}_4$. We also determine the probability that $\mathcal{T}_3$ is acute-angled.

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Random sets appear very early in probability theory, in the context of geometrical probability. The random segments (in Buffon's terms "needles") in Buffon's experiment are one of the first examples of how the idea of randomness can be applied to elements that are geometric objects: a geometric element, of *fixed shape and dimensions*, can have a *"random position"* in the plane or in the space. The notion of random sets appears also in the famous book of Kolmogorov [1], where the mean area of a bounded random set is calculated by means of the Fubini theorem. Unlike the previously quoted Buffon's problem, here the idea of randomness is associated to the shape of the studied object. In Kolmogorov's own words: "We shall give an application of this theorem (i.e. the Fubini theorem) to one example in geometric probability. Let G be a measurable region of the plane whose *shape depends on chance*; in other words, let us assign to every elementary event ξ of a field of probability a definite measurable region G. We shall denote by J the area of the region G, and by $p\,(x,y)$ the probability that the point (x,y) belongs to the region G. Then $E\,(J) = \iint p\,(x,y)\,dxdy$."

In this paper we shall be concerned with this latter notion of randomness. More precisely, we consider systems of three and four points, that are supposed to be stochastically independent and normally distributed in the Euclidean space E_3, and we look at these points as the vertices of a triangle $\mathcal{T}_3$ and of a tetrahedron $\mathcal{T}_4$. Obviously the shapes of $\mathcal{T}_3$ and $\mathcal{T}_4$ depend on the random positions of their vertices, and the area of $\mathcal{T}_3$ and the volume of $\mathcal{T}_4$ are the most natural random variables that we can study in order to get some information about these "regions depending on chance".

As important application fields we just mention geometrical statistics and image analysis, materials sciences, fuzzy data analysis and particle statistics.

1. On a random triangle with normally and independently distributed vertices

Let $\{P_1, P_2, P_3\}$ be independent and identically distributed random points in the Euclidean space E_3, with common distribution

$$f(x,y,z) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{x^2+y^2+z^2}{2}} \tag{1}$$

where $(x,y,z) \in \mathbb{R}^3$ are the Cartesian coordinates in E_3. We denote by $\mathcal{T}_3$ the triangle of vertices $P_i \equiv (x_i, y_i, z_i)$, $i = 1,2,3$, and by $\mathcal{A}$ the area of $\mathcal{T}_3$.

If we define the set $\Delta_{\mathcal{T}_3}$ as

$$\Delta_{\mathcal{T}_3} = \{P_3 \in E_3 : \quad \mathcal{A} = area(\mathcal{T}_3) < \alpha\}$$

then the distribution function of $\mathcal{A}$ can be written:

$$\begin{aligned} F(\alpha) &= P\{\mathcal{A} < \alpha\} \\ &= \frac{1}{(2\pi)^{\frac{9}{2}}} \int\limits_{\mathbb{R}^3 \times \mathbb{R}^3} e^{-\frac{\sum_{i=1}^{2}\left(x_i^2+y_i^2+z_i^2\right)}{2}} dP_1 dP_2 \int\limits_{\Delta_{\mathcal{T}_3}} e^{-\frac{x_3^2+y_3^2+z_3^2}{2}} dP_3 \end{aligned}$$

where $dP_i = dx_i \wedge dy_i \wedge dz_i$, $i = 1,2,3$. From now on we choose $\alpha > 0$, as for $\alpha \leq 0$ we obviously have $F(\alpha) \equiv 0$.

In order to compute $F(\alpha)$ we consider the following change of variables:

$$\begin{cases} x_i = p\cos\varphi\sin\vartheta + q_i\cos\varphi\cos\vartheta + r_i\sin\varphi, & \vartheta \in [0,\pi], \ \varphi \in [0,2\pi] \\ y_i = p\sin\varphi\sin\vartheta + q_i\sin\varphi\cos\vartheta - r_i\cos\varphi, & q_i, r_i \in \mathbb{R}, \ i = 1,2,3 \\ z_i = p\cos\vartheta - q_i\sin\vartheta, & p \in \mathbb{R}^+ \end{cases} \tag{2}$$

