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Fixing and Hindering Systems in Combinatorial Geometry*

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Abstract. In the first and in the second section of the article we give a short survey of former results and new observations on fixing and hindering systems for compact, convex bodies. In the third part, we prove some results on fixing and hindering systems for d-convex bodies in Minkowski spaces. In particular, finite-dimensional normed spaces are considered, for which the unit ball is the polar body of a belt body (or, particularly, of a zonoid).

Keywords: convex body, primitive fixing system, primitive hindering system, Minkowski space, equatorial subspace, belt vector, belt body, zonoid.

1. Fixing systems for compact, convex bodies

Let $M \subset \mathbf{R}^n$ be a compact, convex body and F be a subset of its boundary $\operatorname{bd} M$. We say that a vector $v \neq 0$ *removes* F *from the interior of* M if for any positive, real number λ the equality

$$(\lambda v + F) \cap \operatorname{int} M = \emptyset \tag{1}$$

holds, i.e., the set $t_{\lambda v}(F)$, obtained from F by the translation $t_{\lambda v}(x) = \lambda v + x$, has no common point with the interior of M.

The set $F \subset \operatorname{bd} M$ is said to be a *fixing system* for the body M (against translations) if there is no vector $v \neq 0$ which removes F from $\operatorname{int} M$. Visually, F is a fixing system if it is impossible to translate M in any direction, when we put "fixing nails" at the points of F.

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A fixing system is *primitive* if no proper subsystem $F' \subset F$ is a fixing system for M. Evidently, a fixing system of the least cardinality is primitive. By $\varrho_{\min}(M)$ we denote the *smallest* cardinality of fixing systems for M. Analogously, by $\varrho_{\max}(M)$ we denote the *largest* cardinality of primitive fixing systems for M; we write $\varrho_{\max}(M) = \infty$ if there are primitive fixing systems for M with arbitrarily large cardinality.

The notions of fixing system and primitive fixing system for a compact, convex body in $\mathbf{R}^n$ were introduced by L. Fejes Tóth [9]. He investigated the case of planar, convex sets. In particular, $\varrho_{\min}(M) = 4$ if $M \subset \mathbf{R}^2$ is a parallelogram and $\varrho_{\min}(M) = 3$ otherwise. Furthermore, $\varrho_{\max}(M) \leq 6$ for any compact, convex body $M \subset \mathbf{R}^2$. The case $\varrho_{\max}(M) = 6$ is realized for any hexagon with pairwise parallel sides, F being the set of its vertices. B. Tomor [18] and S. Fudali [10] proved that these and only these plane figures are attaining the upper bound. Moreover, L. Fejes Tóth [9] showed that fixing systems for smooth, convex bodies are strongly connected with the classification of so-called primitive polytopes; see also [15] for extensions and related notions.

B. Grünbaum [12] proved that for every compact, convex body $M \subset \mathbf{R}^n$ the inequalities $n + 1 \leq \varrho_{\min}(M) \leq 2n$ hold, and he posed the task to show that only the parallelotopes reach this upper bound. The positive answer for this task was obtained in [4], cf. Theorem 1 below. Furthermore, P. Mani [14] constructed a convex n-polytope $P \subset \mathbf{R}^n$, $n \geq 3$, with $2n + 2$ vertices which is fixed by its vertex set, and he proved that there is no polytope with less than $2n+2$ vertices which can be fixed by its vertex set. In view of containment problems for convex bodies, a reformulation of Mani's result was given in [13]. For further results see S. Fudali [11].

We now formulate recent results obtained in [4] and [7]. These results use the functional $\operatorname{md} M$ introduced in [2]. We give here its description.

Definition 1. *Vectors $a_0, a_1, \dots, a_k \in \mathbf{R}^n$ are minimally dependent if there is a dependence $\lambda_0 a_0 + \dots + \lambda_k a_k = 0$ with positive coefficients and, moreover, every k of the vectors $a_0, \dots, a_k$ are linearly independent. Furthermore, for a system $H \subset \mathbf{R}^n$ consisting of nonzero vectors, the number $\operatorname{md} H$ is the largest of the integers k such that there are $k+1$ minimally dependent vectors in H (if there is no subsystem of vectors in H which are minimally dependent, then $\operatorname{md} H = 0$). For a compact, convex body $M \subset \mathbf{R}^n$ denote by $H(M)$ the set of all unit outward normals of the body M at its regular boundary points; the integer $\operatorname{md} H(M)$ is denoted by $\operatorname{md} M$. In other words, $\operatorname{md} M$ is the greatest of the integers k such that there are $k + 1$ regular boundary points of the body M for which the corresponding unit outward normals are minimally dependent. We remark that for any compact, convex body $M \subset \mathbf{R}^n$ the inequalities $1 \leq \operatorname{md} M \leq n$ hold.*

Theorem 1. *For every compact, convex body $M \subset \mathbf{R}^n$ the inequalities*

$$n + \frac{n}{\operatorname{md} M} \leq \varrho_{\min}(M) \leq 2n + 1 - \operatorname{md} M \tag{2}$$

hold. These estimates are sharp; more precisely, for every integers m, ϱ satisfying $1 \leq m \leq n$ and $n + \frac{n}{m} \leq \varrho \leq 2n + 1 - m$ there exists a compact, convex body $M \subset \mathbf{R}^n$ with $\operatorname{md} M = m$ and $\varrho_{\min}(M) = \varrho$.