with

$$\begin{aligned} |J| &= \left| \frac{D(x_1,y_1,z_1,x_2,y_2,z_2,x_3,y_3,z_3)}{D(p,q_1,q_2,q_3,r_1,r_2,r_3,\vartheta,\varphi)} \right| \\ &= \left| \sum_{i \neq j = 1}^{2} (-1)^i (q_i - q_3)(r_j - r_3) \right| \sin\vartheta \end{aligned} \tag{3}$$

so that

$$
\begin{aligned}
F(\alpha) &= \frac{1}{(2\pi)^{\frac{9}{2}}} \int_0^{\pi} \sin\vartheta d\vartheta \int_0^{2\pi} d\varphi \int_0^{+\infty} e^{-\frac{3}{2}p^2} dp \\
&\quad \cdot \int_{\Delta'_{\mathcal{T}_3}} \left| \sum_{i \neq j = 1}^{2} (-1)^i (q_i - q_3)(r_j - r_3) \right| e^{-\frac{\sum_{i=1}^{3}\left(q_i^2 + r_i^2\right)}{2}} d\vec{q} d\vec{r} \\
&= \frac{1}{\sqrt{3}(2\pi)^3} \int_{\Delta'_{\mathcal{T}_3}} \left| \sum_{i \neq j = 1}^{2} (-1)^i (q_i - q_3)(r_j - r_3) \right| e^{-\frac{\sum_{i=1}^{3}\left(q_i^2 + r_i^2\right)}{2}} d\vec{q} d\vec{r}
\end{aligned}
$$

where $d\vec{q} = dq_1 dq_2 dq_3$, $d\vec{r} = dr_1 dr_2 dr_3$ and

$$
\Delta'_{\mathcal{T}_3} = \{q_i, r_i \in \mathbb{R},\ i = 1,2,3 : \qquad |(q_2 - q_3)(r_1 - r_3) - (q_1 - q_3)(r_2 - r_3)| < 2\alpha\}.
$$

Next we let

$$
\begin{aligned}
q_1 &= \frac{u_2 + v_2}{2} + q_3 \qquad & q_2 &= \frac{u_1 + v_1}{2} + q_3 \qquad & q_3 &= q_3 \\
r_1 &= \frac{u_1 - v_1}{2} + r_3 \qquad & r_2 &= \frac{u_2 - v_2}{2} + r_3 \qquad & r_3 &= r_3
\end{aligned}
$$

which yields

$$
\begin{aligned}
F(\alpha) = \frac{1}{2^7 \sqrt{3} \pi^3} \int_{\Delta''_{\mathcal{T}_3}} & \left| u_1^2 - u_2^2 - v_1^2 + v_2^2 \right| e^{-\frac{1}{4} \sum_{i=1}^{2} \left(u_i^2 + v_i^2\right)} \\
& \cdot \left(\int_{-\infty}^{+\infty} e^{-\frac{3q_3^2 + q_3 \sum_{i=1}^{2}(u_i + v_i)}{2}} dq_3 \int_{-\infty}^{+\infty} e^{-\frac{3r_3^2 + r_3 \sum_{i=1}^{2}(u_i - v_i)}{2}} dr_3 \right) du_1 du_2 dv_1 dv_2
\end{aligned}
$$

with

$$
\Delta''_{\mathcal{T}_3} = \{(u_1, u_2, v_1, v_2) \in \mathbb{R}^4 : \quad \left| u_1^2 - u_2^2 - v_1^2 + v_2^2 \right| < 8\alpha\}.
$$

Since

$$
-\frac{3}{2} q_3^2 - \frac{q_3}{2} \sum_{i=1}^{2} (u_i + v_i) = -\frac{3}{2} \left[q_3 + \frac{1}{6} \sum_{i=1}^{2} (u_i + v_i) \right]^2 + \frac{1}{24} \left[\sum_{i=1}^{2} (u_i + v_i) \right]^2
$$

integration with respect to dq_3 gives

$$
\int_{-\infty}^{+\infty} e^{-\frac{3q_3^2 + q_3 \sum_{i=1}^{2}(u_i + v_i)}{2}} dq_3 = \sqrt{\frac{2\pi}{3}} e^{\frac{(u_1 + u_2 + v_1 + v_2)^2}{24}}.
$$

Likewise, integrating dr_3 we find

$$\int_{-\infty}^{+\infty} e^{-\frac{3r_3^2+r_3\sum_{i=1}^{2}(u_i-v_i)}{2}}dr_3=\sqrt{\frac{2\pi}{3}}e^{\frac{(u_1+u_2-v_1-v_2)^2}{24}}$$

and therefore

$$F(\alpha)=\frac{1}{3\sqrt{3}\,(8\pi)^2}\cdot\int_{\Delta''_{\mathcal{T}_3}}\left|u_1^2-u_2^2-v_1^2+v_2^2\right|e^{-\frac{\sum_{i=1}^{2}\left(u_i^2+v_i^2\right)-u_1u_2-v_1v_2}{6}}du_1du_2dv_1dv_2.$$

Finally we let

$$\begin{cases} u_1=\rho_1\cos\vartheta_1+\rho_2\cos\vartheta_2\\ u_2=\rho_1\cos\vartheta_1-\rho_2\cos\vartheta_2\\ v_1=\rho_1\sin\vartheta_1+\rho_2\sin\vartheta_2\\ v_2=\rho_1\sin\vartheta_1-\rho_2\sin\vartheta_2\end{cases}\qquad \vartheta_i\in[0,2\pi]\,,\ \rho_i>0,\ i=1,2$$

and we find

$$F(\alpha)=\frac{1}{3\sqrt{3}\,(2\pi)^2}\int_{\Delta'''_{\mathcal{T}_3}}\rho_1^2\rho_2^2\left|\cos(\vartheta_1+\vartheta_2)\right|e^{-\frac{\rho_1^2+3\rho_2^2}{6}}d\vartheta_1d\vartheta_2d\rho_1d\rho_2$$

where now

$$\Delta'''_{\mathcal{T}_3}=\left\{\vartheta_i\in[0,2\pi]\,,\ \rho_i>0,\ i=1,2:\quad \rho_1\rho_2\left|\cos(\vartheta_1+\vartheta_2)\right|<2\alpha\right\}.$$

The previous expression of $F(\alpha)$ is equivalent to

$$\begin{aligned} F(\alpha)=\frac{2}{3\sqrt{3}\pi}&\int_{\{\rho_1\rho_2<2\alpha\}}\rho_1^2\rho_2^2e^{-\frac{\rho_1^2+3\rho_2^2}{6}}d\rho_1d\rho_2\\ &+\int_{\{\rho_1\rho_2>2\alpha\}}\frac{\rho_1^2\rho_2^2e^{-\frac{\rho_1^2+3\rho_2^2}{6}}}{3\sqrt{3}\pi^2}\int_{\left\{\xi<\vartheta_1+\vartheta_2<\frac{\pi}{2}\right\}}\cos(\vartheta_1+\vartheta_2)\,d\vartheta_1d\vartheta_2d\rho_1d\rho_2\\ &\qquad-\int_{\{\rho_1\rho_2>2\alpha\}}\frac{\rho_1^2\rho_2^2e^{-\frac{\rho_1^2+3\rho_2^2}{6}}}{3\sqrt{3}\pi^2}\int_{\left\{\frac{\pi}{2}<\vartheta_1+\vartheta_2<\pi-\xi\right\}}\cos(\vartheta_1+\vartheta_2)\,d\vartheta_1d\vartheta_2d\rho_1d\rho_2\end{aligned}$$

where $\xi=\arccos\left(\frac{2\alpha}{\rho_1\rho_2}\right)$. Integration with respect to $d\vartheta_1$ and $d\vartheta_2$ yields

$$F(\alpha)=\frac{1}{2}+\frac{1}{3\sqrt{3}\pi}\left[I_1(\alpha)-I_2(\alpha)\right] \tag{4}$$

having denoted by $I_1(\alpha)$ and $I_2(\alpha)$ the integrals

$$\begin{aligned} I_1(\alpha) &= \int_{\{\rho_1\rho_2<2\alpha\}} \rho_1^2\rho_2^2 e^{-\frac{\rho_1^2+3\rho_2^2}{6}} d\rho_1 d\rho_2 \\ I_2(\alpha) &= \int_{\{\rho_1\rho_2>2\alpha\}} \rho_1\rho_2\sqrt{\rho_1^2\rho_2^2-4\alpha^2}\cdot e^{-\frac{\rho_1^2+3\rho_2^2}{6}} d\rho_1 d\rho_2. \end{aligned}$$