It follows from (2) that the above Grünbaum's task has a positive answer. Indeed, if a compact, convex body $M \subset \mathbf{R}$ is distinct from a parallelotope, then $\operatorname{md} M \geq 2$; hence $\varrho_{\min}(M) \leq 2n+1-\operatorname{md} M < 2n$.

For $M \subset \mathbf{R}^n$, $n \geq 3$, the situation with maximal cardinalities of primitive fixing systems is, in a sense, unexpected. In [8], L. Danzer formulated the conjecture that the cardinalities of primitive fixing systems are bounded. More detailed, the rhombic dodecahedron $M \subset \mathbf{R}^3$ (i.e., the vector sum of four segments in general position) has a primitive fixing system consisting of 14 points (its vertices), and Danzer assumed that this is the maximal value of the cardinality of a primitive fixing system for any compact, convex body $M \subset \mathbf{R}^3$, i.e., $\varrho_{\max}(M) \leq 14$. The analogous zonotope in $\mathbf{R}^n$ (i.e., the vector sum of $n+1$ segments in general position) has $2(2^n-1)$ vertices, and Danzer assumed that this body has its vertex set as a primitive fixing system of maximal cardinality, i.e., $\varrho_{\max}(M) \leq 2(2^n-1)$ for any compact, convex body $M \subset \mathbf{R}^n$.

But in contradiction to this assumption, B. Bollobás [1] gave an unexpected solution to this question for $n \geq 3$. He showed that for any positive integer $k \geq 6$ there is a compact, convex body $M \subset \mathbf{R}^3$ with a primitive fixing system $F \subset \operatorname{bd} M$ consisting of k points.

We remark that the bodies constructed in [1] depend essentially on k, i.e., for different k the bodies are different, and for every body M in [1] the number $\varrho_{\max}(M)$ is finite. Moreover, each body constructed in [1] has infinitely many nonregular boundary points.

In [5] some new examples are constructed. Namely, consider in $\mathbf{R}^3$ the body $M = \operatorname{conv}(B \cup \{a\})$, where B is a ball and $a \notin B$ (hence M is a cap body). Then $\varrho_{\max}(M) = \infty$. Moreover, for any integer $k \geq 4$ there is a primitive fixing system $F_k \subset \operatorname{bd} M$ that consists of k points. An analogous example can be easily constructed in $\mathbf{R}^n$. Thus, in contrast to [1], we gave a unique body for any cardinality of a primitive fixing system. We remark that the constructed body M has only one nonregular boundary point a, whereas for any smooth, compact, convex body $N \subset \mathbf{R}^n$ the number $\varrho_{\max}(N)$ is not greater than $2n$, see [12].

A further improvement was obtained by E. Morales Amaya [16]. He proved that for any smooth, strictly convex body $B \subset \mathbf{R}^3$ and any point $a \notin B$ the equality $\varrho_{\max}(\operatorname{conv} B \cup \{a\}) = \infty$ holds.

Furthermore, in [5] a zonotope $Z_m \subset \mathbf{R}^3$, $m \geq 9$, is constructed that has a primitive fixing system consisting of $m+1$ points (an extension to $\mathbf{R}^n$ is easy). Thus already in the class of zonotopes we have counterexamples to the conjecture of L. Danzer. Nevertheless, Danzer's conjecture remains to be interesting in the following sense: Is it true that there is no polytope $P \subset \mathbf{R}^3$ with more than 14 vertices such that the set of all its vertices is a primitive fixing system for P?

2. Hindering and strict hindering systems for compact, convex bodies

Let $M \subset \mathbf{R}^n$ be a compact, convex body and F be a subset of its boundary. We say that a vector $v \neq 0$ *removes F from the body M* if for any positive, real number λ the equality

$$(\lambda v + F) \cap M = \emptyset \tag{3}$$

holds, i.e., the set $t_{\lambda v}(F)$, obtained from F by the translation $t_{\lambda v}(x) = \lambda v + x$, has no common point with M. Thus Definition 1 means that $t_{\lambda v}(F) \cap (\operatorname{int} M) = \emptyset$ for any $\lambda > 0$, whereas in Definition 3 we have $t_{\lambda v}(F) \cap M = \emptyset$.

The set $F \subset \operatorname{bd} M$ is said to be a *hindering system* (against translations) for the body M if there is no vector $v \neq 0$ which removes F from M. Visually, F is a hindering system if it is impossible to translate M in any direction, when we put "fixing nails with friction" at the points of F (in contrast to the visual picture for fixing systems, where the nails are "smooth", without friction).

A hindering system $F \subset \operatorname{bd} M$ is *primitive* if no proper subsystem $F' \subset F$ is a hindering system for M. Evidently, a hindering system of the least cardinality is primitive. By $\sigma_{\min}(M)$ we denote the *smallest* cardinality of hindering systems for M. Analogously, by $\sigma_{\max}(M)$ we denote the *largest* cardinality of primitive hindering systems for M; we write $\sigma_{\max}(M) = \infty$ if there are primitive hindering systems for M with arbitrarily large cardinality.