Let us consider first $I_1(\alpha)$. By letting

$$\begin{cases} \rho_1 = \sqrt{6}r\cos\varphi \\ \rho_2 = \sqrt{2}r\sin\varphi \end{cases} \qquad r>0,\ \varphi\in\left[0,\frac{\pi}{2}\right]$$

we obtain

$$\begin{aligned} I_1(\alpha) &= 24\sqrt{3}\int_0^{\frac{\sqrt{2\alpha}}{\sqrt[4]{3}}} dr \int_0^{\frac{\pi}{2}} r^5 e^{-r^2}\sin^2\varphi\cos^2\varphi d\varphi \\ &\quad +24\sqrt{3}\int_{\frac{\sqrt{2\alpha}}{\sqrt[4]{3}}}^{+\infty} dr\int_0^{\frac{1}{2}\arcsin\frac{2\alpha}{\sqrt{3}r^2}} r^5 e^{-r^2}\sin^2\varphi\cos^2\varphi d\varphi \\ &\quad +24\sqrt{3}\int_{\frac{\sqrt{2\alpha}}{\sqrt[4]{3}}}^{+\infty} dr\int_{\frac{\pi}{2}-\frac{1}{2}\arcsin\frac{2\alpha}{\sqrt{3}r^2}}^{\frac{\pi}{2}} r^5 e^{-r^2}\sin^2\varphi\cos^2\varphi d\varphi \\ &= \frac{3\sqrt{3}}{2}\pi - \left(\sqrt{3}\alpha^2+3\alpha+\frac{3\sqrt{3}}{2}\right)\pi e^{-\frac{2\sqrt{3}}{3}\alpha} \\ &\quad +3\sqrt{3}\int_{\frac{\sqrt{2\alpha}}{\sqrt[4]{3}}}^{+\infty} r^5\arcsin\left(\frac{2\alpha}{\sqrt{3}r^2}\right)e^{-r^2}dr \\ &\quad -3\alpha\int_{\frac{\sqrt{2\alpha}}{\sqrt[4]{3}}}^{+\infty} r^3\sqrt{1-\frac{4\alpha^2}{3r^4}}e^{-r^2}dr \end{aligned}$$

and the final substitution $\frac{\sqrt{3}}{2\alpha}r^2 = u$ gives

$$I_1(\alpha) = \frac{3\sqrt{3}}{2}\pi - \int_1^{+\infty}\frac{2\alpha^3 u\left(u^2-1\right)+2\sqrt{3}\alpha^2u^2+6\alpha u+3\sqrt{3}}{u\sqrt{u^2-1}}e^{-\frac{2\alpha}{\sqrt{3}}u}du.$$

Taking into account the following integral representation for the modified Bessel function of the second kind K_n (see, e.g., [4]):

$$K_n(x) = \frac{\sqrt{\pi}x^n}{2^n\Gamma\left(n+\frac{1}{2}\right)}\int_1^{+\infty}\left(u^2-1\right)^{n-\frac{1}{2}}e^{-xu}du \tag{5}$$

we can write

$$I_1(\alpha) = \frac{3\sqrt{3}}{2}\pi - 3\sqrt{3}\alpha^2K_1\left(\frac{2\alpha}{\sqrt{3}}\right) - 6\alpha K_0\left(\frac{2\alpha}{\sqrt{3}}\right) - 3\sqrt{3}\int_1^{+\infty}\frac{e^{-\frac{2\alpha}{\sqrt{3}}u}}{u\sqrt{u^2-1}}du.$$

As far as $I_2(\alpha)$ is concerned we consider the change of variables:

$$\begin{cases} \rho_1 = u \\ \rho_2 = \frac{1}{u}\sqrt{v^2+4\alpha^2} \end{cases}$$

and we get

$$I_2(\alpha) = \frac{\sqrt{2\pi}}{2}\int_0^{+\infty} u^2 e^{-\left(\frac{u^2}{6}+\frac{2\alpha^2}{u^2}\right)} du.$$

As $\int_0^{+\infty} \exp(-a^2x^2 - b^2/x^2)\, dx = \dfrac{\sqrt{\pi}}{2a}\exp(-2ab)$ for any $a, b > 0$, we find

$$\begin{aligned} I_2(\alpha) &= \int_0^{\alpha} \frac{d}{ds} I_2(s)\, ds + I_2(0) \\ &= \frac{\sqrt{2\pi}}{2}\int_0^{+\infty} u^2 e^{-\frac{u^2}{6}} du - 2\sqrt{2\pi}\int_0^{\alpha} ds \int_0^{+\infty} s e^{-\left(\frac{u^2}{6}+\frac{2s^2}{u^2}\right)} du \\ &= \frac{3\sqrt{3}\pi}{2} - 2\pi\sqrt{3}\int_0^{\alpha} s e^{-\frac{2s}{\sqrt{3}}} ds = \frac{3\pi}{2}\left(2\alpha+\sqrt{3}\right) e^{-\frac{2}{\sqrt{3}}\alpha}. \end{aligned}$$

Thus formula (4) becomes

$$F(\alpha) = 1 - \tfrac{\alpha^2}{\pi} K_1\left(\tfrac{2\alpha}{\sqrt{3}}\right) - \frac{2\alpha}{\sqrt{3}\pi} K_0\left(\tfrac{2\alpha}{\sqrt{3}}\right) - \tfrac{2\alpha+3}{2\sqrt{3}} e^{-\frac{2\alpha}{\sqrt{3}}} - \frac{1}{\pi}\int_1^{+\infty} \frac{e^{-\frac{2}{\sqrt{3}}\alpha}}{u\sqrt{u^2-1}} du \tag{6}$$

and we can state the following:

Theorem 1. *Let $\{P_1, P_2, P_3\}$ be independent and identically distributed random points in the Euclidean space E_3, with normal distribution* (1). *The random variable $\mathcal{A}$, area of the triangle $\mathcal{T}_3$ of vertices P_i, $i = 1, 2, 3$, has density function*

$$f(\alpha) = \frac{2\alpha^2}{\sqrt{3}\pi} K_0\left(\frac{2\alpha}{\sqrt{3}}\right) + \frac{\alpha}{3\pi} K_1\left(\frac{2\alpha}{\sqrt{3}}\right) + \frac{2}{3}\alpha e^{-\frac{2\alpha}{\sqrt{3}}}, \qquad \alpha > 0. \tag{7}$$

The mean value and the variance of $\mathcal{A}$ are:

$$E[\mathcal{A}] = \frac{\sqrt{3}}{2} + \frac{7\sqrt{3}}{4\pi} \qquad Var[\mathcal{A}] = \frac{69}{16} - \frac{21}{4\pi} - \frac{147}{16\pi^2}.$$

Proof. In order to get the density function of $\mathcal{A}$ it suffices to take the derivative of its distribution function $F(\alpha)$ given in (6) and to apply the following properties of the modified Bessel functions of second kind K_0 and K_1:

$$\frac{d}{d\alpha} K_0(\alpha) = -K_1(\alpha) \qquad \frac{d}{d\alpha} K_1(\alpha) = -K_0(\alpha) - \frac{1}{\alpha} K_1(\alpha)$$

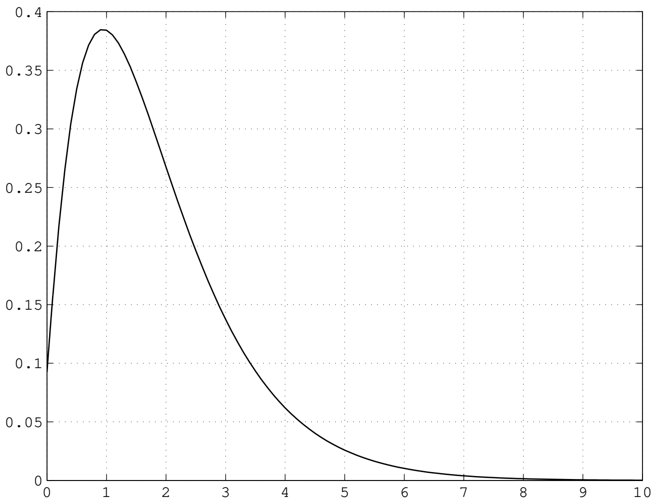

Figure 1: density function of the area $\mathcal{A}$ of $\mathcal{T}_3$

as well as the integral representation (5). The values of $E[\mathcal{A}]$ and $Var[\mathcal{A}]$ are consequences of (7). □

A plot of the density function $f(\alpha)$ is shown in Figure 1.