The notions of hindering system and primitive hindering system for a compact, convex body in $\mathbf{R}^n$ were introduced by P. Mani [14]. He investigated the case of planar, convex sets. In particular, if a compact, convex body $M \subset \mathbf{R}^2$ has a segment I in its boundary and, moreover, there is a regular boundary point of M with the supporting line $l \,||\, I$ (and $I \cap l = \emptyset$), then $\sigma_{\min}(M) = 2$. Otherwise $\sigma_{\min}(M) = 3$.

It is obvious that $\sigma_{\min}(M) \leq \varrho_{\min}(M)$; moreover, every primitive fixing system for M contains a primitive hindering subsystem. For example, let F be the vertex set of a regular hexagon M and F' be obtained from F by removing two opposite vertices. Then F is a primitive fixing system for M, whereas $F' \subset F$ is a primitive hindering system. For a regular pentagon M, the vertex set is its primitive hindering system, i.e., in this case $\sigma_{\max}(M) = 5$. Moreover, $\sigma_{\max}(M) \leq 5$ for every compact, convex body $M \subset \mathbf{R}^2$.

Let again $M \subset \mathbf{R}^n$ be a compact, convex body, F be a subset of its boundary, and $v(t)$, $0 \leq t \leq 1$, be a continuous family of vectors in $\mathbf{R}^n$ with $v(0) = 0$. We say that the family $v(t)$ *removes F from the body M* if for $0 < t \leq 1$ the equality

$$(v(t) + M) \cap F = \emptyset \tag{4}$$

holds, i.e., the set $t_{v(t)}(F)$, obtained from F by the translation $t_{v(t)}(x) = v(t) + x$, contains no point of M.

The set $F \subset \operatorname{bd} M$ is said to be a *strict hindering system* for the body M if there is no continuous family $v(t)$ with $v(0) = 0$ which removes F from M.

Example 1. Let $M \subset \mathbf{R}^2$ be a half-disk, a be the midpoint of its diameter, and $b \in M$ be the point of largest distance to the diameter. Then $F = \{a, b\}$ is a hindering system for M, but F is not a strict hindering system. Indeed, let $b_t \in \operatorname{bd} M$ be the point such that the angle between the vectors $b - a$ and $b_t - a$ is equal to t (all points b_t being situated on the same side of the line ab). We put $v(t) = 2(b_t - b)$. Then for $0 < t \leq 1$ the equality (4) holds, i.e., the family $v(t)$ removes F from M. Hence F is not a strict hindering system for M.

As above, a strict hindering system $F \subset \operatorname{bd} M$ is *primitive* if no proper subsystem $F' \subset F$ is a strict hindering system for M. Evidently, a strict hindering system of the least cardinality is primitive. By $\sigma'_{\min}(M)$ we denote the *smallest* cardinality of strict hindering systems for M. Analogously, by $\sigma'_{\max}(M)$ we denote the *largest* cardinality of primitive, strict hindering systems for M; we write $\sigma'_{\max}(M) = \infty$ if there are primitive, strict hindering systems for M having arbitrarily large cardinality.

Example 2. Let $M \subset \mathbf{R}^3$ be a semicylinder, P_1 and P_2 be its parallel planar facets (i.e., half-disks), and P_0 be its rectangular facet. Choose points $a_i \in \operatorname{ri} P_i$, $i = 0, 1, 2$, and denote by I the segment that has the largest distance from P_0. Choose a point $b_0 \in \operatorname{ri} I$ and points $b_1, b_2 \in (\operatorname{bd} M) \setminus (P_0 \cup P_1 \cup P_2 \cup I)$ which are situated on different sides of I. Then

(i) $\varrho_{\min}(M) = 5$, and $\{a_0, a_1, a_2, b_1, b_2\}$ is a minimal fixing system;
(ii) $\sigma'_{\min}(M) = 3$, and $\{a_0, b_1, b_2\}$ is a minimal strict hindering system;
(iii) $\sigma_{\min}(M) = 2$, and $\{a_0, b_0\}$ is a minimal hindering system.

The notion of strict hindering system is introduced in [3], where the following result is obtained.

Theorem 2. *For every compact, convex body $M \subset \mathbf{R}^n$ the inequalities*

$$2 \le \sigma_{\min}(M) \le \sigma'_{\min}(M) \le \operatorname{md} M + 1$$

hold. Moreover, these estimates are sharp, i.e., there is a body $M \subset \mathbf{R}^n$ with $\sigma_{\min}(M) = \sigma'_{\min}(M) = 2$, and for $2 \le m \le n$ there is a body $M \subset \mathbf{R}^n$ with $\operatorname{md} M = m$ and $\sigma_{\min}(M) = \sigma'_{\min}(M) = m + 1$.

In [4], the following description of fixing systems is established:

Theorem 3. *Let F be contained in the boundary of a compact, convex body $M \subset \mathbf{R}^n$. The set F is a fixing system for M if and only if for every nonzero vector $v \in \mathbf{R}^n$ the direction defined by v illuminates at least one point $a \in F$ (i.e., $a + \lambda v \in \operatorname{int} M$ for $\lambda > 0$ small enough).*

To obtain an analogous description for hindering systems, we introduce a new problem of illumination type:

Definition 2. *Let $M \subset \mathbf{R}^n$ be a compact, convex body and $v \in \mathbf{R}^n$ be a nonzero vector. A boundary point a of the body M is sustained by M with respect to the direction of the vector v if $a + \lambda v \in M$ for $\lambda > 0$ small enough. The sustaining problem is to find the minimal number of directions such that every boundary point of M is sustained by M with respect to at least one of the directions.*

Theorem 4. *Let $M \subset \mathbf{R}^n$ be a compact, convex body and F be a subset of its boundary; the set F is a hindering system for M if and only if with respect to every direction there is at least one point in F that is sustained by M.*

Proof. Admit that F is a hindering system. Then for every nonzero vector $v \in \mathbf{R}^n$ the direction of v does not remove F from M, i.e., there is a point $a \in F$ and a real number $\lambda > 0$ such that $a + \lambda v \in M$ (cf. (3)). By Definition 2, this means that a is sustained by M with respect to the direction of v.

Conversely, assume that for every direction there is at least one point in F sustained by M, i.e., for every $v \neq 0$ there exists a point $a \in F$ such that $a + \lambda v \in M$ for $\lambda > 0$ small enough. By (3), this means that the direction of v does not remove F from M. Consequently, F is a hindering system for M. $\square$

By the following theorem, the lower estimate in Theorem 2 is made more precise.

Theorem 5. *Let $M \subset \mathbf{R}^n$ be a compact, convex body. The number $\sigma_{\min}(M)$ is not smaller than the least of the integers k such that there are k minimally dependent vectors in* $\mathrm{cl}\, H(M)$.

Proof. Let $p \in \mathrm{bd}\, M$ and $q_1, q_2, \ldots$ be a sequence of regular boundary points of M with $\lim_{i\to\infty} q_i = p$. Denote by b_i the unit outward normal of the body M at the point q_i, $i = 1, 2, \ldots$ Passing to a subsequence, if necessary, we may assume that the limit $a = \lim_{i\to\infty} b_i$ exists. The body M is contained in the half-space $\{x : \langle b_i, x - q_i\rangle \le 0\}$; hence, passing to the limit, we conclude that $M \subset \{x : \langle a, x - p\rangle \le 0\}$. Moreover, $a \in \mathrm{cl}\, H(M)$ (since $b_i \in H(M)$ for all i). Thus for every $p \in \mathrm{bd}\, M$ there is a vector $a \in \mathrm{cl}\, H(M)$ such that $M \subset \{x : \langle a, x - p\rangle \le 0\}$.

Let now $F = \{p_1, \ldots, p_s\} \subset \mathrm{bd}\, M$ be a hindering system for the body M. Choose vectors $a_1, \ldots, a_s \in \mathrm{cl}\, H(M)$ such that $M \subset \{x : \langle a_i, x - p_i\rangle \le 0\}$ for every $i = 1, \ldots, k$. If $\mathrm{md}\{a_1, \ldots, a_s\} = 0$, then the vectors $a_1, \ldots, a_s$ are situated in an open hemisphere of the unit sphere $\mathbf{S}^{n-1} \subset \mathbf{R}^n$, i.e., there exists a nonzero vector $v \in \mathbf{R}^n$ such that $\langle a_i, v\rangle > 0$ for all $i = 1, \ldots, s$. Hence $p_i + \lambda v \notin \{x : \langle a_i, x - p_i\rangle \le 0\}$, i.e., $p_i + \lambda v \notin M$, $i = 1, \ldots, s$, contradicting that F is a hindering system for M.

Thus $\mathrm{md}\{a_1, \ldots, a_s\} > 0$, and consequently there are $k \le s$ minimally dependent vectors in $\mathrm{cl}\, H(M)$, i.e., $k \le \sigma_{\min}(M)$. □

Example 3. There are compact, convex bodies for which equality in Theorem 5 holds, as well as bodies for which strong inequality holds. Let $M_1 \subset \mathbf{R}^2$ be the intersection of two circles of radius 2 and centered at the points $(0, \pm 1)$, respectively, and M_2 be the convex hull of the union of those circles. It is easily shown that $\sigma_{\min}(M_1) = 3$, but the minimal number k in Theorem 5 is equal to 2 (since the vectors $(0, \pm 1)$ belong to $H(M_1)$ and consequently to $\mathrm{cl}\, H(M_1)$). Furthermore, the set F consisting of the points $(1, 2)$ and $(-1, -2)$ is a hindering system for M_2. Both the unit outward normals $(\pm 1, 0)$ at the points of F belong to $H(M_2) = \mathrm{cl}\, H(M_2)$, and consequently the minimal number k in Theorem 5 is equal to 2. We also have $\sigma_{\min}(M_2) = 2$.

The above example gives an illustration to the following lemma that is one of the main tools in [3]:

Lemma. *Let $M \subset \mathbf{R}^n$ be a compact, convex body and $p_1, \ldots, p_s$ be regular boundary points of M with the following properties:*

(i) *the corresponding unit outward normals $a_1, \ldots, a_s$ of M are minimally dependent;*

(ii) *denote by N the orthogonal complement of the subspace $L = \mathrm{lin}(a_1, \ldots, a_s)$ and by E_ε the ball $U_\varepsilon(0) \cap N$ in the subspace N; then for $\varepsilon > 0$ small enough the balls $p_1 + E_\varepsilon, \ldots, p_s + E_\varepsilon$ are contained in* $\mathrm{bd}\, M$.

Under these conditions, $F = \{p_1, \ldots, p_s\}$ is a strict hindering system for M.