Remark. If $\{P_1, P_2, P_3\}$ are independent and identically distributed random points in the Euclidean plane E_2, with normal distribution $f(x,y) = \frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}$, then a result of Stoka (see [3], p. 540-541) gives

$$f(\alpha) = 2\sqrt{\frac{2}{\pi}}\alpha \int_0^{+\infty} \frac{1}{q^2} e^{-\left(\frac{q^2}{2}+\frac{\alpha^2}{q^2}\right)} dq, \qquad E[\mathcal{A}] = \frac{1}{\sqrt{2}}, \qquad Var[\mathcal{A}] = \frac{3}{2}.$$

The shape of the triangle $\mathcal{T}_3$ "depends on chance", so it is also interesting to determine whether "$\mathcal{T}_3$ is more likely to be acute-angled or obtuse-angled". The answer is that it is slightly more probable to find an obtuse-angled triangle. More precisely, with the same hypothesis and notation of theorem 1, we have

Theorem 2. *The probability that $\mathcal{T}_3$ is an acute-angled triangle is* $p = \dfrac{3\sqrt{3}}{4\pi^2}$.

Proof. The triangle $\mathcal{T}_3$ is acute-angled if and only if

$$\begin{gathered} |P_1P_2|^2 + |P_1P_3|^2 > |P_2P_3|^2, \qquad |P_1P_2|^2 + |P_2P_3|^2 > |P_1P_3|^2, \\ |P_1P_3|^2 + |P_2P_3|^2 > |P_1P_2|^2, \end{gathered} \tag{8}$$

where $|P_iP_j|$ denotes the euclidean distance between P_i and P_j (see, e.g., [2] p. 99).

If we use the change of variables (2) to express the cartesian coordinates (x_i, y_i, z_i) of P_i, $i = 1, 2, 3$, then condition (8) becomes

$$(q_i - q_j)(q_i - q_k) + (r_i - r_j)(r_i - r_k) > 0 \quad i \neq j \neq k, \quad i, j, k = 1, 2, 3 \tag{9}$$

and the probability p is

$$
\begin{aligned}
p &= P\left\{|P_iP_j|^2 + |P_iP_k|^2 > |P_jP_k|^2 \quad i \neq j \neq k, \quad i,j,k = 1,2,3\right\} \\
&= \frac{1}{(2\pi)^{\frac{9}{2}}} \int_0^{\pi} \sin\vartheta d\vartheta \int_0^{2\pi} d\varphi \int_0^{+\infty} e^{-\frac{3p^2}{2}} dp \\
&\quad \cdot \int_D \left| \sum_{i\neq j=1}^{2} (-1)^i (q_i - q_3)(r_j - r_3) \right| e^{-\frac{\sum_{i=1}^{3}\left(q_i^2+r_i^2\right)}{2}} d\vec{q} d\vec{r} \\
&= \frac{\sqrt{3}}{24\pi^3} \int_D \left| \sum_{i\neq j=1}^{2} (-1)^i (q_i - q_3)(r_j - r_3) \right| e^{-\frac{\sum_{i=1}^{3}\left(q_i^2+r_i^2\right)}{2}} d\vec{q} d\vec{r}
\end{aligned}
$$

where D is the set of real numbers $(q_1, q_2, q_3, r_1, r_2, r_3)$ that satisfies (9).

In order to compute the above integral we let

$$
\begin{cases}
q_1 = \rho_1 \cos\vartheta_1 + \rho_2 \cos\vartheta_2 + q_3 \\
q_2 = \rho_1 \cos\vartheta_1 - \rho_2 \cos\vartheta_2 + q_3 \\
q_3 = q_3 & \vartheta_1, \vartheta_2 \in [0, 2\pi] \\
r_1 = \rho_1 \sin\vartheta_1 + \rho_2 \sin\vartheta_2 + r_3 & \rho_1, \rho_2 > 0 \\
r_1 = \rho_1 \sin\vartheta_1 - \rho_2 \sin\vartheta_2 + r_3 \\
r_3 = r_3
\end{cases}
$$