The above lemma gives a sufficient condition, under which the set $\{p_1, \ldots, p_s\}$ is a strict hindering system for M. An analogous assertion holds for hindering systems. Namely, if instead of (ii) we require that at least one of the balls $p_1 + E_\varepsilon, \ldots, p_s + E_\varepsilon$ is contained in

$\operatorname{bd} M$, then the analogous lemma [3] affirms that $F = \{p_1, \ldots, p_s\}$ is a hindering system for M.

Example 4. Return to the situation in Example 1 and consider two points $p_1 = a$ and $p_2 = b$. The flat L, as in the above lemma, coincides with the line ab and N is the subspace parallel to the diameter of the half-disk. The ball $p_1 + E_\varepsilon$ is contained in $\operatorname{bd} M$ (for $\varepsilon \leq 1$), whereas $p_2 + E_\varepsilon$ is not contained in $\operatorname{bd} M$ for any $\varepsilon > 0$. This means that the sufficient condition given in the lemma is not applicable. And indeed, the set $\{p_1, p_2\}$ is not a strict hindering system for M. However, one of the balls is contained in $\operatorname{bd} M$, and hence the set $\{p_1, p_2\}$ is a hindering system.

The following theorem contains estimates for cardinalities of primitive hindering systems of a compact, convex body which is decomposable into direct vector summands.

Theorem 6. *Let $M \subset \mathbf{R}^n$ be a compact, convex body and $M = M_1 \oplus \cdots \oplus M_s$ be its decomposition into the direct vector sum of convex sets. Then the inequalities*

$$\sigma_{\min}(M) = \min(\sigma_{\min}(M_1), \ldots, \sigma_{\min}(M_s)), \tag{5}$$

$$\sigma_{\max}(M) \geq \prod_{i=1}^{s} \sigma_{\max}(M_i) \tag{6}$$

hold (where $\sigma_{\min}(M_i)$ is the minimal cardinality of hindering systems for M_i with respect to its affine hull, and similarly for $\sigma_{\max}(M_i)$). Analogous inequalities are true for strict hindering systems.

Proof. For an index i, let F_i be a minimal hindering system for M_i (with respect to its affine hull). Furthermore, for all $j \neq i$ choose a point p_j from the relative interior of the convex set M_j. Then $F_i + \sum_{j \neq i} p_j$ is a hindering system for the body M, and this hindering system contains $\sigma_{\min}(F_i)$ points. Consequently $\sigma_{\min}(F) \leq \sigma_{\min}(F_i)$. This is true for every $i = 1, \ldots, s$, and hence

$$\sigma_{\min}(M) \leq \min(\sigma_{\min}(M_1), \ldots, \sigma_{\min}(M_s)). \tag{7}$$

Let now $F = \{q_1, \ldots, q_t\}$ be a hindering system for the body M. Denote by π_i the projection of the space $\mathbf{R}^n$ onto $\operatorname{aff} M_i$ parallel to the vector sum of all other flats $\operatorname{aff} M_j$. Without loss of generality, we may assume that every convex set M_i contains the origin; then $M_i = \pi_i(M)$ for all $i = 1, \ldots, s$. Since q_i is a boundary point of the body M, for at least one $j \in \{1, \ldots, s\}$ the point $\pi_j(q_i)$ is a point from the relative boundary of the convex set M_j (and this is true for every point $q_1, \ldots, q_t$).

Assume that t (i.e., the cardinality of F) is less than the right-hand side of (5), i.e., $t < \sigma_{\min}(M_j)$ for all $j = 1, \ldots, s$. Denote by G_j the set of all points $q_i \in F$ such that $\pi_j(q_i) \in \operatorname{rbd} M_j$. Then $G_1 \cup \cdots \cup G_s = F$. Since the set G_j contains no more than t points, i.e., less than $\sigma_{\min}(M_j)$ points, there is a nonzero vector v_j parallel to $\operatorname{aff} M_j$ that removes $\pi_j(G_j)$ from $M_j = \pi_j(M)$. This is true for every $j \in \{1, \ldots, s\}$. Consequently the vector

$v = v_1 + \cdots + v_s$ removes F from M, contradicting that F is a hindering system for M. This contradiction shows that t is not less than the right-hand side of (5), i.e.,

$$\sigma_{\min}(M) \geq \min(\sigma_{\min}(M_1), \ldots, \sigma_{\min}(M_s)).$$

Together with (7) this proves (5).

We now prove (6). Let F_i be a primitive hindering system of maximal cardinality for M_i, $i = 1, \ldots, s$. Denote by F the set of all points $u_1 + \cdots + u_s$, where $u_i \in F_i$ for all $i = 1, \ldots, s$. The cardinality of F is equal to the right-hand side of (6). Hence to prove (6) it is sufficient to establish that F is a primitive hindering system for M. Let $v \in \mathbf{R}^n$ be a nonzero vector. Represent it in the form $v = v_1 + \cdots + v_s$, where $v_i \in \operatorname{aff} M_i$, $i = 1, \ldots, s$ (again we assume that every convex set M_i contains the origin). If for an index k the vector v_k is distinct from zero, then v_k does not remove F_k from M_k, i.e., there is a point $p_k \in F_k$ such that $p_k + \lambda v_k \in M_k$ for $\lambda > 0$ small enough. We choose such a point p_k for every index k with $v_k \neq 0$. If even $v_k = 0$, then we choose any point $p_k \in F_k$. Thus for every index $k = 1, \ldots, s$ the point $p_k + \lambda v_k$ belongs to M_k, when $\lambda > 0$ is small enough. We denote the point $p_1 + \cdots + p_s$ by p. By definition of the set F, we have $p \in F$. Moreover,

$$p + \lambda v = (p_1 + \lambda v_1) + \cdots + (p_s + \lambda v_s) \in M_1 + \cdots + M_s = M$$

for $\lambda > 0$ small enough. This means that the vector v does not remove the set F from M. This is true for any vector $v \neq 0$, and hence F is a hindering system for M.