and we obtain

$$
\begin{aligned}
p &= \frac{2\sqrt{3}}{9\pi^2} \int_{\left\{\frac{\rho_2}{\rho_1}<1\right\}} \rho_1^2\rho_2^2 e^{-\frac{\rho_1^2}{3} - \rho_2^2} d\rho_1 d\rho_2 \int_{\left\{|\cos(\vartheta_1-\vartheta_2)|<\frac{\rho_2}{\rho_1}\right\}} |\sin(\vartheta_1 - \vartheta_2)| \, d\vartheta_1 d\vartheta_2 \\
&= \frac{16\sqrt{3}}{9\pi} \int_{\left\{\frac{\rho_2}{\rho_1}<1\right\}} \rho_1\rho_2^3 e^{-\frac{\rho_1^2}{3} - \rho_2^2} d\rho_1 d\rho_2 = \frac{3\sqrt{3}}{4\pi}
\end{aligned}
$$

and the proof is complete. □

2. On a random tetrahedron with normally independently distributed vertices

Let us add now a fourth point P_4 to the set $\{P_1, P_2, P_3\}$ which we considered in the previous section and let us suppose that $\{P_i\}_{i=1,\ldots,4}$ are still independent and identically distributed.

We denote by $\mathcal{T}_4$ the tetrahedron of vertices P_i, $i = 1, \ldots, 4$, and by $\mathcal{V}$ the volume of $\mathcal{T}_4$. Once again our aim is to determine the density function, the mean value and the variance of $\mathcal{V}$. In order to be able to exploit the results of Section 1, we define the set

$$\Delta_{\mathcal{T}_4} = \{P_4 \in E_3 : \quad \mathcal{V} = volume(\mathcal{T}_4) < v\}$$

thus for any $v > 0$

$$\begin{aligned} G(v) &= P\{\mathcal{V} < v\} \\ &= \frac{1}{(2\pi)^6} \int\limits_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} e^{-\frac{1}{2}\sum\limits_{i=1}^{3}\left(x_i^2+y_i^2+z_i^2\right)} dP_1 dP_2 dP_3 \int\limits_{\Delta_{\mathcal{T}_4}} e^{-\frac{x_4^2+y_4^2+z_4^2}{2}} dP_4. \end{aligned}$$

Following (2), we introduce the change of variables:

$$\begin{cases} x_i = p\cos\varphi\sin\vartheta + q_i\cos\varphi\cos\vartheta + r_i\sin\varphi & \\ y_i = p\sin\varphi\sin\vartheta + q_i\sin\varphi\cos\vartheta - r_i\cos\varphi & \\ z_i = p\cos\vartheta - q_i\sin\vartheta & \vartheta\in[0,\pi],\ \varphi\in[0,2\pi] \\ x_4 = (p+s)\cos\varphi\sin\vartheta + q_4\cos\varphi\cos\vartheta + r_4\sin\varphi & p\in\mathbb{R}^+,\ s\in\mathbb{R} \\ y_4 = (p+s)\sin\varphi\sin\vartheta + q_4\sin\varphi\cos\vartheta - r_4\cos\varphi & q_i, r_i\in\mathbb{R},\ i=1,2,3,4 \\ z_4 = (p+s)\cos\vartheta - q_4\sin\vartheta & \end{cases}$$

with $|J|$ again given by (3). We have

$$\begin{aligned} G(v) = {} & \frac{1}{(2\pi)^6}\int_0^{\pi}\sin\vartheta d\vartheta\int_0^{2\pi}d\varphi\int_{-\infty}^{+\infty}e^{-\frac{q_4^2}{2}}dq_4\int_{-\infty}^{+\infty}e^{-\frac{r_4^2}{2}}dr_4 \\ & \cdot\int\limits_{\Delta'_{\mathcal{T}_4}}\left|\sum_{i\neq j=1}^{2}(-1)^i(q_i-q_3)(r_j-r_3)\right| e^{-\frac{4p^2+2ps+s^2}{2}-\sum\limits_{i=1}^{3}\frac{q_i^2+r_i^2}{2}} dp\,ds\,d\vec{q}\,d\vec{r} \\ = {} & \frac{1}{8\pi^4}\int\limits_{\Delta'_{\mathcal{T}_4}}\left|\sum_{i\neq j=1}^{2}(-1)^i(q_i-q_3)(r_j-r_3)\right| e^{-2p^2-ps-\frac{s^2}{2}+\sum\limits_{i=1}^{3}\frac{q_i^2+r_i^2}{2}} dp\,ds\,d\vec{q}\,d\vec{r} \end{aligned}$$

where

$$\Delta'_{\mathcal{T}_4} = \{q_i, r_i, s\in\mathbb{R},\ i=1,2,3,\ p\in\mathbb{R}^+ :\ \left|\sum_{i\neq j=1}^{2}(-1)^i(q_i-q_3)(r_j-r_3)\,s\right| < 6v\}.$$

Writing

$$e^{-\left(2p^2+ps+\frac{s^2}{2}\right)} = e^{-\frac{\left(2p+\frac{s}{2}\right)^2}{2}}e^{-\frac{3s^2}{8}}$$

and integrating dp we get

$$G(v) = \frac{1}{(2\pi)^{\frac{7}{2}}}\int\limits_{\Delta''_{\mathcal{T}_4}}\left|\sum_{i\neq j=1}^{2}(-1)^i(q_i-q_3)(r_j-r_3)\right| e^{-\frac{3s^2}{8}-\sum\limits_{i=1}^{3}\frac{\left(q_i^2+r_i^2\right)}{2}} ds\,d\vec{q}\,d\vec{r}$$

having denoted by $\Delta''_{\mathcal{T}_4}$ the set

$$\Delta''_{\mathcal{T}_4} = \left\{ q_i, r_i \in \mathbb{R},\ i = 1,2,3,\ s > 0 : \quad \left| \sum_{i \neq j = 1}^{2} (-1)^i \left(q_i - q_3\right)\left(r_j - r_3\right) \right| s < 6v \right\}.$$

If we compare the sets $\Delta'_{\mathcal{T}_3}$ and $\Delta''_{\mathcal{T}_4}$ it is not difficult to realize that

$$G(v) = \sqrt{\frac{3}{2\pi}} \int_0^{+\infty} F\left(\frac{3v}{s}\right) e^{-\frac{3s^2}{8}} ds$$

where $F(\cdot)$, that is the distribution function of the area of the triangle $\mathcal{T}_3$ of vertices $\{P_1, P_2, P_3\}$, is given by (6).
Using the change of variable $s = \dfrac{6v}{\sqrt{3}u}$ we find

$$\begin{aligned} G(v) = 1 &- \frac{9v}{(2\pi)^{\frac{3}{2}}} \int_0^{+\infty} K_1(u)\, e^{-\frac{9v^2}{2u^2}} du - \frac{6v}{\pi\sqrt{2\pi}} \int_0^{+\infty} \frac{1}{u} K_0(u)\, e^{-\frac{9v^2}{2u^2}} du \\ &- \frac{3v}{\sqrt{2\pi}} \int_0^{+\infty} \frac{1+u}{u^2} e^{-u - \frac{9v^2}{2u^2}} du - \frac{6v}{\pi\sqrt{2\pi}} \int_0^{+\infty} du \int_1^{+\infty} \frac{e^{-ut - \frac{9v^2}{2u^2}}}{u^2 t \sqrt{t^2-1}} dt \end{aligned} \tag{10}$$

and we can now state

Theorem 3. *Let $\{P_1, P_2, P_3, P_4\}$ be independent and identically distributed random points in the Euclidean space E_3, with normal distribution (1). The random variable $\mathcal{V}$, volume of the tetrahedron $\mathcal{T}_4$ of vertices P_i, $i = 1,2,3,4$, has density function*

$$g(v) = \frac{3}{\sqrt{2\pi}} \int_0^{+\infty} \left(\frac{3u}{2\pi} K_0(u) + \frac{1}{2\pi} K_1(u) + e^{-u} \right) e^{-\frac{9v^2}{2u^2}} du \qquad v > 0$$

where $K_0(\cdot)$ and $K_1(\cdot)$ are the modified Bessel functions of second kind of order $n = 0$ and $n = 1$. The mean value and the variance of $\mathcal{V}$ are

$$E[\mathcal{V}] = \frac{1}{3\sqrt{2\pi}} \left(2 + \frac{7}{\pi}\right) \qquad Var[\mathcal{V}] = \frac{3}{4} - \frac{2}{9\pi} - \frac{14}{9\pi^2} - \frac{49}{18\pi^3}.$$

Proof. The expression of the density function $g(v) = \dfrac{d}{dv} G(v)$ is easily obtained from (10). For the computation of $E[\mathcal{V}]$ and $Var[\mathcal{V}]$ it is useful to remind the integral representation (5) of the functions $K_0(u)$ and $K_1(u)$. $\square$

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