Moreover, F is a primitive hindering system. Indeed, let $q \in F$, i.e., $q = q_1 + \cdots + q_s$, where $q_i \in F_i$ for all $i = 1, \ldots, s$. For every $i = 1, \ldots, s$ choose a nonzero vector w_i that removes $F_i \setminus \{q_i\}$ from M_i (such a vector exists, since the hindering system F_i for M_i is primitive). Then the vector $w = w_1 + \cdots + w_s$ removes $F \setminus \{q\}$ from M. This means that the hindering system F for the body M is primitive. □

3. Fixing and hindering systems for compact, *d*-convex bodies

Let $\mathcal{R}^n$ be an n-dimensional Minkowski space and Σ be its unit ball. For distinct points $x, y \in \mathcal{R}^n$, by $d(x, y)$ we denote the *Minkowski distance* between x and y, i.e., the least positive real number λ such that $\frac{1}{\lambda}(x - y) \in \Sigma$. A set $M \subset \mathcal{R}^n$ is said to be *d-convex* (cf. [17], [6]) if for any points $a, c \in M$, $b \in \mathcal{R}^n$, the equality $d(a, c) = d(a, b) + d(b, c)$ implies $b \in M$.

The class of all d-convex sets in $\mathcal{R}^n$ is a subset of the class of all linearly convex sets (obtained if $\mathcal{R}^n$ is considered as a linear n-dimensional space without any metric). More precisely, this subset is a proper one if and only if the unit ball Σ of $\mathcal{R}^n$ is not strictly convex. For that reason, there are some special results on d-convex bodies.

Definition 3. *Let $\Gamma \subset \mathcal{R}^n$ be a subspace (i.e., a flat through the origin). We say that Γ is an equatorial subspace if for any point $x \in \Gamma \cap (\operatorname{bd} \Sigma)$ every segment $I \subset \operatorname{bd} \Sigma$ with $x \in \operatorname{ri} I$ is contained in the subspace Γ.*

We now recall some facts on d-convexity established in [17] (cf. also chapter II of the monograph [6]):

Theorem 7. *A flat $L \subset \mathcal{R}^n$ is d-convex if and only if the subspace $\Gamma \,||\, L$ is equatorial.*

Theorem 8. *Let $M \subset \mathcal{R}^n$ be a compact, convex body. The body M is d-convex if and only if its supporting hyperplane at any regular boundary point is d-convex, i.e., parallel to an equatorial hypersubspace.*

Let now $M \subset \mathbf{R}^n$ be a compact, convex body, m be a supporting line of M, and $a \in m \cap M$. We say that m is a *half-tangent* for M if at least one ray defined on m by the point a is a tangential ray for M, i.e., is contained in the *supporting cone* of M at a which is given by

$$\sup \operatorname{cone}_a M = \operatorname{conv} \bigcup_{\lambda > 0} (a + \lambda(-a + M)) \,.$$

Definition 4. *Let $M \subset \mathbf{R}^n$ be a compact, convex body and $l \subset \mathbf{R}^n$ be a line. We say that l has a belt direction with respect to M if every supporting line $m \,||\, l$ of the body M is a half-tangent for M.*

Definition 5. *Let $\Gamma \subset \mathcal{R}^n$ be a linear hypersubspace. The union of all linear functionals $\mathcal{R}^n \to \mathbf{R}$ with kernel Γ and zero functional $0 : \mathcal{R}^n \to \mathbf{R}$ is a line in the conjugate space $(\mathcal{R}^n)^*$. We denote this line by Γ^*. (More exactly, for any hyperplane $G \subset \mathcal{R}^n$ not passing through the origin, its polar image is a point in the conjugate space; if even $G = \Gamma$ passes through the origin, then this point is situated at infinity. Instead of considering that point at infinity, we use the line Γ^* through the origin representing that point.)*

The following theorem establishes a connection between properties of the unit ball $\Sigma \subset \mathcal{R}^n$ and its polar body

$$\Sigma^* = \{f \in (\mathcal{R}^n)^* : \ f(x) \le 1 \quad \text{for all} \quad x \in \Sigma\}.$$

Theorem 9. *A linear hypersubspace $\Gamma \subset \mathcal{R}^n$ is equatorial with respect to Σ if and only if the line Γ^* has a belt direction with respect to the polar body Σ^*.*

Proof. Admit that a linear hypersubspace $\Gamma \subset \mathcal{R}^n$ is not equatorial with respect to Σ. This means that there is a point $x \in \Gamma \cap (\operatorname{bd} \Sigma)$ and a segment $I \subset \operatorname{bd} \Sigma$ with $x \in \operatorname{ri} I$ such that I is not contained in the subspace Γ. Denote the endpoints of the segment I by p_1 and p_2. Let Ξ be a supporting hyperplane of Σ containing the segment $[p_1, p_2]$. Then Ξ^* is a boundary point of the polar body Σ^*. The polar image x^* of the point x is a supporting hyperplane x^* of the polar body Σ^*, and the point Ξ^* belongs to that hyperplane x^*. Furthermore, the polar image of the segment $[0, p_i]$ is a supporting half-space P_i, and its boundary p_i^* is a supporting hyperplane of Σ^*, $i = 1, 2$. Each of the supporting hyperplanes p_1^*, p_2^* contains Ξ^*. The intersection $P_1 \cap P_2$ is a convex cone containing Σ^*, and Ξ^* belongs to the boundary of that cone. Consequently $P_1 \cap P_2$ contains the supporting cone of Σ^* at the point Ξ^*. Moreover, $p_1^* \cap p_2^*$ is the polar image of the line l through p_1 and p_2. Since the segment $[p_1, p_2]$ is not contained in the hypersubspace Γ, and x is a relatively interior point of $[p_1, p_2]$, the $(n-2)$-dimensional flat $p_1^* \cap p_2^*$ does not contain the line $\Xi^* + \Gamma^*$. (We recall that Γ^* is a one-dimensional *subspace* in $(\mathcal{R}^n)^*$, cf. Definition 5). Thus one of the rays defined on the line $\Xi^* + \Gamma^*$ by the point Ξ^* is not contained in the half-space P_1, whereas the second ray is not contained in the half-space P_2. This means that none of the rays is contained in the supporting cone of Σ^* at the point Ξ^*, i.e., the line $\Xi^* + \Gamma^*$ is not a half-tangent for the body Σ^*. Consequently Γ^* does not have a belt direction with respect to Σ^*.

Conducting the above reasoning in the opposite direction, we prove the converse assertion: If the line $\Gamma^* \subset (\mathcal{R}^n)^*$ does not have a belt direction with respect to the polar body Σ^*, then the linear hypersubspace $\Gamma \subset \mathcal{R}^n$ is not equatorial. $\square$

The following definition of the Helly dimension of a Minkowski space is more explicit than that given in [17] and [6], Section 14.

Definition 6. *Denote by B the set of all equatorial hypersubspaces of the Minkowski space $\mathcal{R}^n$ and by B^* the set of all polar lines Γ^* for $\Gamma \in B$. In other words, B^* is the set of all one-dimensional subspaces in $(\mathcal{R}^n)^*$ which have belt directions with respect to Σ^*. For every line $\Gamma^* \in B^*$ consider two points at which Γ^* intersects* $\operatorname{bd}\Sigma^*$ *and denote by $H(\mathcal{R}^n)$ the union of all these point pairs. The number* $\operatorname{md} H(\mathcal{R}^n)$ *is said to be the Helly dimension of the Minkowski space $\mathcal{R}^n$ and is denoted by* $\operatorname{him}\mathcal{R}^n$.

Theorem 10. *For every compact, d-convex body $M \subset \mathcal{R}^n$ the inequality* $\operatorname{md} M \le \operatorname{him}\mathcal{R}^n$ *holds.*

Proof. In the space $\mathcal{R}^n$, we introduce a Euclidean metric (besides the Minkowski metric $d(x, y)$). Then it is possible to consider orthogonality of vectors and scalar product. This allows to use unit outward normals at regular boundary points of a convex body M and to define $H(M)$ and $\operatorname{md} M$.

Let $M \subset \mathcal{R}^n$ be a compact, d-convex body, p be a regular boundary point of M, and a be the unit outward normal of M at the point p. Denote by L the supporting hyperplane of M at the point p and by $\Gamma \subset \mathcal{R}^n$ the hypersubspace parallel to L. Since the body M is d-convex and the point $p \in \operatorname{bd} M$ is regular, the hyperplane L is d-convex (Theorem 8) and hence the hypersubspace Γ is equatorial (Theorem 7). Now $f_a(x) = \langle a, x\rangle$ is a linear functional with the kernel Γ defined on $\mathcal{R}^n$ (since $a \perp L$, i.e., $a \perp \Gamma$). Consequently f belongs to the one-dimensional subspace $\Gamma^* \subset \mathcal{R}^n$. This means that for a suitable positive factor μ the inclusion $\mu f_a \in H(\mathcal{R}^n)$ holds (cf. Definition 6).

Denote the number $\operatorname{md} M$ by m and let $p_0, p_1, \dots, p_m$ be regular boundary points of M such that the corresponding unit outward normals $a_0, a_1, \dots, a_m$ are minimally dependent. For suitable positive factors $\mu_0, \dots \mu_m$, the functionals $\mu_0 f_{a_0}, \dots, \mu_m f_{a_m}$ belong to $H(\mathcal{R}^n)$. Since the vectors $a_0, \dots, a_m$ are minimally dependent, the corresponding functionals $f_{a_0}, \dots, f_{a_m}$ also are minimally dependent (with the same positive dependence). Consequently the functionals $\mu_0 f_{a_0}, \dots, \mu_m f_{a_m}$ are minimally dependent, too (since positive multipliers do not influence minimal dependence). Thus we have $m + 1$ minimally dependent functionals $\mu_0 f_{a_0}, \dots, \mu_m f_{a_m}$ in $H(\mathcal{R}^n)$. But $\operatorname{md} H(\mathcal{R}^n)$ is the greatest of the integers k such that there are $k+1$ minimally dependent vectors in $H(\mathcal{R}^n)$. Hence $m \le \operatorname{md} H(\mathcal{R}^n)$, i.e., $\operatorname{md} M \le \operatorname{him}\mathcal{R}^n$. $\square$

The following assertion is a Helly type theorem for Minkowski spaces and explains the name "Helly dimension" for $\operatorname{him}\mathcal{R}^n$.

Theorem 11. *Let $M_1, \dots, M_s \subset \mathcal{R}^n$ be d-convex bodies. If every* $\operatorname{him}\mathcal{R}^n + 1$ *of them have a common point, then $M_1 \cap \dots \cap M_s \neq \emptyset$.*

Proof. Let $M \subset \mathcal{R}^n$ be a d-convex body, p be a regular boundary point of M, and a be the corresponding unit outward normal of M. Then the functional $f_a(x)$ (multiplied by a suitable positive factor) belongs to $H(\mathcal{R}^n)$ (cf. the previous proof). This is true for every regular boundary point of M. Now the theorem follows immediately from the main result of [2] (cf. also Theorem 22.1 in the monograph [6]). □

Now we again pay attention to fixing and hindering systems for d-convex bodies.

Theorem 12. *For every compact, d-convex body $M \subset \mathcal{R}^n$ the inequalities*

$$2 \le \sigma_{\min}(M) \le \sigma'_{\min}(M) \le \operatorname{him}\mathcal{R}^n + 1 \le n + 1, \tag{8}$$

$$n + 1 \le n + \frac{n}{\operatorname{him}\mathcal{R}^n} \le \varrho_{\min}(M) \le 2n \tag{9}$$

hold.

This follows immediately from Theorems 1, 2, and 10.

We remark that not every Minkowski space has compact, d-convex bodies. For example, if $\mathcal{R}^n$ is the Minkowski space with a cube as the unit ball, then (for $n \ge 3$) there is no compact, d-convex body in $\mathcal{R}^n$ (cf. Example 9.3 in the monograph [6]). A Minkowski space in which there are compact, d-convex bodies is said to be an α-space. Thus the above Theorem 12 is interesting only for α-spaces.

We recall that a compact, convex body $M \subset \mathbf{R}^n$ is said to be a *belt body* if for any vector $v \in H(M)$ and an arbitrary positive number ε there are $n-1$ linearly independent belt vectors $a_1, \dots, a_{n-1}$ of the body M (i.e., vectors defining belt directions) such that the hyperplane spanned by them has a unit normal p satisfying $\| p - v \| < \varepsilon$, cf. [4]. In particular, each zonoid is a belt body.

Theorem 13. *Let $\mathcal{R}^n$ be a Minkowski space and Σ be its unit ball. Assume that the polar body $\Sigma^* \subset (\mathcal{R}^n)^*$ be a belt body (in particular, a zonoid), and let $\Sigma^* = \Xi_1 \oplus \dots \oplus \Xi_s$ be its decomposition into the direct vector sum of indecomposable, convex sets. Then $\mathcal{R}^n$ is an α-space and for every compact, d-convex body $M \subset \mathcal{R}^n$ the inequalities*

$$2 \le \sigma_{\min}(M) \le \sigma'_{\min}(M) \le \operatorname{mid}\mathcal{R}^n + 1 \tag{10}$$

hold, where

$$\operatorname{mid}\mathcal{R}^n = \min(\dim \Xi_1, \dots, \dim \Xi_s).$$

Proof. Since Σ^* is a belt body, the set B^* of its belt directions is not situated in a hyperplane of the space $(\mathcal{R}^n)^*$. Consequently there are n linearly independent belt vectors, which implies that $\mathcal{R}^n$ is an α-space.

Furthermore, the decomposition $\Sigma^* = \Xi_1 \oplus \dots \oplus \Xi_s$ gives the dual representation $\Sigma = \operatorname{conv}(\Xi_1^* \cup \dots \cup \Xi_s^*)$, where $\dim \Xi_i^* = \dim \Xi_i$ for every $i = 1, \dots, s$ and $\mathcal{R}^n = \operatorname{aff}\Xi_1^* \oplus \dots \oplus \operatorname{aff}\Xi_s^*$. Every compact, d-convex body $M \subset \mathcal{R}^n$ has the form $M = M_1 \oplus \dots \oplus M_s$, where M_i is a

d-convex body in the subspace $\mathcal{R}_i = \operatorname{aff} \Xi_i^*$ (considered as the Minkowski space with the unit ball Ξ_i^*). By Theorem 6, this implies

$$\sigma'_{\min}(M) = \min(\sigma'_{\min}(M_1), \dots, \sigma'_{\min}(M_s)) \leq$$
$$\min(\dim M_1 + 1, \dots, \dim M_s + 1) =$$
$$\min(\dim \Xi_1 + 1, \dots, \dim \Xi_s + 1) = \operatorname{mid} \mathcal{R}^n + 1,$$

which proves (10). □

We remark that in the notation of Theorem 13, the Helly dimension $\operatorname{him} \mathcal{R}^n$ is equal to $\max(\operatorname{him} \mathcal{R}_1, \dots, \operatorname{him} \mathcal{R}_s)$, where $\mathcal{R}_i$ is the Minkowski space $\operatorname{aff} \Xi_i^*$ with the unit ball Ξ_i^*. By (9) this gives a lower estimate for $\varrho_{\min}(M)$.

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